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ABSTRACT. We give a corrected proof of the main Lemma 2 from the paper in the title (our Corollary 7).

1. INTRODUCTION

The paper [H] was concerned with the problem of describing the closure (in the topology of uniform convergence on the unit ball) of the algebra $\mathcal{A}_n(X)$ of polynomials generated by all polynomials of degree at most n on the Banach space X. Of course, if X is finite-dimensional, then this situation is covered by the classical theorem of Stone and Weierstraß, so our interest lied with the case of infinite-dimensional Banach spaces X. This natural problem was suggested to us by Richard Aron, but its origin can be traced back to Shilov ([A1], [A2], [S]), and some early partial results on it were obtained in [NS].

The principal tool for obtaining our results in [H] was the finite-dimensional quantitative Lemma 2, which was obtained as a by-product of a new theory of algebraic bases for algebras of sub-symmetric polynomials on $\mathbb{R}^N$.

Unfortunately, the arguments in [H] contain a serious gap, which was recently spotted by the third one of the present authors. More precisely, the power series on top of page 213 should have been correctly centred at the point $(x_1^0, \ldots, x_n^0)$, rather than at the origin. It is not clear to us at the present moment if this problem can be fixed, and so the theory of algebraic bases developed in [H] remains to be only a conjecture.

In the present note we give a different proof of the above mentioned lemma, which corresponds to our Corollary 7. As a result, all the infinite dimensional applications stated in [H], as well as in several papers by various authors which have relied on our previous work (e.g. [DD], [DG]) remain valid. In fact, the strongest results concerning polynomial algebras are contained in the paper [DAH], which is also based on the lemma in question.

For more detailed introduction to the subject and more references we refer to [DAH]. Let us now proceed with the corrected proof of Corollary 7.

2. PROOF OF THE MAIN RESULT

By $\mathbb{N}_0$ we denote the set $\mathbb{N} \cup \{0\}$, i.e. the non-negative integers. The canonical basis of $\mathbb{R}^N$ will be denoted by $\{e_j\}_{j=1}^N$. By $Df(x)$ we denote the Fréchet derivative of the function f at the point x. By $\mathcal{P}(^d\mathbb{R}^N)$, $\mathcal{P}^d(\mathbb{R}^N)$, and $\mathcal{P}(\mathbb{R}^N)$ we denote the space of real d-homogeneous polynomials, polynomials of degree at most d, and all polynomials on $\mathbb{R}^N$, respectively. We say that a polynomial $P \in \mathcal{P}(\mathbb{R}^N)$ is sub-symmetric if

$$P\left(\sum_{j=1}^k x_j e_{n_j}\right) = P\left(\sum_{j=1}^k x_j e_j\right)$$

whenever $1 \le k < N$, $x_1, \ldots, x_k \in \mathbb{R}$, and $1 \le n_1 < \cdots < n_k \le N$.

Let $n \in \mathbb{N}$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ we denote its order by $|\alpha| = \sum_{j=1}^n \alpha_j$. Further, we denote the set of multi-indices of length k and order $d \in \mathbb{N}_0$ by

$$\mathcal{I}(n, d) = \{\alpha \in \{0, \ldots, d\}^n;\ |\alpha| = d\}.$$

For $n, d \in \mathbb{N}$ we denote $\mathcal{I}^+(n, d) = \{\alpha \in \mathcal{I}(n, d);\ \alpha_j > 0,\ j = 1, \ldots, n\}$ and $\mathcal{I}^+(d) = \bigcup_{n=1}^d \mathcal{I}^+(n, d)$.

