SMOOTH APPROXIMATIONS

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ABSTRACT. We prove, among other things, that a Lipschitz (or uniformly continuous) mapping $f : X \to Y$ can be approximated (even in a fine topology) by smooth Lipschitz (resp. uniformly continuous) mapping, if X is a separable Banach space admitting a smooth Lipschitz bump and either X or Y is a separable $C(K)$ space (resp. super-reflexive space). Further, we show how smooth approximation of Lipschitz mappings is closely related to a smooth approximation of C^1-smooth mappings together with their first derivatives. As a corollary we obtain new results on smooth approximation of C^1-smooth mappings together with their first derivatives.

1. INTRODUCTION

The theory of approximation of continuous mappings between infinite-dimensional Banach spaces by smooth mappings, which goes back to Kurzweil [K] and Bonic and Frampton [BF], is nowadays well understood and provides satisfactory results, see for example [DGZ].

The related problem, whether the smooth approximation of Lipschitz (or uniformly continuous) mappings can retain the Lipschitz (or uniform continuity) property is much less studied, and so far the results available are not very general. One of the reasons is that most of the results on approximation of continuous mappings use the notion of smooth partition of unity, and it is very difficult, if not impossible, to keep some uniformity in the partition.

In the present paper we introduce some new techniques and prove several new results concerning smooth approximation of Lipschitz mappings and smooth approximation of C^1-smooth mappings together with their first derivatives. In Section 2 we show how approximation of Lipschitz functions (i.e. mappings into reals) relates to bi-Lipschitz embeddings into $c_0(\Gamma)$.

In Section 3 using the bi-Lipschitz embeddings into $c_0(\Gamma)$ we develop some more general theorems concerning uniform approximation of Lipschitz mappings and then apply the results of Lindenstrauss on absolute retracts (see e.g. [BL, Theorem I.1.6, Theorem I.1.26]). Thus we obtain one of the main results of this paper, namely we prove that a Lipschitz (or uniformly continuous) mapping $f : X \to Y$ can be approximated by smooth Lipschitz mapping (Corollary 9), resp. uniformly continuous mapping (Corollary 11), if X is a separable Banach space admitting a smooth Lipschitz bump and either X or Y is a separable $C(K)$ space (resp. super-reflexive space). These two results complement the presently known theorems (see below), for example we remove the assumption on X having a basis from Theorem I but unfortunately we have to restrict the type of the target space.

However, since the main ingredient of this technique relies on integral convolutions in $c_0(\Gamma)$, we obtain only uniform approximation. To achieve the fine approximation we need to introduce a new approach concerning smooth partitions of unity. This is done in Section 4.

In Sections 5 and 6 we show how this new approach can be used to translate some "separable" techniques into general (non-separable) setting. Namely, in Section 5 we prove that uniform approximation of Lipschitz mappings implies fine approximation (Theorem 15) and thus in combination with the results of Section 3 we obtain stronger versions of those theorems (Corollary 16).

Finally, in Section 6 we prove the next of the main results of this paper, which shows how smooth approximation of Lipschitz mappings is closely related to a smooth approximation of C^1-smooth mappings together with their first derivatives. In particular we generalise the result of Moulis (Theorem C) into arbitrary (non-separable) spaces (Theorem 19). As a corollary we obtain results on approximation of C^1-smooth mappings together with their first derivatives (Corollary 20, Corollary 21). Moreover our techniques also allow us to prove a result dual to the result of Moulis (Theorem A): The approximation result also holds for X being arbitrary (separable) and Y having an unconditional basis. Thus Corollary 20 exhibits an interesting symmetry in its hypotheses.

To put our results into perspective, we summarise the current state of the theory below.

But first, we need to fix some notation. Let B_X (U_X) denote a closed (open) unit ball of a normed linear space X. Further, for a metric space (P, ρ), we denote $B(x, r) = \{y \in P;\ \rho(x, y) \leq r\}$ and $U(x, r) = \{y \in P;\ \rho(x, y) < r\}$ the closed and open ball in P centred at $x \in P$ with radius $r \geq 0$. Let $A \subset P$. A neighbourhood $U \subset P$ of A is called an *r-uniform neighbourhood* if there is $r > 0$ such that $\bigcup_{x\in A} U(x, r) \subset U$. A neighbourhood is called a *uniform neighbourhood* if it is r-uniform for some $r > 0$. For a set $M \in P$ and $\varepsilon > 0$ we denote $M_\varepsilon = \{x \in M;\ \operatorname{dist}(x, P \setminus M) > \varepsilon\}$. For a function f into reals we denote $\operatorname{supp} f = f^{-1}(\mathbb{R} \setminus \{0\})$.

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Now we list the known results, in the order as they appeared in the literature:

Theorem A (Moulis). *Let X be a Banach space with an unconditional Schauder basis that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$, and Y be a Banach space. For any open $\Omega \subset X$, any mapping $f \in C^1(\Omega, Y)$, and any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ there is $g \in C^k(\Omega, Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

This theorem immediately follows from the following two results:

Theorem B (Moulis). *Let X be a Banach space with a monotone unconditional Schauder basis $\{e_i\}_{i=1}^{\infty}$ that admits a C^k-smooth Lipschitz bump function. Denote $X_n = \operatorname{span}\{e_i\}_{i=1}^{n}$. There is a constant $C > 0$ such that if Y is a Banach space, $M \subset X$ such that $P_n M \subset M$ for all $n \in \mathbb{N}$, Ω a uniform open neighbourhood of M, $f\colon \Omega \to Y$ an L-Lipschitz Gâteaux differentiable mapping such that the mappings $x \mapsto f'(x)e_i$ are uniformly continuous on $\Omega \cap X_n$ for each $i, n \in \mathbb{N}$, and $\varepsilon > 0$, then there is $g \in C^k(X, Y)$ such that $\|g'(x)\| \le C(1 + \varepsilon)L$ for all $x \in M_\varepsilon$ and $\|f(x) - g(x)\| < \varepsilon$ for all $x \in M_\varepsilon$.*

Theorem C (Moulis). *Let X, Y be normed linear spaces, X separable, and $k \in \mathbb{N} \cup \{\infty\}$. Suppose there is $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f \in C^1(2U_X, Y)$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X, Y)$ such that $\sup_{x \in U_X} \|f(x) - g(x)\| \le \varepsilon$. Then for any open $\Omega \subset X$, any mapping $f \in C^1(\Omega, Y)$, and any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ there is $g \in C^k(\Omega, Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

We note, that Theorem B is actually formulated as [M, Lemme fondamental 1] under much stronger assumptions, namely for ℓ_p spaces and mappings C^1-smooth on some ball. However, the proof in [M] works also for spaces with unconditional basis with only formal modifications. Denote $f_n = f\restriction_{X_n}$. Then the assumptions of Theorem B imply that f_n' are uniformly continuous on $\Omega \cap X_n$. Noticing this, the proof in [M] works also almost verbatim under the relaxed differentiability assumptions.

The following two theorems use the infimal convolution techniques, hence they provide only C^1-smooth approximation of functions. Nevertheless, they are the first non-separable results.

Theorem D (Lasry-Lions, [LL]). *Let X be a Hilbert space, $f\colon X \to \mathbb{R}$ an L-Lipschitz function, and $\varepsilon > 0$. Then there is an L-Lipschitz function $g \in C^{1,1}(X)$ such that $\sup_{x \in X} |f(x) - g(x)| \le \varepsilon$.*

Theorem E (Cepedello, [C, Corollary 3]). *Let X be a super-reflexive Banach space, $f\colon X \to \mathbb{R}$ a Lipschitz function, and $\varepsilon > 0$. Then there is a function $g \in C^1(X)$ which is Lipschitz on bounded sets and such that $\sup_{x \in X} |f(x) - g(x)| \le \varepsilon$.*

We note that the original formulation in [LL] is for bounded functions, however in the Lipschitz case the boundedness is not needed.

If we put no assumptions on the smoothness of the source space, we obtain only a uniformly Gâteaux differentiable approximation.

Theorem F (Johanis, [J]). *Let X be a separable Banach space, Y be a Banach space, $f\colon X \to Y$ be an L-Lipschitz mapping, and let $\varepsilon > 0$. Then there is a mapping $g\colon X \to Y$ which is L-Lipschitz, uniformly Gâteaux differentiable, and satisfies $\sup_{x \in X} \|f(x) - g(x)\| \le \varepsilon$.*

The following theorem gives smooth approximations of bounded Lipschitz functions.

Theorem G (Fry). *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. For each $\varepsilon > 0$ there is a constant $K \in \mathbb{R}$ such that if $f\colon X \to [0, 1]$ is 1-Lipschitz, then there is a K-Lipschitz function $g \in C^k(X)$ such that $\sup_{x \in X} |f(x) - g(x)| \le \varepsilon$.*

By obvious adjustments of the proof of [F1, Theorem 1] we obtain this more general Theorem G, see also the proof of Theorem 3, (i)⇒(ii). We note that the subsequent attempt to generalise Theorem G for WCG spaces in [F3] appears to be seriously flawed and it is unknown at present if the result holds.

