A REMARK ON THE APPROXIMATION THEOREMS OF WHITNEY AND CARLEMAN-SCHEINBERG

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ABSTRACT. We show that a C^k-smooth mapping on an open subset of $\mathbb{R}^n$, $k \in \mathbb{N} \cup \{0, \infty\}$, can be approximated in a fine topology and together with its derivatives by a restriction of a holomorphic mapping with explicitly described domain. As a corollary we obtain a generalisation of the Carleman-Scheinberg theorem on approximation by entire functions.

First we fix some notation. By $B(x, r)$ we denote the closed ball centred at x with radius $r > 0$. Let X, Y be normed linear spaces and $U \subset X$ an open set. Recall that a norm of a polynomial $P\colon X \to Y$ is defined by $\|P\| = \sup_{h \in B_X} \|P(h)\|$. For a mapping $f\colon U \to Y$ we denote by $d^k f(x)$ the kth Fréchet differential of f at $x \in U$, i.e. the k-homogeneous polynomial associated to the kth Fréchet derivative of f at x. For convenience we denote $d^0 f = f$ and accordingly by C^0-smooth mapping we mean a continuous mapping.

Let $G \subset \mathbb{C}^n$ be an open set and let Z be a complex Banach space. Recall that a mapping $f\colon G \to Z$ is holomorphic if it is Fréchet differentiable in G, and that this is equivalent to f being analytic, i.e. locally expandable into a power series (see e.g. [Hi]). By $H(G; Z)$ we denote the space of holomorphic mappings from G to Z. By $C^\omega(U; Y)$ we denote the space of real-analytic mappings from an open $U \subset \mathbb{R}^n$ to a real Banach space Y. For a real Banach space Y we denote by $\tilde{Y}$ its Taylor complexification, i.e. a complex Banach space which can be described as $Y + iY$ and with a norm $\|z\| = \|\operatorname{Re} z + i \operatorname{Im} z\| = \sup_{t \in [0, 2\pi]} \|\cos(t) \operatorname{Re} z + \sin(t) \operatorname{Im} z\|$; see e.g. [FHHMZ, Section 2.1]. Clearly, $\widetilde{\mathbb{R}^n}$ is isomorphic to $\mathbb{C}^n$. Throughout the paper we will be using the Euclidean norm on $\mathbb{C}^n$.

In his seminal paper [W] Hassler Whitney proved among other things that a C^k-smooth function on an open subset of $\mathbb{R}^n$ can be approximated in a fine topology and together with its derivatives by a real-analytic function. His proof however is quite technical and settles for real analyticity of the approximating functions, not dealing with the domain of their holomorphic extensions. Earlier, Torsten Carleman in [C] showed that any function continuous on $\mathbb{R}$ can be uniformly approximated by a restriction of an entire holomorphic function. The usual proofs of this result employ some relatively advanced techniques of complex analysis, e.g. Runge's approximation theorem. Surprisingly, this kind of proof appears even in recent books, although for example Stephen Scheinberg in [S] gave an ingenious short proof of the Carleman theorem and moreover generalised it for continuous functions on $\mathbb{R}^n$ (and the approximation is in a fine topology); the methods are used also earlier by Lothar Hoischen and in fact go even back to Karl Weierstraß. When comparing the results of Whitney and Carleman-Scheinberg, an immediate question arises: Can the approximation by entire functions (or even mappings) be done so that simultaneously also the derivatives are approximated? What about the domain of the approximation when the continuous function is not defined on the whole $\mathbb{R}^n$? The first question is treated e.g. in [K], where the approximation on $\mathbb{R}$ of the first derivative is achieved, and further in [Ho] and [FG], where the approximation of higher derivatives is proved on $\mathbb{R}$, resp. $\mathbb{R}^n$. The second question is studied e.g. in [N], where the so-called Carleman sets in $\mathbb{C}$ are studied.