Given $k, N \in \mathbb{N}$, $k \le N$, and $\alpha \in \mathcal{I}^+(k, d)$ we define $P_\alpha^N \in \mathcal{P}(^d\mathbb{R}^N)$ by

$$P_\alpha^N(x) = \sum_{1 \le \rho_1 < \cdots < \rho_k \le N} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k}. \tag{1}$$

For $N \ge d$ the polynomials $\{P_\alpha^N;\ \alpha \in \mathcal{I}^+(d)\}$ form a linear basis of the space of sub-symmetric d-homogeneous polynomials on $\mathbb{R}^N$. An important special case of these polynomials are the power sum symmetric polynomials $s_n^N(x) = P_{(n)}^N(x) = x_1^n + \cdots + x_N^n$.

Our main result concerns the properties of sub-symmetric polynomials, however in its proof we need to work also with partial derivatives of the polynomials P_α^N and for this reason we consider also the polynomials P_α^N given by the formula (1) where $\alpha \in \mathcal{I}(k, d)$, $k \le N$, using the convention that $x^0 = 1$ for every $x \in \mathbb{R}$. We denote by $H^{n,K}(\mathbb{R}^N)$ the subspace of $\mathcal{P}^n(\mathbb{R}^N)$

generated by the polynomials P_α^N, $\alpha \in \bigcup_{d=0}^n \bigcup_{k=1}^K \mathcal{I}(k,d)$. For formal reasons we also put $P_\alpha^N = 0$ if $k > N$ and $P_{()}^N = 1$, both even for $N = 0$, further $\mathcal{I}(0,0) = \{()\}$, and $\mathbb{R}^0 = \{0\}$. Note that these definitions are consistent with (1), using the convention that a sum over an empty set is zero and a product over an empty set is equal to 1.

The following fact describes an important relation between the restriction of P_α^M to the first N coordinates and P_α^N. Note that for $M > N$ we consider canonically $\mathbb{R}^N$ as a subspace of $\mathbb{R}^M$.

Fact 1. *Let $M, N, k, d \in \mathbb{N}_0$, $N < M$, and $\alpha \in \mathcal{I}(k,d)$ be such that $\alpha_m > 0$ and $\alpha_{m+1} = \cdots = \alpha_k = 0$ for some $0 \le m \le k$. Then*

$$P_\alpha^M(x) = \sum_{j=m}^k \binom{M-N}{k-j} P_{(\alpha_1,\ldots,\alpha_j)}^N(x)$$

for every $x \in \mathbb{R}^N$. Conversely,

$$P_\alpha^N(x) = \sum_{j=m}^k (-1)^{k-j} \binom{M-N+k-j-1}{k-j} P_{(\alpha_1,\ldots,\alpha_j)}^M(x)$$

for every $x \in \mathbb{R}^N$.

Proof. The first relation follows from the following (recall that $x \in \mathbb{R}^N$, i.e. $x_{N+1} = \cdots = x_M = 0$ as per the aforementioned convention):

$$P_\alpha^M(x) = \sum_{1 \le \rho_1 < \cdots < \rho_k \le M} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m} = \sum_{\substack{1 \le \rho_1 < \cdots < \rho_k \le M \\ \rho_m \le N}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m}$$

$$= \sum_{j=m}^k \sum_{\substack{1 \le \rho_1 < \cdots < \rho_k \le M \\ \rho_j \le N < \rho_{j+1}}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_m}^{\alpha_m} = \sum_{j=m}^k \binom{M-N}{k-j} P_{(\alpha_1,\ldots,\alpha_j)}^N(x).$$

The second relation can be proved by induction on $k - m$. For $k - m = 0$ it follows immediately from the first one. For the induction step we use the first relation together with the inductive hypothesis to obtain

$$P_\alpha^N(x) = P_\alpha^M(x) - \sum_{j=m}^{k-1} \binom{M-N}{k-j} P_{(\alpha_1,\ldots,\alpha_j)}^N(x)$$

$$= P_\alpha^M(x) - \sum_{j=m}^{k-1} \binom{M-N}{k-j} \sum_{l=m}^j (-1)^{j-l} \binom{M-N+j-l-1}{j-l} P_{(\alpha_1,\ldots,\alpha_l)}^M(x)$$