Finally, there is a recent result on approximation of Lipschitz (or more generally uniformly continuous) mappings on $c_0(\Gamma)$.

Theorem H (Hájek-Johanis). *Let Γ be an arbitrary set, Y be a Banach space, $M \subset c_0(\Gamma)$, $U \subset c_0(\Gamma)$ be a uniform neighbourhood of M, $f\colon U \to Y$ be a uniformly continuous mapping with modulus of continuity ω and let $\varepsilon > 0$. Then there is a mapping $g \in C^\infty(c_0(\Gamma), Y)$ which locally depends on finitely many coordinates, such that $\sup_M \|f(x) - g(x)\| \le \varepsilon$, and g is uniformly continuous on M with modulus of continuity dominated by ω. In particular, if f is L-Lipschitz, then g is L-Lipschitz on M.*

This stronger version of [HJ, Theorem 1] follows by not very difficult modification of the proof.

If a uniformly continuous mapping $f\colon X \to Y$ is uniformly Gâteaux differentiable, then the mappings $x \mapsto f'(x)h$ are uniformly continuous on X (see e.g. [HJ, Lemma 4]). Thus combining Theorem F and Theorem B we immediately obtain the following corollary:

Theorem I. *Let X be a Banach space with an unconditional Schauder basis that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. There is a constant $C > 0$ such that if Y is a Banach space, $f\colon X \to Y$ an L-Lipschitz mapping, and $\varepsilon > 0$, then there is a $C(1 + \varepsilon)L$-Lipschitz mapping $g \in C^k(X, Y)$ such that $\|f(x) - g(x)\| < \varepsilon$ for all $x \in X$.*

This result was first stated by R. Fry in [F2] but with an incorrect proof.

2. APPROXIMATION OF FUNCTIONS AND EMBEDDINGS INTO $c_0(\Gamma)$

First, although not directly related to our results, we show the following observation, which basically says that to approximate Lipschitz functions it only suffices to consider approximation of *bounded* functions, and moreover we gain control over the Lipschitz constant of the approximation.

Proposition 1. *Let $k \in \mathbb{N} \cup \{\infty\}$ and X be a normed linear space with the following property: There is a $C \in \mathbb{R}$ such that for each $A \subset X$ there is a C-Lipschitz function $h_A \in C^k(X, [0, 1])$ satisfying $h_A(x) = 0$ for all $x \in A$ and $h_A(x) = 1$ for all $x \in X$ such that* $\operatorname{dist}(x, A) \geq 1$.

Then for each $\varepsilon > 0$ and an arbitrary L-Lipschitz function $f : X \to \mathbb{R}$ there is a CL-Lipschitz function $g \in C^k(X)$ such that $|g(x) - f(x)| \leq \varepsilon$ for each $x \in X$.

Proof. Let us define a function $\tilde{f} : X \to \mathbb{R}$ by $\tilde{f}(x) = \frac{1}{\varepsilon} f(\frac{\varepsilon}{L} x)$. This function is obviously 1-Lipschitz. Next, let us define sets $A_n = \{x \in X;\ \tilde{f}(x) \geq n\}$ for $n \in \mathbb{Z}$. Clearly, $A_{n+1} \subset A_n$ for all $n \in \mathbb{Z}$, and using the 1-Lipschitz property of $\tilde{f}$ it is easy to check that

$$\operatorname{dist}(X \setminus A_n, A_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{Z}. \tag{1}$$

Further, denote $h_n(x) = 1 - h_{A_{n+1}}(x)$ for $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, $h_n \in C^k(X, [0, 1])$, h_n is C-Lipschitz, $h_n(x) = 1$ for all $x \in A_{n+1}$, and by (1), $h_n(x) = 0$ for all $x \in X \setminus A_n$.

Now, put

$$h(x) = \sum_{n=0}^{\infty} h_n(x) - \sum_{n=-\infty}^{-1} \big(1 - h_n(x)\big). \tag{2}$$

Fix an arbitrary $x \in X$. Then there is $m \in \mathbb{Z}$ such that $x \in A_m \setminus A_{m+1}$. It follows, that $h_n(x) = 0$ for all $n > m$ and $h_n(x) = 1$ for all $n < m$. Hence (2) defines a function $h : X \to \mathbb{R}$. Moreover, by (1), the sums in (2) are even locally finite, therefore $h \in C^k(X)$. Further, it is easy to check that $h(x) = m + h_m(x)$. This implies that $h(x) \in [m, m + 1]$, while $\tilde{f}(x) \in [m, m + 1)$ and hence $\left|h(x) - \tilde{f}(x)\right| \leq 1$.

It remains to show that h is C-Lipschitz. To this end, choose $x, y \in X$ and find $n, l \in \mathbb{Z}$ such that $x \in A_n \setminus A_{n+1}$ and $y \in A_{n+l} \setminus A_{n+l+1}$. Without loss of generality we may assume that $l \geq 0$. If $l = 0$, then clearly $|h(x) - h(y)| = |n + h_n(x) - n - h_n(y)| \leq C \|x - y\|$.

We prove the case $l > 0$ by induction on l. As the first step of the induction assume that $l = 1$. Denote by $[x, y]$ the line segment between the points x and y. Since $[x, y]$ is connected, there is a point $z \in [x, y] \cap A_{n+1} \cap \overline{(X \setminus A_{n+1})}$. From the properties of h_n and h_{n+1}, and from the continuity of h_{n+1} it follows that $h_n(z) = 1$ and $h_{n+1}(z) = 0$. Thus

$$\begin{aligned}|h(y) - h(x)| &= |n + 1 + h_{n+1}(y) - n - h_n(x)| = |h_{n+1}(y) + 1 - h_n(x)| = |h_{n+1}(y) - h_{n+1}(z) + h_n(z) - h_n(x)| \\ &\leq |h_{n+1}(y) - h_{n+1}(z)| + |h_n(z) - h_n(x)| \leq C \|y - z\| + C \|z - x\| = C \|y - x\| .\end{aligned}$$

To prove the general induction step let us assume that $l > 1$. By the continuity of $\tilde{f}$ there is a point $z \in [x, y]$ such that $z \in A_{n+1} \setminus A_{n+2}$. Using the induction hypothesis on the pair x, z and again on the pair z, y we obtain $|h(x) - h(y)| \leq |h(x) - h(z)| + |h(z) - h(y)| \leq C \|x - z\| + C \|z - y\| = C \|x - y\|$.

Finally, let $g(x) = \varepsilon h(\frac{L}{\varepsilon} x)$. It is straightforward to check that g satisfies the conclusion of our theorem.

□

Combining Proposition 1 and Theorem G we would obtain a smooth approximation of Lipschitz functions on smooth separable normed linear spaces. However, we skip the details, since we will show much more, see Corollary 16.

In the sequel we will be using smooth bi-Lipschitz homeomorphisms into $c_0(\Gamma)$. The following two results show how they can be constructed and how they are related to smooth approximation of Lipschitz functions. First we define some notions useful in this context.

For a metric space P, we denote $\mathcal{U}(r) = \{U(x, r);\ x \in P\}$.

Let X be a set. A collection $\{\psi_\alpha\}_{\alpha \in \Lambda}$ of functions on X is called a *sup-partition of unity* if

- $\psi_\alpha : X \to [0, 1]$ for all $\alpha \in \Lambda$,
- for each $x \in X$ the set $\{\alpha \in \Lambda;\ \psi_\alpha(x) > 0\}$ is finite,
- for each $x \in X$ there is $\alpha \in \Lambda$ such that $\psi_\alpha(x) = 1$.

Let $\mathcal{U}$ be a covering of X. We say that the sup-partition of unity $\{\psi_\alpha\}_{\alpha \in \Lambda}$ is subordinated to $\mathcal{U}$ if $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in \Lambda}$ refines $\mathcal{U}$.

Fact 2. *Let Γ be an infinite set, $r > 0$, and $0 < \delta < \frac{r}{2}$. There is an open point-finite uniform refinement $\mathcal{V} = \{V_\gamma\}_{\gamma \in \Gamma}$ of the uniform covering $\mathcal{U}(r)$ of $c_0(\Gamma)$ such that $\mathcal{U}(\frac{r}{2} - \delta)$ refines $\mathcal{V}$. Moreover, $\mathcal{V}$ is formed by the translates of the open ball $U(0, r - \delta)$. Further, there is a C^∞-smooth, locally dependent on finitely many coordinate functionals, and $(\frac{2}{r} + \delta)$-Lipschitz sup-partition of unity $\{\psi_\gamma\}_{\gamma \in \Gamma}$ on $c_0(\Gamma)$ subordinated to $\mathcal{U}(r)$.*

The first part of this fact was already shown in [P, Proposition 2.3], but with more complicated proof.

Proof. Notice that, by homogeneity, it suffices to prove all the statements only for $r = 1$.

Let $\{a_\gamma\}_{\gamma \in \Gamma}$ be the set of all vectors in $c_0(\Gamma)$ with coordinates in $\mathbb{Z}$. (Notice that the cardinality of such set is $|\Gamma|$ and so we may index its points by Γ.) We claim that $\mathcal{V} = \{U(a_\gamma, 1 - \delta)\}_{\gamma \in \Gamma}$ is the desired refinement.