We combine the techniques of Whitney and Scheinberg and by explicitly estimating the domain of holomorphy of the extension of the real-analytic approximating mappings we prove the following slight generalisation of the Whitney approximation theorem:

Theorem 1. *Let Y be a real Banach space, $\Omega \subset \mathbb{R}^n$ open, $k \in \mathbb{N} \cup \{0, \infty\}$, $f \in C^k(\Omega; Y)$, and $\varepsilon \in C(\Omega; \mathbb{R}^+)$. Put $G = \{z \in \mathbb{C}^n;\ \|\operatorname{Im} z\| < \operatorname{dist}(\operatorname{Re} z, \mathbb{R}^n \setminus \Omega)\}$. Then there is a mapping $g \in H(G; \tilde{Y})$ such that $g\restriction_\Omega \in C^\omega(\Omega; Y)$ and $\|d^j f(x) - d^j(g\restriction_\Omega)(x)\| < \varepsilon(x)$ for all $x \in \Omega$, $0 \le j \le \min\{k, 1/\varepsilon(x)\}$.*

In particular if $\Omega = \mathbb{R}^n$, then g is an entire mapping and we obtain the Frih-Gauthier version of the Carleman-Scheinberg theorem with approximations also for the derivatives.

For the proof we introduce some more notation. Let X be a set, Y a normed linear space, $f\colon X \to Y$, and $S \subset X$. We denote $\|f\|_S = \sup_{x \in S} \|f(x)\|$. Let X, Y be normed linear spaces, $\Omega \subset X$ open, and $f \in C^k(\Omega; Y)$ for some $k \in \mathbb{N} \cup \{0\}$. For $S \subset \Omega$ we define

$$\|f\|_{S,k} = \sum_{j=0}^{k} \sup_{x \in S} \|d^j f(x)\|.$$

Clearly $\|\cdot\|_{S,k}$ is a semi-norm on the subspace of $C^k(\Omega; Y)$ consisting of mappings with all derivatives up to k bounded on S.

Lemma 2. *Let X, Y be normed linear spaces, $\Omega \subset X$ open, $k \in \mathbb{N} \cup \{0\}$, $\varphi \in C^k(\Omega)$, $f \in C^k(\Omega; Y)$, and $S \subset \Omega$. Then*

$$\|\varphi f\|_{S,k} \le \binom{k}{\left[\frac{k}{2}\right]} \|\varphi\|_{S,k} \|f\|_{S,k}.$$

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Proof. Fix $x \in \Omega$ and $0 \le j \le k$. By the Leibniz formula

$$\|d^j(\varphi f)(x)\| \le \sum_{l=0}^{j}\binom{j}{l}\|d^{j-l}\varphi(x)\cdot d^l f(x)\| \le \sum_{l=0}^{j}\binom{j}{l}\|d^{j-l}\varphi(x)\|\cdot\|d^l f(x)\| \le \binom{j}{\left[\frac{j}{2}\right]}\sum_{l=0}^{j}\|d^{j-l}\varphi(x)\|\cdot\|d^l f(x)\|.$$

Therefore

$$\begin{aligned}\|\varphi f\|_{S,k} &\le \sum_{j=0}^{k}\binom{j}{\left[\frac{j}{2}\right]}\sum_{l=0}^{j}\|d^{j-l}\varphi\|_S\|d^l f\|_S \le \binom{k}{\left[\frac{k}{2}\right]}\sum_{j=0}^{k}\sum_{l=0}^{j}\|d^{j-l}\varphi\|_S\|d^l f\|_S\\ &\le \binom{k}{\left[\frac{k}{2}\right]}\sum_{j=0}^{k}\sum_{l=0}^{k}\|d^{j}\varphi\|_S\|d^l f\|_S = \binom{k}{\left[\frac{k}{2}\right]}\|\varphi\|_{S,k}\|f\|_{S,k}.\end{aligned}$$

□

To slightly shorten our notation we denote $\bar g = g\restriction_{M\cap\mathbb{R}^n}$ for $g\colon M \to Y$, where $M \subset \mathbb{C}^n$ and Y is a Banach space.