$$= P_\alpha^M(x) - \sum_{l=m}^{k-1} \left(\sum_{j=l}^{k-1} (-1)^{j-l} \binom{M-N}{k-j} \binom{M-N+j-l-1}{j-l} \right) P_{(\alpha_1,\ldots,\alpha_l)}^M(x)$$

and the result now follows from the identity $\sum_{j=l}^k (-1)^{j-l} \binom{M-N}{k-j} \binom{M-N+j-l-1}{j-l} = 0$. Adding or removing a couple of zero summands, this is equivalent to $\sum_{p=0}^{M-N} (-1)^{k-l-p} \binom{M-N}{p} \binom{M-N+k-l-p-1}{M-N-1} = 0$, which is the Fréchet formula for the polynomial $t \mapsto \binom{M-N+k-l-t-1}{M-N-1}$ of degree $M - N - 1$ ([F], or [HK] for a more recent proof). $\square$

It is very important to notice that the previous fact covers all the special cases like $N < k \le M$, $k > M$, $N = 0$, $m = 0$, or $k = 0$. Observe also that in particular in the sub-symmetric case (i.e. $\alpha \in \mathcal{I}^+(d)$) we have $P_\alpha^M \restriction_{\mathbb{R}^N} = P_\alpha^N$. Hence for sub-symmetric polynomials the superscript N can be dropped. We will use this simplification for the polynomials $s_n^N = s_n$.

The next fact deals with the situation when we fix the first N coordinates of P_α^M.

Fact 2. *Let $N, d \in \mathbb{N}_0$, $M, k \in \mathbb{N}$, $N < M$, $k \le M$, $\alpha \in \mathcal{I}(k,d)$, and $y \in \mathbb{R}^N$. Then the polynomial $(x_1,\ldots,x_{M-N}) \mapsto P_\alpha^M(y_1,\ldots,y_N,x_1,\ldots,x_{M-N})$ belongs to $H^{d,\min\{k,M-N\}}(\mathbb{R}^{M-N})$.*

Proof.

$$P_\alpha^M(y_1,\ldots,y_N,x_1,\ldots,x_{M-N}) = \sum_{j=0}^k \sum_{\substack{1 \le \rho_1 < \cdots < \rho_k \le M \\ \rho_j \le N < \rho_{j+1}}} y_{\rho_1}^{\alpha_1} \cdots y_{\rho_j}^{\alpha_j} x_{\rho_{j+1}-N}^{\alpha_{j+1}} \cdots x_{\rho_k-N}^{\alpha_k}$$

$$= \sum_{\substack{0 \le j \le k \\ k-(M-N) \le j \le N}} P_{(\alpha_1,\ldots,\alpha_j)}^N(y) P_{(\alpha_{j+1},\ldots,\alpha_k)}^{M-N}(x_1,\ldots,x_{M-N}).$$

$\square$

Let $k, d \in \mathbb{N}$, $\alpha \in \mathcal{I}(k,d)$, $k \leq N$, $x \in \mathbb{R}^N$, and $1 \leq l \leq N$. Then

$$\frac{\partial P_\alpha^N}{\partial x_l}(x) = \frac{\partial}{\partial x_l}\left(\sum_{j=1}^{k} \sum_{\substack{1 \leq \rho_1 < \cdots < \rho_k \leq N \\ \rho_j = l}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_k}^{\alpha_k}\right) = \sum_{\substack{j=1 \\ \alpha_j > 0}}^{k} \alpha_j \sum_{\substack{1 \leq \rho_1 < \cdots < \rho_{j-1} < l \\ l < \rho_{j+1} < \cdots < \rho_k \leq N}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_l^{\alpha_j - 1} x_{\rho_{j+1}}^{\alpha_{j+1}} \cdots x_{\rho_k}^{\alpha_k} \tag{2}$$

$$= \sum_{\substack{j=1 \\ \alpha_j > 0}}^{k} \alpha_j P_{(\alpha_1,\ldots,\alpha_{j-1})}^{l-1}(x_1,\ldots,x_{l-1}) x_l^{\alpha_j - 1} P_{(\alpha_{j+1},\ldots,\alpha_k)}^{N-l}(x_{l+1},\ldots,x_N).$$