Clearly, $\mathcal{V}$ is an open refinement of $\mathcal{U}(1)$. To see that it is point-finite, pick any $x \in c_0(\Gamma)$ and find a finite $F \subset \Gamma$ such that $|x(\gamma)| < \delta$ whenever $\gamma \in \Gamma \setminus F$. Suppose that $\alpha \in \Gamma$ is such that $x \in U(a_\alpha, 1-\delta)$. Then for $\gamma \notin F$ we have $|a_\alpha(\gamma)| \leq |a_\alpha(\gamma) - x(\gamma)| + |x(\gamma)| < 1$ and so $a_\alpha(\gamma) = 0$. From $|x(\gamma) - a_\alpha(\gamma)| < 1-\delta$ and $a_\alpha(\gamma) \in \mathbb{Z}$ it follows that there are at most two possibilities for $a_\alpha(\gamma)$ for each $\gamma \in F$. From this we can conclude that $\left|\{\alpha;\ x \in U(a_\alpha, 1-\delta)\}\right| \leq 2^{|F|}$.

Finally, we show that $\mathcal{U}\big(\frac{1}{2} - \delta\big)$ refines $\mathcal{V}$. Choose any $x \in c_0(\Gamma)$ and find $\beta \in \Gamma$ such that $\|x - a_\beta\| \leq \frac{1}{2}$. This is always possible, since there is a finite $F \subset \Gamma$ such that $|x(\gamma)| < \frac{1}{2}$ whenever $\gamma \notin F$, and so $a_\beta(\gamma) = 0$ for such γ. Suppose $z \in U(x, \frac{1}{2} - \delta)$. Then $\|a_\beta - z\| \leq \|a_\beta - x\| + \|x - z\| < \frac{1}{2} + \frac{1}{2} - \delta = 1 - \delta$, which implies $U(x, \frac{1}{2} - \delta) \subset U(a_\beta, 1-\delta)$.

To construct the sup-partition of unity subordinated to the uniform covering $\mathcal{U}(1)$, find $\varepsilon > 0$ and $0 < \eta < \frac{1}{2}$ such that $0 < 1/\big(1 - \eta - \frac{1+\varepsilon}{2}\big) < 2 + \frac{\delta}{4}$ and $(1+\varepsilon)\big(2 + \frac{\delta}{2}\big) \leq 2 + \delta$. Let $\mathcal{W} = \{U(a_\gamma, 1-\eta)\}_{\gamma\in\Gamma}$ be the point-finite refinement of $\mathcal{U}(1)$ from the first part of the proof such that $\mathcal{U}\big(\frac{1}{2} - \eta\big)$ refines $\mathcal{W}$. Further, let $\|\cdot\|$ be an equivalent C^∞-smooth norm on $c_0(\Gamma)$ which locally depends on finitely many of the coordinate functionals $\{e^*_\gamma\}_{\gamma\in\Gamma}$ (away from the origin) and such that $\|x\|_\infty \leq \|x\| \leq (1+\varepsilon)\|x\|_\infty$ for all $x \in c_0(\Gamma)$. (To construct such a norm, take for example the Minkowski functional of the set $\big\{x \in c_0(\Gamma);\ \sum_{\gamma\in\Gamma} \varphi(x_\gamma) \leq 1\big\}$, where $\varphi \in C^\infty(\mathbb{R})$, φ is convex and even, $\varphi(1) = 1$, and $\varphi(t) = 0$ for $t \in [-\frac{1}{1+\varepsilon}, \frac{1}{1+\varepsilon}]$.)

For each $\gamma \in \Gamma$ we put $\psi_\gamma(x) = q(\|x - a_\gamma\|)$, where $q \in C^\infty(\mathbb{R}, [0,1])$, q is $\big(2 + \frac{\delta}{2}\big)$-Lipschitz, $q(t) = 0$ for $t \geq 1 - \eta$, and $q(t) = 1$ for $t \leq \frac{1+\varepsilon}{2}$. The collection $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a sup-partition of unity. Indeed, it is easy to see, that $\operatorname{supp}\psi_\gamma \subset U(a_\gamma, 1-\eta)$ for each $\gamma \in \Gamma$, and consequently the set $\{\gamma \in \Gamma;\ \psi_\gamma(x) > 0\}$ is finite for each $x \in X$. Further, fix any $x \in X$. There is an $\alpha \in \Gamma$ such that $U(x, \frac{1}{2} - \eta) \subset U(a_\alpha, 1-\eta)$, which gives $\|x - a_\alpha\|_\infty \leq \frac{1}{2}$. Hence $\|x - a_\alpha\| \leq (1+\varepsilon)\|x - a_\alpha\|_\infty \leq \frac{1+\varepsilon}{2}$, which in turn implies $\psi_\alpha(x) = 1$.

As the function q is $\big(2 + \frac{\delta}{2}\big)$-Lipschitz and the function $\|\cdot\|$ is $(1+\varepsilon)$-Lipschitz (with respect to the norm $\|\cdot\|_\infty$), the functions ψ_γ are $(2+\delta)$-Lipschitz according to the choice of ε. The rest of the properties of the functions ψ_γ is obvious.

□

Theorem 3. *Let X be a normed linear space, Γ an infinite set, and $k \in \mathbb{N} \cup \{0, \infty\}$. Then the following are equivalent:*

(i) There is $M \in \mathbb{R}$ such that there is a C^k-smooth and M-Lipschitz sup-partition of unity $\{\phi_\gamma\}_{\gamma\in\Gamma}$ on X subordinated to $\mathcal{U}(1)$.

(ii) X is uniformly homeomorphic to a subset of $c_0(\Gamma)$ and for each $\varepsilon > 0$ there is $K > 0$ such that for each 1-Lipschitz function $f\colon X \to [0,1]$ there is a K-Lipschitz function $g \in C^k(X)$ such that $\sup_{x\in X} |g(x) - f(x)| \leq \varepsilon$.

(iii) There is a bi-Lipschitz homeomorphism $\varphi\colon X \to c_0(\Gamma)$ such that the coordinate functions $e^*_\gamma \circ \varphi \in C^k(X)$ for every $\gamma \in \Gamma$.

Proof. First we show that (i) implies (iii). From the properties of the sup-partition of unity there is $\beta \in \Gamma$ such that $\phi_\beta(0) = 1$. By scaling and composing ϕ_β with a suitable function we construct a C-Lipschitz function $h \in C^k(X, [0,1])$ such that $h = 0$ on $B(0, r)$ and $h = 1$ outside $U(0,1)$ for some constants $C, r \in \mathbb{R}$, $r > 0$. (We may for example choose r such that $1 - 2Mr > 0$ and take $h(x) = q(\phi_\beta(2x))$, where $q \in C^k(\mathbb{R})$, q is Lipschitz, $q([0,1]) = [0,1]$, $q(0) = 1$, and $q(s) = 0$ for $s \geq 1 - 2Mr$.)

Choose $t > 1$ and for each $n \in \mathbb{Z}$ and $\gamma \in \Gamma$ define functions $\phi^n_\gamma \in C^k(X)$ by

$$\phi^n_\gamma(x) = t^n \phi_\gamma\left(\frac{x}{t^n}\right) h\left(\frac{x}{t^n}\right).$$

The properties of the functions ϕ_γ and h guarantee that each ϕ^n_γ is $(M+C)$-Lipschitz. Let $d\colon \mathbb{Z} \times \Gamma \to \Gamma$ be some one-to-one mapping and define $\varphi\colon X \to \mathbb{R}^\Gamma$ by $\varphi(x)_\alpha = \phi^n_\gamma(x)$ if $\alpha = d(n, \gamma)$ for some $n \in \mathbb{Z}$, $\gamma \in \Gamma$; $\varphi(x)_\alpha = 0$ otherwise.

We show that φ actually maps into $c_0(\Gamma)$. Choose an arbitrary $x \in X$ and $\varepsilon > 0$. There is $n_0 \in \mathbb{Z}$ such that $t^n < \varepsilon$ for all $n < n_0$ and $n_1 \in \mathbb{Z}$ such that $\|x\| \leq rt^n$ for all $n > n_1$. It follows that $\left|\phi^n_\gamma(x)\right| < \varepsilon$ for all $n < n_0$ and $\gamma \in \Gamma$, and, by the properties of h, $\phi^n_\gamma(x) = 0$ for all $n > n_1$ and $\gamma \in \Gamma$. As for each $n_0 \leq n \leq n_1$, $\phi_\gamma(x/t^n) \neq 0$ only for finitely many $\gamma \in \Gamma$, we can conclude that $\varphi\colon X \to c_0(\Gamma)$.

Since each ϕ^n_γ is $(M+C)$-Lipschitz, the mapping φ is $(M+C)$-Lipschitz as well.