Lemma 3. *Let Y be a real Banach space, $k \in \mathbb{N}\cup\{0\}$, $f \in C^k(\mathbb{R}^n;Y)$ with compact support, and $\varepsilon > 0$. Then there is an entire mapping $g \in H(\mathbb{C}^n;\tilde Y)$ such that $\bar g$ maps into Y, $\|f-\bar g\|_{\mathbb{R}^n,k} < \varepsilon$, and $\|g\|_G < \varepsilon$, where*

$$G = \left\{z \in \mathbb{C}^n;\ \|\operatorname{Im} z\|^2 < \operatorname{dist}(\operatorname{Re} z, \operatorname{supp} f)^2 - \varepsilon^2\right\}.$$

Proof. Put $\Psi_\kappa(z) = \exp\left(-\kappa\sum_{j=1}^{n} z_j^2\right)$ for $z \in \mathbb{C}^n$ and for $\kappa > 0$ define $g_\kappa\colon \mathbb{C}^n \to \tilde Y$ by the Bochner integral

$$g_\kappa(z) = \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\Psi_\kappa(z-u)f(u)\,\mathrm{d}u,$$

where $c_\kappa = \int_{\mathbb{R}^n}\Psi_\kappa(u)\,\mathrm{d}u = \left(\frac{\pi}{\kappa}\right)^{\frac{n}{2}}$. Using standard theorems on integrals dependent on a parameter we obtain $g_\kappa \in H(\mathbb{C}^n;\tilde Y)$. Obviously $\bar{g_\kappa}$ maps into Y. Using the uniform continuity of $d^j f$ on $\mathbb{R}^n$ for $0 \le j \le k$ it is standard to check that $\|f-\bar{g_\kappa}\|_{\mathbb{R}^n,k} < \varepsilon$ for κ large enough.

For any $z \in G$ we can estimate

$$\begin{aligned}\|g_\kappa(z)\| &\le \frac{1}{c_\kappa}\int_{\mathbb{R}^n}|\Psi_\kappa(z-u)|\|f(u)\|\,\mathrm{d}u = \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\exp\Bigg(-\kappa\sum_{j=1}^{n}\operatorname{Re}(z_j-u_j)^2\Bigg)\|f(u)\|\,\mathrm{d}u\\ &= \frac{1}{c_\kappa}\int_{\operatorname{supp} f}\exp\Bigg(-\kappa\sum_{j=1}^{n}\left((\operatorname{Re} z_j-u_j)^2-(\operatorname{Im} z_j)^2\right)\Bigg)\|f(u)\|\,\mathrm{d}u\\ &= \frac{1}{c_\kappa}\int_{\operatorname{supp} f}\exp\left(-\kappa\left(\|\operatorname{Re} z-u\|^2-\|\operatorname{Im} z\|^2\right)\right)\|f(u)\|\,\mathrm{d}u \le \frac{1}{c_\kappa}\exp\left(-\kappa\varepsilon^2\right)\int_{\operatorname{supp} f}\|f(u)\|\,\mathrm{d}u.\end{aligned}$$

Clearly, $\|g_\kappa\|_G < \varepsilon$ is satisfied for κ large enough.

□

Proof of Theorem 1. Define $K_{-1} = K_0 = \emptyset$, $K_j = \{x \in \mathbb{R}^n;\ \operatorname{dist}(x,\mathbb{R}^n\setminus\Omega) \ge 2^{-j}\}\cap B(0,j)$, $L_j = K_j\setminus\operatorname{Int} K_{j-1}$, and $U_j = (\operatorname{Int} K_{j+1})\setminus K_{j-2}$ for $j \in \mathbb{N}$. Note that $K_j \subset K_{j+1}$, L_j is compact, $U_j \subset \Omega$ is an open neighbourhood of L_j, $\Omega = \bigcup_{j=1}^{\infty} L_j$, and $L_j \cap U_l = \emptyset$ for $l > j+1$. There are functions $\varphi_j \in C^\infty(\mathbb{R}^n;[0,1])$, $j \in \mathbb{N}$, satisfying $\operatorname{supp}\varphi_j \subset U_j$ (hence $\operatorname{supp}\varphi_j$ is compact) and $\varphi_j = 1$ on a neighbourhood of L_j.