These partial derivatives have the following useful property:

Fact 3. *Let $k, d, N \in \mathbb{N}$, $\alpha \in \mathcal{I}(k,d)$, $k \leq N$. Then $\sum_{l=1}^{N} \frac{\partial P_\alpha^N}{\partial x_l} \in H^{d-1,k}(\mathbb{R}^N)$.*

Proof.

$$\sum_{l=1}^{N} \frac{\partial P_\alpha^N}{\partial x_l}(x) = \sum_{l=1}^{N} \sum_{\substack{j=1 \\ \alpha_j > 0}}^{k} \alpha_j \sum_{\substack{1 \leq \rho_1 < \cdots < \rho_{j-1} < l \\ l < \rho_{j+1} < \cdots < \rho_k \leq N}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_l^{\alpha_j - 1} x_{\rho_{j+1}}^{\alpha_{j+1}} \cdots x_{\rho_k}^{\alpha_k}$$

$$= \sum_{\substack{j=1 \\ \alpha_j > 0}}^{k} \alpha_j \sum_{l=1}^{N} \sum_{\substack{1 \leq \rho_1 < \cdots < \rho_k \leq N \\ \rho_j = l}} x_{\rho_1}^{\alpha_1} \cdots x_{\rho_{j-1}}^{\alpha_{j-1}} x_{\rho_j}^{\alpha_j - 1} x_{\rho_{j+1}}^{\alpha_{j+1}} \cdots x_{\rho_k}^{\alpha_k} = \sum_{\substack{j=1 \\ \alpha_j > 0}}^{k} \alpha_j P_{(\alpha_1,\ldots,\alpha_{j-1},\alpha_j-1,\alpha_{j+1},\alpha_k)}^{N}(x).$$

$\square$

We note that this fact does not hold with $\mathcal{I}^+(k,d)$ and the space of sub-symmetric polynomials in place of $\mathcal{I}(k,d)$ and $H^{d-1,k}(\mathbb{R}^N)$, and this is the sole reason for considering the larger spaces $H^{n,K}(\mathbb{R}^N)$.

For each $x \in \mathbb{R}^N$ we naturally identify $DP_\alpha^N(x)$ with the vector $\left(\frac{\partial P_\alpha^N}{\partial x_1}(x),\ldots,\frac{\partial P_\alpha^N}{\partial x_N}(x)\right) \in \mathbb{R}^N$.

Fact 4. *Let $M, N, k, d \in \mathbb{N}$, $M > N$, $\alpha \in \mathcal{I}(k,d)$, $k \leq N$, and $x \in \mathbb{R}^N$. Then $DP_\alpha^N(x)$ is a linear combination of vectors $DP_\beta^M(x) \restriction_N = \left(\frac{\partial P_\beta^M}{\partial x_1}(x),\ldots,\frac{\partial P_\beta^M}{\partial x_N}(x)\right) \in \mathbb{R}^N$, $\beta \in \bigcup_{m=1}^{k} \mathcal{I}(m,d)$.*

Proof. Let $1 \leq m \leq k$ be such that $\alpha_m > 0$ and $\alpha_{m+1} = \cdots = \alpha_k = 0$. Fix $1 \leq l \leq N$. If $\alpha_j > 0$, then $m \geq j$ and hence by Fact 1

$$P_{(\alpha_{j+1},\ldots,\alpha_k)}^{N-l}(x_{l+1},\ldots,x_N) = \sum_{s=m}^{k} c_s P_{(\alpha_{j+1},\ldots,\alpha_s)}^{M-l}(x_{l+1},\ldots,x_N,0,\ldots,0),$$