To prove that φ is one-to-one and φ^{-1} is Lipschitz too, choose any two points $x, y \in X$, $x \neq y$, and find $m \in Z$ such that $2t^m \leq \|x - y\| < 2t^{m+1}$. Without loss of generality we may assume that $\|x\| \geq t^m$. Then $h(x/t^m) = 1$ and so there is $\gamma \in \Gamma$ such that $\phi^m_\gamma(x) = t^m$. Now suppose there is $z \in X$ such that $\phi^m_\gamma(z) > 0$. As $\operatorname{supp}\phi_\gamma \subset U(w, 1)$ for some $w \in X$, $\left\|\frac{x}{t^m} - \frac{z}{t^m}\right\| < 2$ and consequently $\|x - z\| < 2t^m$. But this means that $\phi^m_\gamma(y) = 0$ and therefore

$$\|\varphi(x) - \varphi(y)\|_\infty \geq \left|\phi^m_\gamma(x) - \phi^m_\gamma(y)\right| = \phi^m_\gamma(x) = t^m > \frac{1}{2t}\|x - y\|.$$

(iii)⇒(i): Let $A, B \in \mathbb{R}$ are such that $A\|x - y\| \leq \|\varphi(x) - \varphi(y)\|_\infty \leq B\|x - y\|$. By Fact 2, there is a $C > 0$ and a C^∞-smooth, locally dependent on finitely many coordinate functionals, and C-Lipschitz sup-partition of unity $\{\psi_\gamma\}_{\gamma\in\Gamma}$ on $c_0(\Gamma)$ subordinated to $\mathcal{U}(A)$. Putting $\phi_\gamma = \psi_\gamma \circ \varphi$, $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is a BC-Lipschitz sup-partition of unity subordinated to $\mathcal{U}(1)$. Fix $\gamma \in \Gamma$. To see that $\phi_\gamma \in C^k(X)$, pick any $x \in X$. There is a neighbourhood V of $\varphi(x)$ such that $\psi_\gamma(w) = G(f_1(w), \dots, f_n(w))$ for each $w \in V$, where $f_1, \dots, f_n \in \{e^*_\gamma\}_{\gamma\in\Gamma}$ and $G \in C^\infty(\Omega)$ for some $\Omega \subset \mathbb{R}^n$ open. Let U be an open neighbourhood of x such that $\varphi(U) \subset V$. Then $\phi_\gamma(y) = \psi_\gamma(\varphi(y)) = G(f_1(\varphi(y)), \dots, f_n(\varphi(y)))$ for each $y \in U$. Since, by the assumption, $f_i \circ \varphi \in C^k(X)$ for each $i = 1, \dots, n$, and $G \in C^\infty(\Omega)$, ϕ_γ is C^k-smooth on U.

(i)⇒(ii): We already know that (iii) holds and from this the first part of (ii) follows immediately. To prove the second part of (ii), let $\varepsilon > 0$. The basic idea of the proof is that Lipschitz functions are stable under the operation of pointwise supremum. To preserve the smoothness, we will use a "smoothened supremum", or an equivalent smooth norm on $c_0(\Gamma)$. Let $\|\cdot\|$ be an equivalent C^∞-smooth norm on $c_0(\Gamma)$ which locally depends on finitely many of the coordinate functionals $\{e_\gamma^*\}_{\gamma\in\Gamma}$ (away from the origin), and let $C > 0$ be such that $\|x\|_\infty \le \|x\| \le C\,\|x\|_\infty$ for all $x \in c_0(\Gamma)$ (see the proof of Fact 2). We will show that $K = 2C^2(C+1)M/\varepsilon$ satisfies our claim.

By adding the constant 1 we may and do assume that f maps into $[1,2]$. Put $\delta = \frac{\varepsilon}{C}$ and $\psi_\gamma(x) = \phi_\gamma(\frac{x}{\delta})$ for all $x \in X$, $\gamma \in \Gamma$. It follows, that $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a C^k-smooth and M/δ-Lipschitz sup-partition of unity subordinated to $\mathcal{U}(\delta)$. Since the sets $\{\gamma \in \Gamma;\ \psi_\gamma(x) > 0\}$ are finite, $(\psi_\gamma(x))_{\gamma\in\Gamma} \in c_0(\Gamma)$ for each $x \in X$. For each $\gamma \in \Gamma$ there is a point $x_\gamma \in X$ such that $\operatorname{supp}\psi_\gamma \subset U(x_\gamma, \delta)$. The boundedness of the function f guarantees that also $\big(f(x_\gamma)\psi_\gamma(x)\big)_{\gamma\in\Gamma} \in c_0(\Gamma)$ for each $x \in X$. Therefore we can define the function $g\colon X \to \mathbb{R}$ by

$$g(x) = \frac{\big\|\big(f(x_\gamma)\psi_\gamma(x)\big)_{\gamma\in\Gamma}\big\|}{\big\|\big(\psi_\gamma(x)\big)_{\gamma\in\Gamma}\big\|}.$$

As

$$\big\|(\psi_\gamma(x))\big\| \ge \big\|(\psi_\gamma(x))\big\|_\infty = \sup_{\gamma\in\Gamma}\psi_\gamma(x) = 1 \quad \text{for each } x \in X, \tag{3}$$

the function g is well defined on all of X.

The mapping $x \mapsto \big(\psi_\gamma(x)\big)$ and, by the boundedness of f, also the mapping $x \mapsto \big(f(x_\gamma)\psi_\gamma(x)\big)$ are Lipschitz mappings from X into $c_0(\Gamma) \setminus U(0,1)$. (Notice that for each $x \in X$ there is $\gamma \in \Gamma$ such that $\psi_\gamma(x) = 1$ and $f(x_\gamma)\psi_\gamma(x) \ge 1$.) Since $\|\cdot\|$ is C^∞-smooth and depends locally on finitely many coordinates away from the origin, and since $\psi_\gamma \in C^k(X)$ and $f(x_\gamma)\psi_\gamma \in C^k(X)$ for each $\gamma \in \Gamma$, similarly as in the proof of (iii)⇒(i) we infer that $g \in C^k(X)$.

Using the facts that f maps into $[1,2]$, the functions ψ_γ are M/δ-Lipschitz and map into $[0,1]$, and $\|\cdot\|$ is C-Lipschitz as a function on $(c_0, \|\cdot\|_\infty)$, we obtain that the function $x \mapsto \big\|\big(f(x_\gamma)\psi_\gamma(x)\big)\big\|$ is $2CM/\delta$-Lipschitz and bounded by $2C$. Similarly, the function $x \mapsto \big\|\big(\psi_\gamma(x)\big)\big\|$ is CM/δ-Lipschitz and bounded below by 1. It follows that the function g is K-Lipschitz.

Finally, to show that g approximates f, choose an arbitrary $x \in X$. Applying successively the inequality (3) and the facts that $\operatorname{supp}\psi_\gamma \subset U(x_\gamma, \delta)$ and f is 1-Lipschitz, we obtain

$$\begin{aligned}
|g(x) - f(x)| &= \left|\frac{\big\|\big(f(x_\gamma)\psi_\gamma(x)\big)\big\|}{\big\|\big(\psi_\gamma(x)\big)\big\|} - f(x)\frac{\big\|\big(\psi_\gamma(x)\big)\big\|}{\big\|\big(\psi_\gamma(x)\big)\big\|}\right| \le \frac{\big\|\big((f(x_\gamma) - f(x))\psi_\gamma(x)\big)\big\|}{\big\|\big(\psi_\gamma(x)\big)\big\|} \le C\,\big\|\big((f(x_\gamma) - f(x))\psi_\gamma(x)\big)\big\|_\infty \\
&= C\sup_{\gamma\in\Gamma}\big\{\big|f(x_\gamma) - f(x)\big|\,\psi_\gamma(x)\big\} = C \sup_{\substack{\gamma\in\Gamma\\ x\in U(x_\gamma,\delta)}}\big\{\big|f(x_\gamma) - f(x)\big|\,\psi_\gamma(x)\big\} \le C \sup_{\substack{\gamma\in\Gamma\\ x\in U(x_\gamma,\delta)}}\big\{\|x_\gamma - x\|\big\} \le C\delta = \varepsilon.
\end{aligned}$$

(ii)⇒(i): It is not difficult to construct a point-finite base of the uniform coverings of $c_0(\Gamma)$ and pull it back onto X via the uniform homeomorphism (see e.g. [P, Proposition 2.3]). So let $\mathcal{V} = \{V_\gamma\}_{\gamma\in\Gamma}$ be an open point-finite uniform refinement of the covering $\mathcal{U}(1)$ of X. (We note that such refinement can be chosen so that $|\mathcal{V}| = |\Gamma|$ and so we can indeed index it by the set Γ.) Let $0 < \delta \le 1$ be such that $\mathcal{U}(\delta)$ refines $\mathcal{V}$. For each $\gamma \in \Gamma$ we define the function $f_\gamma\colon X \to [0,1]$ by $f_\gamma(x) = \min\{\operatorname{dist}(x, X \setminus V_\gamma), \delta\}$.