Further, we put $\varepsilon_0 = 1$, $\varepsilon_j = \min\{\varepsilon_{j-1}, \min_{x\in L_j}\varepsilon(x)\}$, $k_0 = 0$, $k_j = k$ if $k < \infty$, and finally $k_j = \max\{k_{j-1}, [\max_{x\in L_j}\frac{1}{\varepsilon(x)}]\}$ if $k = \infty$. Notice that the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ is non-increasing, while the sequence $\{k_j\}_{j=1}^{\infty}$ is non-decreasing. Put $M_j = \nu_{k_j}\|\varphi_j\|_{\mathbb{R}^n,k_j}$, where $\nu_l = \binom{l}{\left[\frac{l}{2}\right]}$. For each $j \in \mathbb{N}$ let $\delta_j > 0$ be such that

$$\delta_j(1+M_{j+1}) < \frac{\varepsilon_j}{2^j}. \tag{1}$$

For each $j \in \mathbb{N}$ we define inductively mappings $f_j \in C^k(\mathbb{R}^n;Y)$ and $g_j \in H(\mathbb{C}^n;\tilde Y)$ such that $\bar{g_j}$ maps into Y as follows: We put $f_j = 0$ on $\mathbb{R}^n\setminus\Omega$ and

$$f_j = \varphi_j\cdot\Bigg(f-\sum_{l=1}^{j-1}\bar{g_l}\Bigg) \tag{2}$$

on Ω. Then $f_j \in C^k(\mathbb{R}^n;Y)$ and since $\operatorname{supp}\varphi_j$ is compact, so is $\operatorname{supp} f_j$. By Lemma 3 there is a mapping $g_j \in H(\mathbb{C}^n;\tilde Y)$ such that $\bar{g_j}$ maps into Y,

$$\|f_j-\bar{g_j}\|_{\mathbb{R}^n,k_j} < \delta_j, \tag{3}$$

and $\|g_j\|_{G_j} < \frac{1}{2^j}$, where $G_j = \left\{z \in \mathbb{C}^n;\ \|\operatorname{Im} z\|^2 < \operatorname{dist}(\operatorname{Re} z,\operatorname{supp} f_j)^2-\frac{1}{4^j}\right\}$.

Put

$$g = \sum_{j=1}^{\infty} g_j. \tag{4}$$

Fix any $z \in G$ and put $\delta = \frac{1}{2}\min\{\operatorname{dist}(\operatorname{Re} z, \mathbb{R}^n \setminus \Omega) - \|\operatorname{Im} z\|, 1\}$. (We note that the minimum here is to cater for the case when $\operatorname{dist}(\operatorname{Re} z, \mathbb{R}^n \setminus \Omega) = +\infty$, i.e. $\Omega = \mathbb{R}^n$.) Further, put $V = \{w \in \mathbb{C}^n;\ \|\operatorname{Re} w - \operatorname{Re} z\| + \|\operatorname{Im} w - \operatorname{Im} z\| < \delta\}$, which is a neighbourhood of z. Let $j_0 \in \mathbb{N}$ be such that $2^{-j_0} < \frac{\delta}{2}$ and $\|\operatorname{Re} z\| + \|\operatorname{Im} z\| + \frac{3}{2}\delta \le j_0$. We claim that $V \subset G_j$ for all $j \ge j_0 + 2$. Indeed, pick any $w \in V$. Since j_0 is chosen so that $U_{\mathbb{R}^n}\big(\operatorname{Re} z, \|\operatorname{Im} z\| + \frac{3}{2}\delta\big) \subset K_{j_0}$, we have

$$\begin{aligned}\|\operatorname{Im} w\| &\le \|\operatorname{Im} z\| + \|\operatorname{Im} w - \operatorname{Im} z\| < \|\operatorname{Im} z\| + \delta - \|\operatorname{Re} w - \operatorname{Re} z\| \\ &\le \|\operatorname{Im} z\| + \delta + \operatorname{dist}(\operatorname{Re} w, \mathbb{R}^n \setminus K_{j_0}) - \operatorname{dist}(\operatorname{Re} z, \mathbb{R}^n \setminus K_{j_0}) \le \operatorname{dist}(\operatorname{Re} w, \mathbb{R}^n \setminus K_{j_0}) - \frac{\delta}{2}.\end{aligned}$$