where $c_s = (-1)^{k-s}\binom{M-N+k-s-1}{k-s}$. Therefore using (2) and the fact that $\alpha_{s+1} = \cdots = \alpha_k = 0$ if $m \leq s \leq k$ we obtain

$$\frac{\partial P_\alpha^N}{\partial x_l}(x) = \sum_{\substack{j=1 \\ \alpha_j > 0}}^{k} \alpha_j P_{(\alpha_1,\ldots,\alpha_{j-1})}^{l-1}(x_1,\ldots,x_{l-1}) x_l^{\alpha_j - 1} \sum_{s=m}^{k} c_s P_{(\alpha_{j+1},\ldots,\alpha_s)}^{M-l}(x_{l+1},\ldots,x_N,0,\ldots,0) = \sum_{s=m}^{k} c_s \frac{\partial P_{(\alpha_1,\ldots,\alpha_s)}^{M}}{\partial x_l}(x),$$

from which the statement follows.

$\square$

We will also make use of the following version of the Lagrange multipliers theorem.

Theorem 5. *Let $G \subset \mathbb{R}^n$ be an open set, $f \in C^1(G)$, $F \in C^1(G;\mathbb{R}^m)$, and assume that F has a constant rank. If the function f has a local extremum with respect to $M = \{x \in G;\ F(x) = 0\}$ at $a \in M$, then $Df(a)$ is a linear combination of $DF_1(a),\ldots,DF_m(a)$, where $F_1,\ldots,F_m$ are the components of the mapping F.*

Proof. Let $k = \operatorname{rank} F(x)$ for $x \in G$. Since DF is continuous, we may without loss of generality assume that $DF_1(x),\ldots,DF_k(x)$ are linearly independent for each $x \in G$. From the Rank theorem it follows that there are C^1-smooth functions g_j of k variables, $j = k+1,\ldots,m$, and a neighbourhood U of a such that $F_j(x) = g_j(F_1(x),\ldots,F_k(x))$ for each $x \in U$, $j = k+1,\ldots,m$ (see e.g. [Z, Proposition 8.6.3.1]). Notice that $g_j(0,\ldots,0) = g_j(F_1(a),\ldots,F_k(a)) = F_j(a) = 0$, $j = k+1,\ldots,m$. Therefore $M \cap U = \{x \in U;\ F_1(x) = 0,\ldots,F_k(x) = 0\}$ and we may use the classical version of the Lagrange multipliers theorem

$\square$

Now we are ready to prove the key lemma.

Lemma 6. *For every $n, K \in \mathbb{N}$ there are $N \in \mathbb{N}$ and $u, v \in \mathbb{R}^N$ such that $P(u) = P(v)$ for every $P \in H^{n,K}(\mathbb{R}^N)$ but $s_{n+1}(u) \neq s_{n+1}(v)$.*

Proof. The proof is based on the observation that $\sum_{l=1}^{N} \frac{\partial s_{n+1}}{\partial x_l}(x) = (n+1)s_n(x)$, which together with Fact 3 leads to an inductive proof. For each fixed $K \in \mathbb{N}$ we prove the statement by induction on n. So fix $K \in \mathbb{N}$ and denote $\mathcal{M}(n) = \bigcup_{1 \le d \le n} \bigcup_{1 \le k \le K} \mathcal{I}(k, d)$. The space $H^{n,K}(\mathbb{R}^N)$ is generated by a constant function and polynomials P_α^N, $\alpha \in \mathcal{M}(n)$. For $n = 1$ the functions P_α^N, $\alpha \in \mathcal{M}(n)$ are linear and so there is $N \in \mathbb{N}$ large enough such that $\bigcap_{\alpha \in \mathcal{M}(n)} \ker P_\alpha^N$ contains a non-zero element u. Then it suffices to take $v = 2u$.