Choose some $0 < \theta < \frac{\delta}{2}$. For each $\gamma \in \Gamma$, the function f_γ is 1-Lipschitz and so, by (ii), there is a K-Lipschitz function $g_\gamma \in C^k(X)$ such that $\sup_{x\in X}\big|g_\gamma(x) - f_\gamma(x)\big| \le \theta$. Let $q \in C^k(\mathbb{R}, [0,1])$ be a C-Lipschitz function for some $C \in \mathbb{R}$, such that $q(t) = 0$ for $t \le \theta$ and $q(t) = 1$ for $t \ge \delta - \theta$. Finally, we let $\phi_\gamma(x) = q(g_\gamma(x))$ for each $\gamma \in \Gamma$. Clearly, each function ϕ_γ belongs to $C^k(X, [0,1])$ and is M-Lipschitz, where $M = CK$. Further, for any $x \in X$ there is $\alpha \in \Gamma$ such that $U(x,\delta) \subset V_\alpha$, hence $f_\alpha(x) = \delta$ and consequently $\phi_\alpha(x) = 1$. As $\operatorname{supp}\phi_\gamma \subset V_\gamma$ for all $\gamma \in \Gamma$ and $\mathcal{V}$ is point-finite, $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is a sup-partition of unity subordinated to $\mathcal{U}(1)$.

□

We note, that the proof could be made considerably shorter by proving (iii)⇒(ii) directly using Theorem H (see the proof of Theorem 8) instead of (i)⇒(ii) and (iii)⇒(i). However, the reasons for our strategy of the proof were two: First, we do not need the full generality (and associated machinery) of Theorem H and second, the proof of (i)⇒(ii) shows an interesting technique for constructing smooth Lipschitz approximations (due to Fry, [F1]), and in fact shows the reason for the definition of the notion of sup-partition of unity.

Corollary 4. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Then there is a bi-Lipschitz homeomorphism $\varphi\colon X \to c_0$ such that the coordinate functions $e_j^* \circ \varphi \in C^k(X)$ for every $j \in \mathbb{N}$.*

Proof. Fry in [F1] has constructed a C^k-smooth M-Lipschitz sup-partition of unity $\{\psi_j\}_{j=1}^\infty$ on X that is subordinated to $\mathcal{U}(1)$, so Theorem 3 applies.

□

We note that this corollary is a Lipschitz counterpart to the separable case of [DGZ, Theorem VIII.3.2(vi)⇒(v)], see also [DGZ, p. 360].

3. Approximation of mappings

To be able to use Theorem H, we need to "extend" Lipschitz mappings from subsets of $c_0(\Gamma)$. To this end we introduce some additional notions.

Let (X, ρ) be a metric space, $A \subset X$. For $\varepsilon > 0$, a mapping $r_\varepsilon \colon X \to A$ such that $\rho(r_\varepsilon(x), x) < \varepsilon$ for each $x \in A$ is called an *ε-retraction*.

A is called a *Lipschitz approximate retract* (LAR), if there is $K > 0$ such that for any $\varepsilon > 0$ there is a K-Lipschitz ε-retraction of X into A. A is called a *Lipschitz approximate uniform neighbourhood retract* (LAUNR), if there is $K > 0$ such that for any $\varepsilon > 0$ there is a uniform open neighbourhood $U \subset X$ of A and a K-Lipschitz ε-retraction of U into A.

A metric space is called an *absolute Lipschitz approximate uniform neighbourhood retract* (ALAUNR) if it is a LAUNR of every metric space containing it as a subspace.

Example. Let X be a Banach space with an unconditional basis $\{e_n\}_{n=1}^\infty$ and let $X_\infty = \operatorname{span}\{e_n\}_{n=1}^\infty$ be its linear subspace consisting of finitely supported vectors. Then X_∞ is a Lipschitz approximate retract of X.

Indeed, let $C = 2\operatorname{ubc}\{e_n\}$ and $D = \operatorname{bc}\{e_n\}$, put $K = C(5 + 4D)$ and choose an arbitrary $\varepsilon > 0$. Let $\varphi \colon \mathbb{R} \to [0, 1]$ be defined as $\varphi(t) = 0$ for $t \le \varepsilon/(2C)$, $\varphi(t) = 1$ for $t \ge \varepsilon/C$, and φ is affine on $[\varepsilon/(2C), \varepsilon/C]$. Notice that φ is $2C/\varepsilon$-Lipschitz. Denote $R_n = I - P_n$. Define the ε-retraction $r \colon X \to X_\infty$ by $r(x) = \sum_{n=1}^\infty \varphi(\|R_{n-1}x\|)x_n e_n$. We claim that $\|x - r(x)\| < \varepsilon$ for all $x \in X$. To see this, fix $x \in X$ and find $n_0 \in \mathbb{N} \cup \{0\}$ such that $\|R_{n_0}x\| < \varepsilon/C$ and $\|R_n x\| \ge \varepsilon/C$ for all $0 \le n < n_0$. Then

$$\begin{aligned}\|x - r(x)\| &= \left\|\sum_{n=1}^\infty \bigl(1 - \varphi(\|R_{n-1}x\|)\bigr)x_n e_n\right\| = \left\|\sum_{n>n_0} \bigl(1 - \varphi(\|R_{n-1}x\|)\bigr)x_n e_n\right\| \\ &\le C\left\|\sum_{n>n_0} x_n e_n\right\| = C\,\|R_{n_0}x\| < C\frac{\varepsilon}{C} = \varepsilon.\end{aligned}$$

To show that r is K-Lipschitz, choose any $x, y \in X$. We may without loss of generality assume that $\|x - y\| \le \varepsilon/(C(1 + D))$. (It is an easy fact, that mappings on normed linear spaces that are Lipschitz on short distances are Lipschitz globally with the same Lipschitz constant.) Find $n_0 \in \mathbb{N} \cup \{0\}$ such that $\|R_{n_0}y\| < 2\varepsilon/C$ and $\|R_n y\| \ge 2\varepsilon/C$ for all $0 \le n < n_0$. Then $\|R_n x\| \ge \|R_n y\| - \|R_n(x - y)\| \ge 2\varepsilon/C - (1 + D)\varepsilon/(C(1 + D)) = \varepsilon/C$ for all $0 \le n < n_0$. It follows that $\varphi(\|R_n x\|) = \varphi(\|R_n y\|) = 1$ for all $0 \le n < n_0$. Using this fact, we can estimate

$$\begin{aligned}\|r(x) - r(y)\| &= \left\|\sum_{n=1}^\infty \bigl(\varphi(\|R_{n-1}x\|)x_n - \varphi(\|R_{n-1}y\|)y_n\bigr)e_n\right\| \\ &\le \left\|\sum_{n=1}^\infty \varphi(\|R_{n-1}x\|)(x_n - y_n)e_n\right\| + \left\|\sum_{n=1}^\infty \bigl(\varphi(\|R_{n-1}x\|) - \varphi(\|R_{n-1}y\|)\bigr)y_n e_n\right\| \\ &\le C\,\|x - y\| + \left\|\sum_{n>n_0} \bigl(\varphi(\|R_{n-1}x\|) - \varphi(\|R_{n-1}y\|)\bigr)y_n e_n\right\| \\ &\le C\,\|x - y\| + C\sup_{n>n_0}\bigl|\varphi(\|R_{n-1}x\|) - \varphi(\|R_{n-1}y\|)\bigr|\left\|\sum_{n>n_0} y_n e_n\right\| \\ &\le C\,\|x - y\| + C\frac{2C}{\varepsilon}\sup_{n>n_0}\bigl|\|R_{n-1}x\| - \|R_{n-1}y\|\bigr|\,\|R_{n_0}y\| \\ &< C\,\|x - y\| + C\frac{2C}{\varepsilon}(1 + D)\,\|x - y\|\,\frac{2\varepsilon}{C} = K\,\|x - y\|.\end{aligned}$$

We note that in fact the mapping r locally maps into finite-dimensional subspaces of X. Similarly (replacing $\varphi(\|R_n x\|)$ by $\phi(R_n x)$, where ϕ is a suitable smooth function) we can prove the following lemma (cf. [M, pp. 297–300]), which will be useful later.

Lemma 5 (Moulis)**.** *Let X be a Banach space with an unconditional Schauder basis $\{e_i\}_{i=1}^\infty$ that admits a C^k-smooth Lipschitz bump function. Denote $X_n = \operatorname{span}\{e_i\}_{i=1}^n$. Then there is a constant $K > 0$ such that for any $\varepsilon > 0$ there is a K-Lipschitz mapping $\psi \in C^k(X, X)$ such that for each $x \in X$ there is a neighbourhood U of x and $n \in \mathbb{N}$ such that $\psi(U) \subset X_n$ and $\|x - \psi(x)\| < \varepsilon$.*

The following proposition shows how the notion of ALAUNR relates to "approximate extensions" of Lipschitz mappings.

Proposition 6. *Let (X, ρ) be a metric space. The following are equivalent:*

(i) X is an ALAUNR.
(ii) There is $K > 0$ such that for each $\varepsilon > 0$ there is $\delta > 0$ such that for any metric space P, $X \subset P$, there is $U \subset P$ a δ-uniform open neighbourhood of X such that X is a K-Lipschitz ε-retraction of U (i.e. the Lipschitz constant K and the "sizes" of the uniform neighbourhoods do not depend on the metric space which X is a subspace of).