Hence, using the fact that $(a-b)^2 \le a^2 - b^2$ whenever $a, b \in \mathbb{R}$, $a - b \ge 0$, and $b \ge 0$,

$$\begin{aligned}\|\operatorname{Im} w\|^2 &\le \operatorname{dist}(\operatorname{Re} w, \mathbb{R}^n \setminus K_{j_0})^2 - \frac{\delta^2}{4} \le \operatorname{dist}(\operatorname{Re} w, \mathbb{R}^n \setminus K_{j-2})^2 - \frac{\delta^2}{4} \le \operatorname{dist}(\operatorname{Re} w, U_j)^2 - \frac{\delta^2}{4} \\ &\le \operatorname{dist}(\operatorname{Re} w, \operatorname{supp} f_j)^2 - \frac{\delta^2}{4} < \operatorname{dist}(\operatorname{Re} w, \operatorname{supp} f_j)^2 - \frac{1}{4^j}\end{aligned}$$

and the claim follows. This means that $\|g_j\|_V < \frac{1}{2^j}$ for $j \ge j_0 + 2$. Therefore the series (4) converges absolutely locally uniformly on G and so $g \in H(G; \tilde{Y})$. Obviously since each $\bar{g_j}$ maps into Y, so does $\bar{g}$.

To show the approximation property of the mapping $\bar{g}$ fix $x \in \Omega$ and $0 \le l \le \min\{k, 1/\varepsilon(x)\}$. There is $p \in \mathbb{N}$ such that $x \in L_p$. Hence $l \le k_p$ and $\varepsilon_p \le \varepsilon(x)$. Since $\varphi_p = 1$ on a neighbourhood of L_p, by (2) and (3) we have

$$\Big\| f - \sum_{j=1}^{p} \bar{g_j} \Big\|_{L_p, k_p} = \|f_p - \bar{g_p}\|_{L_p, k_p} < \delta_p. \tag{5}$$

From Lemma 2, the fact that the sequences $\{k_j\}$ and $\{\nu_j\}$ are non-decreasing, (3), and (5) we obtain

$$\begin{aligned}\|\overline{g_{p+1}}\|_{L_p,k_p} &\le \|\overline{g_{p+1}} - f_{p+1}\|_{L_p,k_p} + \|f_{p+1}\|_{L_p,k_p} \le \|\overline{g_{p+1}} - f_{p+1}\|_{L_p,k_p} + \nu_{k_p}\|\varphi_{p+1}\|_{L_p,k_p}\Big\| f - \sum_{j=1}^{p}\bar{g_j}\Big\|_{L_p,k_p} \\ &\le \|\overline{g_{p+1}} - f_{p+1}\|_{\mathbb{R}^n,k_{p+1}} + \nu_{k_{p+1}}\|\varphi_{p+1}\|_{\mathbb{R}^n,k_{p+1}}\Big\| f - \sum_{j=1}^{p}\bar{g_j}\Big\|_{L_p,k_p} < \delta_{p+1} + M_{p+1}\delta_p.\end{aligned}$$

Finally, for $j > p + 1$ we have $U_j \cap L_p = \emptyset$ and since $\operatorname{supp} f_j \subset U_j$, $f_j = 0$ on a neighbourhood of L_p. Hence

$$\|\bar{g_j}\|_{L_p,k_p} = \|\bar{g_j} - f_j\|_{L_p,k_p} \le \|\bar{g_j} - f_j\|_{\mathbb{R}^n,k_j} < \delta_j.$$

Putting all this together with (1) yields

$$\begin{aligned}\|d^l f(x) - d^l \bar{g}(x)\| &= \Big\| d^l f(x) - \sum_{j=1}^{\infty} d^l \bar{g_j}(x)\Big\| \le \Big\| d^l f(x) - \sum_{j=1}^{p} d^l \bar{g_j}(x)\Big\| + \sum_{j=p+1}^{\infty}\|d^l \bar{g_j}(x)\| \\ &\le \Big\| f - \sum_{j=1}^{p}\bar{g_j}\Big\|_{L_p,k_p} + \sum_{j=p+1}^{\infty}\|\bar{g_j}\|_{L_p,k_p} < \delta_p(1 + M_{p+1}) + \sum_{j=p+1}^{\infty}\delta_j \\ &\le \sum_{j=p}^{\infty}\delta_j(1 + M_{j+1}) < \sum_{j=p}^{\infty}\frac{\varepsilon_j}{2^j} \le \sum_{j=p}^{\infty}\frac{\varepsilon_p}{2^j} \le \varepsilon_p \le \varepsilon(x).\end{aligned}$$

We note that the first equality follows from the fact that the series (4) is a locally uniformly convergent series of holomorphic mappings.

□

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