The inductive step from $n - 1$ to n will be proved by contradiction. So assume that for each $N \ge K$ and each $u, v \in \mathbb{R}^N$ satisfying $P_\alpha^N(u) = P_\alpha^N(v)$ for all $\alpha \in \mathcal{M}(n)$ we have $s_{n+1}(u) = s_{n+1}(v)$. Now let $F^N \colon \mathbb{R}^N \to \mathbb{R}^{|\mathcal{M}(n)|}$ be the mapping whose components are the polynomials P_α^N, $\alpha \in \mathcal{M}(n)$ in some fixed order and let $\mathbb{A}_N(x)$ be its Jacobi matrix at $x \in \mathbb{R}^N$, i.e. $\mathbb{A}_N(x) = \left(\frac{\partial P_\alpha^N}{\partial x_l}(x)\right)_{\substack{\alpha \in \mathcal{M}(n) \\ l=1,\dots,M}}$. Note that the number of rows of the matrix of functions $\mathbb{A}_N$ does not depend on N. Thus there is $N \ge K$ and $y \in \mathbb{R}^N$ such that $\operatorname{rank} \mathbb{A}_N(y) = r = \max_{M \ge K, x \in \mathbb{R}^M} \operatorname{rank} \mathbb{A}_M(x)$.

By the inductive hypothesis there are $M > N$ and $g, h \in \mathbb{R}^{M-N}$ such that $P(g) = P(h)$ for all $P \in H^{n-1,K}(\mathbb{R}^{M-N})$ but $s_n(g) \ne s_n(h)$. If we denote by $\mathbb{A}_M(x) \restriction_N$ the matrix consisting of the first N columns of the matrix $\mathbb{A}_M(x)$, then $r = \operatorname{rank} \mathbb{A}_N(y) \le \operatorname{rank} \mathbb{A}_M(y) \restriction_N \le \operatorname{rank} \mathbb{A}_M(y) \le r$, where the first inequality follows from Fact 4. Let $w_1^M, \dots, w_r^M$ be the rows of $\mathbb{A}_M$ such that $w_1^M(y) \restriction_N, \dots, w_r^M(y) \restriction_N$ are linearly independent. Using the continuity of the entries of $\mathbb{A}_M$ it is easy to see that there is a neighbourhood $U \subset \mathbb{R}^M$ of y such that for each $x \in U$ the vectors $w_1^M(x) \restriction_N, \dots, w_r^M(x) \restriction_N$ are linearly independent and so they form a basis of the space spanned by the rows of $\mathbb{A}_M(x) \restriction_N$. Clearly the same holds for $w_1^M(x), \dots, w_r^M(x)$ and $\mathbb{A}_M(x)$.

Fix an arbitrary $z \in U$ and put $S = \{x \in U; \ P_\alpha^M(x) = P_\alpha^M(z), \alpha \in \mathcal{M}(n)\}$. By our assumption s_{n+1} is constant on S and so Theorem 5 implies that $Ds_{n+1}(z)$ is a linear combination of the rows of $\mathbb{A}_M(z)$. It follows that for each $z \in U$ the vector $Ds_{n+1}(z)$ is a linear combination of $w_1^M(z), \dots, w_r^M(z)$.

Next, we put $u = y + c \sum_{j=1}^{M-N} g_j e_{N+j}$, $v = y + c \sum_{j=1}^{M-N} h_j e_{N+j}$ for some suitable $c \ne 0$ so that $u, v \in U$. Notice that since $H^{n-1,K}(\mathbb{R}^{M-N})$ is generated by homogeneous polynomials, we still have $P(cg) = P(ch)$ for all $P \in H^{n-1,K}(\mathbb{R}^{M-N})$ but $s_n(cg) \ne s_n(ch)$. For a fixed $\alpha \in \mathcal{M}(n)$ and $1 \le l \le N$ consider the polynomial $P(x) = \frac{\partial P_\alpha^M}{\partial x_l}(y_1, \dots, y_N, x_1, \dots, x_{M-N})$. Then by (2) and Fact 2 we have $P \in H^{n-1,K}(\mathbb{R}^{M-N})$ and so $P(cg) = P(ch)$. Therefore

$$w_j^M(u) \restriction_N = w_j^M(v) \restriction_N, \quad j = 1, \dots, r. \tag{3}$$