(iii) *There is $K > 0$ such that for each $\varepsilon > 0$ there is $\delta > 0$ such that for any metric spaces $Q \subset P$ and every L-Lipschitz mapping $f: Q \to X$ there is $U \subset P$ a δ/L-uniform open neighbourhood of Q and a KL-Lipschitz mapping $g: U \to X$ such that $\rho(f(x), g(x)) < \varepsilon$ for all $x \in Q$.*
(iv) *For any metric spaces $Q \subset P$ and every L-Lipschitz mapping $f : Q \to X$ there is $K > 0$ such that for any $\varepsilon > 0$ there is $U \subset P$ a uniform open neighbourhood of Q and a KL-Lipschitz mapping $g: U \to X$ such that $\rho(f(x), g(x)) < \varepsilon$ for all $x \in Q$.*
(v) *There is $K > 0$ such that for each $\varepsilon > 0$ there is $\delta > 0$ such that for any metric space P, $X \subset P$, there is $U \subset P$ a δ-uniform open neighbourhood of X such that for any metric space (Q, σ) and every L-Lipschitz mapping $f : X \to Q$ there is a KL-Lipschitz mapping $g: U \to Q$ such that $\sigma(f(x), g(x)) < L\varepsilon$ for all $x \in X$.*
(vi) *For any metric spaces P and (Q, σ), $X \subset P$, and every L-Lipschitz mapping $f : X \to Q$ there is $K > 0$ such that for any $\varepsilon > 0$ there is $U \subset P$ a uniform open neighbourhood of X and a KL-Lipschitz mapping $g: U \to Q$ such that $\sigma(f(x), g(x)) < \varepsilon$ for all $x \in X$.*

Proof. (ii)⇒(i), (iii)⇒(iv), and (v)⇒(vi) are trivial.

(i)⇒(iii): Embed X isometrically into $\ell_\infty(\Gamma)$. There is $K > 0$ such that X is a K-Lipschitz approximate neighbourhood retract of $\ell_\infty(\Gamma)$. Choose $\varepsilon > 0$ and let $\delta > 0$ be such that there is a K-Lipschitz ε-retraction $r: V \to X$ for some δ-uniform open neighbourhood V of X in $\ell_\infty(\Gamma)$. Let $Q \subset P$ be metric spaces and $f : Q \to X$ be an L-Lipschitz mapping. Since $\ell_\infty(\Gamma)$ is an absolute Lipschitz retract, there is an L-Lipschitz extension $h: P \to \ell_\infty(\Gamma)$ of $f : Q \to X \subset \ell_\infty(\Gamma)$. Put $U = h^{-1}(V)$. Then U is open in P, and it is a δ/L-uniform neighbourhood of Q. Indeed, if $y \in U(z, \delta/L)$ for some $z \in Q$, then $h(y) \in U(h(z), \delta)$, where $h(z) \in X$; hence $h(y) \in V$. Finally, put $g(x) = r(h(x))$ for any $x \in U$. Then $\rho(f(x), g(x)) = \rho\big(f(x), r(h(x))\big) = \rho\big(f(x), r(f(x))\big) < \varepsilon$ whenever $x \in Q$.

(iii)⇒(ii), (v)⇒(ii) (and (iv)⇒(i), (vi)⇒(i) similarly): Let X be a subspace of a metric space P, we put $Q = X$ and $f = \mathrm{id}$. For any $\varepsilon > 0$, the K-Lipschitz mapping g is the desired retraction r_ε.

(ii)⇒(v): Let $\varepsilon > 0$. Find $\delta > 0$ from (ii). Let $X \subset P$ and $r: U \to X$ be the K-Lipschitz ε-retraction from some δ-uniform neighbourhood $U \subset P$ of X. Put $g = f \circ r$. Then $\sigma(f(x), g(x)) = \sigma\big(f(x), f(r(x))\big) \le L\rho(x, r(x)) < L\varepsilon$ for any $x \in X$. □

Corollary 7. *Let (X, ρ) be an ALAUNR.*
(a) If (Z, σ) is bi-Lipschitz homeomorphic to X, then Z is an ALAUNR.
(b) If Z is a LAUNR of X, then Z is an ALAUNR.

Proof. (a): Let $\varphi: Z \to X$ be a bi-Lipschitz homeomorphism such that $A\sigma(x, y) \le \rho(\varphi(x), \varphi(y)) \le B\sigma(x, y)$. We show that (iv) of Proposition 6 holds. Let $Q \subset P$ be metric spaces and $f : Q \to Z$ an L-Lipschitz mapping. Let $\tilde{f} : Q \to X$ be defined as $\tilde{f} = \varphi \circ f$ and let K_0 be the constant in Proposition 6(iv) for $\tilde{f}$. Put $K = K_0B/A$. Choose any $\varepsilon > 0$. There is a uniform open neighbourhood $U \subset P$ of Q and a K_0BL-Lipschitz mapping $\tilde{g}: U \to X$ such that $\rho(\tilde{f}(x), \tilde{g}(x)) < A\varepsilon$ for all $x \in Q$. Then $g: U \to Z$, $g = \varphi^{-1} \circ \tilde{g}$ is a K_0BL/A-Lipschitz mapping such that $\sigma(f(x), g(x)) = \sigma\big(f(x), \varphi^{-1}(\tilde{g}(x))\big) = \sigma\big(\varphi^{-1}(\tilde{f}(x)), \varphi^{-1}(\tilde{g}(x))\big) \le (1/A)\rho(\tilde{f}(x), \tilde{g}(x)) < A\varepsilon/A = \varepsilon$ whenever $x \in Q$.

(b): Let K_0 be the Lipschitz constant of the ε-retractions into X (as X is ALAUNR) and K_1 be the Lipschitz constant of the ε-retractions from $U \subset X$ into Z. We show that (iv) of Proposition 6 holds. Let $Q \subset P$ be metric spaces and $f : Q \to Z \subset X$ an L-Lipschitz mapping. Put $K = K_1K_0$. Choose any $\varepsilon > 0$. There is a δ-uniform open neighbourhood $V \subset X$ of Z and a K_1-Lipschitz $(\varepsilon/2)$-retraction $r: V \to Z$. Further, there is an η-uniform open neighbourhood $W \subset P$ of Q and a K_0L-Lipschitz mapping $h: W \to X$ such that $\rho(f(x), h(x)) < \min\{\varepsilon/(2K_1), \delta/2\}$ for all $x \in Q$.

Let $U = h^{-1}(V)$. Then $U \subset W$ is open in W and hence in P, and it is a uniform neighbourhood of Q. Indeed, let $\zeta = \min\{\delta/(2K_0L), \eta\}$. If $y \in U(z, \zeta)$ for some $z \in Q$, then $y \in W$ and so $h(y) \in U(h(z), \delta/2)$. From this we obtain $h(y) \in U(f(z), \delta)$, and since $f(z) \in Z$, it follows that $h(y) \in V$.

Finally, put $g(x) = r(h(x))$ for any $x \in U$. Then $g: U \to Z$ is a K_1K_0L-Lipschitz mapping such that $\rho(f(x), g(x)) = \rho\big(f(x), r(h(x))\big) \le \rho\big(f(x), r(f(x))\big) + \rho\big(r(f(x), r(h(x))\big) < \varepsilon/2 + K_1\rho(f(x), h(x)) < \varepsilon$ whenever $x \in Q$. □

Finally we can prove one of our main approximation theorems.

Theorem 8. *Let Y be a Banach space, $k \in \mathbb{N} \cup \{\infty\}$, and X be a normed linear space such that there is a set Γ and a bi-Lipschitz homeomorphism $\varphi: X \to c_0(\Gamma)$ such that the coordinate functions $e^*_\gamma \circ \varphi \in C^k(X)$ for every $\gamma \in \Gamma$. Assume further that X or Y is an ALAUNR. There is a constant $C \in \mathbb{R}$ such that if $f : X \to Y$ is L-Lipschitz and $\varepsilon > 0$, then there is a CL-Lipschitz mapping $g \in C^k(X, Y)$ such that $\sup_{x \in X} \|f(x) - g(x)\| \le \varepsilon$.*

Moreover, if $C_1, C_2 \in \mathbb{R}$ are such that φ is C_1-Lipschitz and φ^{-1} is C_2-Lipschitz, and if K is the Lipschitz constant of the ALAUNR, then $C = C_1C_2K$.

Proof. We define $\tilde{f} : \varphi(X) \to Y$ by $\tilde{f}(z) = f(\varphi^{-1}(z))$ for any $z \in \varphi(X)$. The mapping $\tilde{f}$ is C_2L-Lipschitz. If Y is a K-Lipschitz ALAUNR, then by Proposition 6(iii) there is a uniform open neighbourhood U of $\varphi(X)$ in $c_0(\Gamma)$ and a mapping $\hat{f} : U \to Y$ such that $\hat{f}$ is KC_2L-Lipschitz and $\big\|\hat{f}(z) - \tilde{f}(z)\big\| < \frac{\varepsilon}{2}$ for each $z \in \varphi(X)$. In case that X is a K-Lipschitz ALAUNR, we come to the same conclusion by using Proposition 6(iii) to a mapping φ^{-1} to obtain a uniform open neighbourhood

U of $\varphi(X)$ in $c_0(\Gamma)$ and a KC_2 Lipschitz mapping $q\colon U \to X$ such that $\|q(z) - \varphi^{-1}(z)\| < \frac{\varepsilon}{2L}$ for all $z \in \varphi(X)$, and then putting $\hat{f} = f \circ q$. (Using Corollary 7 and Proposition 6(v) to $\tilde{f}$ instead, we would arrive to a worse Lipschitz constant $KC_1C_2^2L$.)