We have $Ds_{n+1}(u) = \sum_{j=1}^{r} \lambda_j w_j^M(u)$ and $Ds_{n+1}(v) = \sum_{j=1}^{r} \mu_j w_j^M(v)$ for some $\lambda_j, \mu_j \in \mathbb{R}$ and of course the same holds when we restrict to the first N coordinates of all of these vectors. But since $Ds_{n+1}(u) \restriction_N = (n+1)(y_1^n, \dots, y_N^n) = Ds_{n+1}(v) \restriction_N$, combined with (3) and the fact that $w_1^M(u) \restriction_N, \dots, w_r^M(u) \restriction_N$ are linearly independent we obtain $\mu_j = \lambda_j$, $j = 1, \dots, r$. Finally, from Fact 3 and Fact 2 it follows that $x \mapsto \sum_{l=1}^{M} w_j^M \left(y + \sum_{j=1}^{M-N} x_j e_{N+j}\right)_l \in H^{n-1,K}(\mathbb{R}^{M-N})$, $j = 1, \dots, r$. Therefore

$$(n+1)s_n(u) = \sum_{l=1}^{M} \frac{\partial s_{n+1}}{\partial x_l}(u) = \sum_{j=1}^{r} \lambda_j \sum_{l=1}^{M} w_j^M(u)_l = \sum_{j=1}^{r} \lambda_j \sum_{l=1}^{M} w_j^M(v)_l = \sum_{l=1}^{M} \frac{\partial s_{n+1}}{\partial x_l}(v) = (n+1)s_n(v).$$

Since $s_n(u) = s_n(y) + s_n(cg)$ and $s_n(v) = s_n(y) + s_n(ch)$, we get $s_n(cg) = s_n(ch)$, which is a contradiction. $\square$

Recall that an algebra of polynomials on $\mathbb{R}^N$ is a subspace of $\mathcal{P}(\mathbb{R}^N)$ that is closed with respect to pointwise multiplication. Given an algebra $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^N)$ we say that the set $B \subset \mathcal{A}$ generates the algebra $\mathcal{A}$ if for every $p \in \mathcal{A}$ there is a subset $\{b_1, \dots, b_k\} \subset B$ and a polynomial $P \in \mathcal{P}(\mathbb{R}^k)$ such that $p = P \circ (b_1, \dots, b_k)$.

Corollary 7. *For every $n \in \mathbb{N}$ there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ such that for every $M \ge N$*

$$\sup_{x \in B_{\ell_1^M}} |p(x) - s_{n+1}(x)| \ge \varepsilon$$

for every p from the algebra generated by the sub-symmetric polynomials on $\mathbb{R}^M$ of degree at most n.

Proof. Applying Lemma 6 to $K = n$ we obtain $N \in \mathbb{N}$ and $u, v \in B_{\ell_1^N}$ such that $P(u) = P(v)$ for every $P \in H^{n,n}(\mathbb{R}^N)$ but $s_{n+1}(u) \ne s_{n+1}(v)$. We put $\varepsilon = \frac{1}{2}|s_{n+1}(u) - s_{n+1}(v)|$. Let $M \ge N$. Since all sub-symmetric polynomials from $\mathcal{P}^n(\mathbb{R}^N)$ are contained in $H^{n,n}(\mathbb{R}^N)$, from the remark after Fact 1 it follows that in particular $P(u) = P(v)$ for every sub-symmetric $P \in \mathcal{P}^n(\mathbb{R}^M)$. We conclude that $p(u) = p(v)$ for every p from the algebra generated by the sub-symmetric polynomials from $\mathcal{P}^n(\mathbb{R}^M)$. The statement now easily follows. $\square$

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