By Theorem H there is a mapping $\hat{g} \in C^\infty(c_0(\Gamma), Y)$ locally dependent on finitely many coordinates and such that it is C_2KL-Lipschitz on $\varphi(X)$ and $\|\hat{g}(z) - \hat{f}(z)\| \le \frac{\varepsilon}{2}$ for all $z \in \varphi(X)$. We define the mapping $g\colon X \to Y$ by $g = \hat{g} \circ \varphi$. Similarly as in the proof of Theorem 3, (iii)⇒(i), we obtain that $g \in C^k(X, Y)$. Clearly, g is C_1C_2KL-Lipschitz. To see that g approximates f, choose any $x \in X$. Then

$$\begin{aligned}\|g(x) - f(x)\| &= \left\|\hat{g}(\varphi(x)) - f\left(\varphi^{-1}(\varphi(x))\right)\right\| = \left\|\hat{g}(\varphi(x)) - \tilde{f}(\varphi(x))\right\| \\ &\le \left\|\hat{g}(\varphi(x)) - \hat{f}(\varphi(x))\right\| + \left\|\hat{f}(\varphi(x)) - \tilde{f}(\varphi(x))\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

We note that the notion of ALAUNR is necessary for our approach to Theorem 8 (at least in the case of the source space X): For any Banach space Y and any Lipschitz mapping $f : \varphi(X) \to Y$ we need to find a Lipschitz "approximate extension" to a uniform neighbourhood U of $\varphi(X)$. Now, consider $Y = X$ and a mapping $\varphi^{-1} : \varphi(X) \to X$, find an "approximate extension" $q\colon U \to X$ and put $r = \varphi \circ q$. Then r is a Lipschitz ε-retraction of U into $\varphi(X)$.

Let V be a topological space, let $v_0 \in V$. By $B_0(V)$ we denote the space of all bounded real-valued functions f on V for which $f(v) \to 0$ whenever $v \to v_0$, considered with the supremum norm. Let P be a metric space; by $C_u(P)$ we denote the space of all bounded, uniformly continuous, real-valued functions on P with the supremum norm. By the result of Lindenstrauss, [L, Theorem 6] (see also [BL]), both $B_0(V)$ and $C_u(P)$ are absolute Lipschitz retracts.

Now using Corollary 4 and Theorem 8 we obtain the following:

Corollary 9. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Y be a Banach space. If at least one of the spaces X or Y is equal to either $B_0(V)$ for some topological space V, or $C_u(P)$ for some metric space P, then there is a constant $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f : X \to Y$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(X, Y)$ for which $\sup_{x\in X} \|f(x) - g(x)\| \le \varepsilon$.*

The above approach can be modified to deal with uniformly continuous mappings. However, we must be somewhat careful in the formulation of the result (notice the necessity of a sub-additive modulus of the embedding in Theorem 10). We skip the details, as the proofs are almost identical to the ones already given.

A modulus is a non-decreasing function $\omega\colon [0, +\infty) \to [0, +\infty)$ continuous at 0 such that $\omega(0) = 0$. The set of all moduli will be denoted by $\mathcal{M}$. The subset of $\mathcal{M}$ of all moduli that are sub-additive will be denoted by $\mathcal{M}_s \subset \mathcal{M}$. A modulus of continuity of a mapping f is denoted by ω_f.

Theorem 10. *Let Y be a Banach space, $k \in \mathbb{N} \cup \{\infty\}$, and X be a normed linear space such that there is a set Γ and a uniform homeomorphism $\varphi\colon X \to c_0(\Gamma)$ such that $\omega_{\varphi^{-1}} \le \omega_1 \in \mathcal{M}_s$ and the coordinate functions $e_\gamma^* \circ \varphi \in C^k(X)$ for every $\gamma \in \Gamma$. Assume further that X or Y is an absolute uniform approximate uniform neighbourhood retract. If $f : X \to Y$ is uniformly continuous and $\varepsilon > 0$, then there is a function $\omega \in \mathcal{M}$ and a mapping $g \in C^k(X, Y)$ such that $\omega_g \le \omega$ and $\sup_{x\in X} \|f(x) - g(x)\| \le \varepsilon$.*

Moreover, if X is AUAUNR with modulus ω_0, then $\omega = \omega_f \circ \omega_0 \circ \omega_1 \circ \omega_\varphi$. If Y is AUAUNR with modulus ω_0, then $\omega = \omega_0 \circ \omega_f \circ \omega_1 \circ \omega_\varphi$.

By the result of Lindenstrauss, [L, Theorem 8] (see also [BL]), super-reflexive Banach spaces are absolute uniform uniform (sic) neighbourhood retracts. Hence, using Corollary 4 and Theorem 10 we obtain the following:

Corollary 11. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Y be a Banach space. If X or Y is a super-reflexive Banach space, then there is a constant $C \in \mathbb{R}$ and a modulus $\omega_0 \in \mathcal{M}$ such that for any uniformly continuous mapping $f : X \to Y$ and any $\varepsilon > 0$ there is a mapping $g \in C^k(X, Y)$ for which $\sup_{x\in X} \|f(x) - g(x)\| \le \varepsilon$ and $\omega_g(\delta) \le \omega_f(\omega_0(C\delta))$ (if X is super-reflexive) or $\omega_g(\delta) \le \omega_0(\omega_f(C\delta))$ (if Y is super-reflexive) for $\delta \in [0, +\infty)$.*

4. Smooth Lipschitz partitions of unity

Recall that a (locally finite) partition of unity on a topological space X is a collection $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of functions on X if

- $\psi_\alpha : X \to [0, 1]$ for all $\alpha \in \Lambda$,
- $\sum_{\alpha\in\Lambda} \psi_\alpha(x) = 1$ for each $x \in X$,
- for each $x \in X$ there is a neighbourhood $U \subset X$ of x such that the set $\{\alpha \in \Lambda;\ \operatorname{supp} \psi_\alpha \cap U \neq \emptyset\}$ is finite.

Let $\mathcal{U}$ be a covering of X. We say that a partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is *subordinated to* $\mathcal{U}$ if $\{\operatorname{supp} \psi_\alpha\}_{\alpha\in\Lambda}$ refines $\mathcal{U}$.

A family of subsets of a topological space is called *discrete* if for each point $x \in X$ there is a neighbourhood of x that meets at most one member of this family. We say that a partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is σ-discrete if the family $\{\operatorname{supp} \psi_\alpha\}_{\alpha\in\Lambda}$ is σ-discrete, that is it can be decomposed into countably many discrete families.

First we need some finer information about refinements of open coverings.

Lemma 12 (M.E. Rudin, [R]). *Let P be a metric space, $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of P. Then there are open refinements $\{V_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, $\{W_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$ of $\mathcal{U}$ that satisfy the following:*

- *$V_{n\alpha} \subset W_{n\alpha} \subset U_\alpha$ for all $n \in \mathbb{N}$, $\alpha \in \Lambda$,*
- *$\operatorname{dist}(V_{n\alpha}, P \setminus W_{n\alpha}) \geq 2^{-n}$ for all $n \in \mathbb{N}$, $\alpha \in \Lambda$,*
- *$\operatorname{dist}(W_{n\alpha}, W_{n\beta}) \geq 2^{-n}$ for any $n \in \mathbb{N}$ and $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$,*
- *for each $x \in P$ there is an open ball $U_x \in P$ with centre x and a number $n_x \in \mathbb{N}$ such that*
  (i) if $i > n_x$, then $U_x \cap W_{i\alpha} = \emptyset$ for any $\alpha \in \Lambda$,
  (ii) if $i \leq n_x$, then $U_x \cap W_{i\alpha} \neq \emptyset$ for at most one $\alpha \in \Lambda$.

Using some refinement of the ideas in [JTZ] we can prove the following key lemma:

Lemma 13. *Let X be a normed linear space and $k \in \mathbb{N} \cup \{\infty\}$. Suppose that there is an $L \in \mathbb{R}$ such that for any $V \subset X$ bounded there is an L-Lipschitz function $\varphi \in C^k(X, [0,1])$ satisfying $V \subset \varphi^{-1}(\{1\})$ and $\varphi(x) = 0$ whenever $\operatorname{dist}(x, V) \geq 1$. Let $\Omega \subset X$ be open. Then for any open covering $\mathcal{U}$ of Ω there is a Lipschitz and C^k-smooth σ-discrete partition of unity on Ω subordinated to $\mathcal{U}$.*

Proof. Without loss of generality we may assume that all the sets in $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ are bounded, $U_\alpha \subset \Omega$ for each $\alpha \in \Lambda$, and $L \geq 1$. By Lemma 12 there are open refinements $\{V_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, $\{W_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$ of $\mathcal{U}$ such that $V_{n\alpha} \subset W_{n\alpha} \subset U_\alpha$, $\operatorname{dist}(V_{n\alpha}, \Omega \setminus W_{n\alpha}) \geq 2^{-n}$, $\operatorname{dist}(W_{n\alpha}, W_{n\beta}) \geq 2^{-n}$ for $\alpha \neq \beta$, the family $\{W_{n\alpha}\}_{\alpha\in\Lambda}$ is discrete in Ω for all $n \in \mathbb{N}$, and the family $\{W_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$ is locally finite in Ω. By the assumption (using a scaling argument) for each $n \in \mathbb{N}$, $\alpha \in \Lambda$ there is an L_n-Lipschitz function $\tilde{\varphi}_{n\alpha} \in C^k(X, [0,1])$ satisfying $V_{n\alpha} \subset \tilde{\varphi}_{n\alpha}^{-1}(\{1\})$ and $\tilde{\varphi}_{n\alpha}(x) = 0$ whenever $\operatorname{dist}(x, V_{n\alpha}) \geq 2^{-n}$, where $L_n = 2^n L$. Put $\varphi_{n\alpha} = \tilde{\varphi}_{n\alpha}\restriction_\Omega$. Then $\varphi_{n\alpha} \in C^k(\Omega, [0,1])$, $V_{n\alpha} \subset \varphi_{n\alpha}^{-1}(\{1\})$, and $\operatorname{supp}\varphi_{n\alpha} \subset W_{n\alpha}$ for all $n \in \mathbb{N}$, $\alpha \in \Lambda$. Moreover, each function $\varphi_{n\alpha}$ is L_n-Lipschitz.

For each $n \in \mathbb{N}$ define $\varphi_n\colon \Omega \to [0,1]$ by $\varphi_n(x) = \varphi_{n\alpha}(x)$ whenever there is $\alpha \in \Lambda$ such that $x \in W_{n\alpha}$, $\varphi_n(x) = 0$ otherwise. Notice that by the discreteness of $\{W_{n\alpha}\}_{\alpha\in\Lambda}$ the functions φ_n are well defined and also C^k-smooth. It is easy to check that for each $n \in \mathbb{N}$ the function φ_n is L_n-Lipschitz. Indeed, let $x, y \in \Omega$ and suppose there are $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$ such that $x \in W_{n\alpha}$, $y \in W_{n\beta}$. Then $\|x - y\| \geq 2^{-n}$ and hence $|\varphi_n(x) - \varphi_n(y)| \leq 1 \leq 2^n \|x - y\| \leq L_n \|x - y\|$. The other cases follow from the fact that $\varphi_{n\alpha}$ are L_n-Lipschitz.

Now for $n \in \mathbb{N}$ let $\psi_n = \varphi_n \prod_{j=1}^{n-1}(1 - \varphi_j)$. Then $\psi_n \in C^k(\Omega, [0,1])$ and each function ψ_n is Lipschitz. Further, $\{\psi_n\}_{n\in\mathbb{N}}$ is a (locally finite) partition of unity on Ω. Indeed, for any $x \in \Omega$ there is $m \in \mathbb{N}$ and $\beta \in \Lambda$ such that $x \in V_{m\beta}$. Choose any $y \in V_{m\beta}$. Then $\varphi_m(y) = \varphi_{m\beta}(y) = 1$ and hence $\psi_n(y) = 0$ for $n > m$. Since

$$(1 - \varphi_1)(1 - \varphi_2)\cdots(1 - \varphi_m) = 1 - \psi_1 - \cdots - \psi_m,$$

it follows that $\sum_{n=1}^{\infty} \psi_n(y) = \sum_{n=1}^{m} \psi_n(y) = 1$.

Finally, for $n \in \mathbb{N}$ and $\alpha \in \Lambda$ let $\psi_{n\alpha} = \chi_{W_{n\alpha}} \cdot \psi_n$. Using the fact that $\operatorname{supp}\psi_n \subset \operatorname{supp}\varphi_n \subset \bigcup_{\alpha\in\Lambda} W_{n\alpha}$ and the discreteness of $\{W_{n\alpha}\}_{\alpha\in\Lambda}$ it follows that all the functions $\psi_{n\alpha}$ are C^k-smooth and Lipschitz (using similar argument as above), and that $\sum_{\alpha\in\Lambda} \psi_{n\alpha} = \psi_n$. As moreover $\operatorname{supp}\psi_{n\alpha} \subset W_{n\alpha}$, we can conclude that $\{\psi_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$ is a locally finite, σ-discrete Lipschitz C^k-smooth partition of unity on Ω subordinated to $\mathcal{U}$.

□

Notice that to satisfy the requirements of Lemma 13 it suffices that we are able to approximate the distance functions by smooth Lipschitz functions. Namely we obtain the following corollary:

Corollary 14. *Let X, Y be normed linear spaces and $k \in \mathbb{N} \cup \{\infty\}$. Suppose that there is a $C \in \mathbb{R}$ such that for each 1-Lipschitz mapping $f\colon 2U_X \to Y$ and $\varepsilon > 0$ there is a C-Lipschitz mapping $g \in C^k(U_X, Y)$ satisfying $\sup_{x\in U_X} \|f(x) - g(x)\| \leq \varepsilon$. Let $\Omega \subset X$ be open. Then for any open covering $\mathcal{U}$ of Ω there is a Lipschitz and C^k-smooth σ-discrete partition of unity on Ω subordinated to $\mathcal{U}$.*

Proof. It is sufficient to notice that approximation of mappings into Y gives us also approximations of functions. Indeed, if $f\colon 2U_X \to \mathbb{R}$ is 1-Lipschitz, then choose some $y \in S_Y$ and consider the mapping $\tilde{f}\colon 2U_X \to Y$, $\tilde{f}(x) = f(x) \cdot y$. Let $\tilde{g} \in C^k(U_X, Y)$ be an approximation of $\tilde{f}$ provided by our assumption and $F \in Y^*$ be a Hahn-Banach extension of the norm-one functional $ty \mapsto t$ defined on $\operatorname{span}\{y\}$. Then $g = F \circ \tilde{g}$ is the desired approximation of the function f.

□

5. Approximation of Lipschitz mappings revisited

In subsequent proofs we use the following convention: If X, Y are normed linear spaces, $F \in X^*$, and $y \in Y$, we denote by $y \cdot F$ or yF the bounded linear operator $yF \in \mathcal{B}(X, Y)$ given by $(yF)h = (Fh) \cdot y$ for $h \in X$. Let $g\colon X \to Y$, $\psi\colon X \to \mathbb{R}$, and both g and ψ be Fréchet differentiable at $x \in X$. Then the mapping $g\psi = g \cdot \psi$ is Fréchet differentiable at x and using the convention above, the formula for the derivative of the product can be written as $(g\psi)'(x) = \psi(x)g'(x) + g(x)\psi'(x)$.

Armed with the Lipschitz partitions of unity constructed in the previous section we can extend our results a little bit further. First we prove a result that allows us to pass from uniform approximations to fine approximations.

Theorem 15. *Let X, Y be normed linear spaces and $k \in \mathbb{N} \cup \{\infty\}$. Suppose that there is a $C \geq 1$ such that for each L-Lipschitz mapping $f : 2U_X \to Y$ and $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X, Y)$ satisfying $\sup_{x \in U_X} \|f(x) - g(x)\| \leq \varepsilon$. Let $\Omega \subset X$ be open. Then for any L-Lipschitz mapping $f : \Omega \to Y$, any continuous function $\varepsilon : \Omega \to (0, +\infty)$, and any $\eta > 1$ there is an ηCL-Lipschitz mapping $g \in C^k(\Omega, Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

Proof. First notice that from approximations on U_X by translating and scaling we immediately obtain approximations on any open ball in X. For each $x \in \Omega$ find $r(x) > 0$ such that $U(x, 4r(x)) \subset \Omega$ and

$$\varepsilon(y) > \frac{\varepsilon(x)}{3} \quad \text{for each } y \in U(x, r(x)). \tag{4}$$

By Corollary 14 there is a σ-discrete Lipschitz C^k-smooth partition of unity on Ω subordinated to $\{U(x, r(x));\ x \in \Omega\}$. We may assume that the partition of unity is of the form $\{\psi_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, where for each $n \in \mathbb{N}$ the family $\{\operatorname{supp}\psi_{n\alpha}\}_{\alpha\in\Lambda}$ is discrete in Ω. For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ let $U_{n\alpha} = U(x_{n\alpha}, r(x_{n\alpha}))$ be such that $\operatorname{supp}\psi_{n\alpha} \subset U_{n\alpha}$. Let $L_{n\alpha}$ be the Lipschitz constant of $\psi_{n\alpha}$, and without loss of generality assume that $L_{n\alpha} \geq 1$. Further, denote $V_{n\alpha} = U(x_{n\alpha}, 2r(x_{n\alpha}))$.

For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ we approximate f on $V_{n\alpha}$ by CL-Lipschitz mapping $g_{n\alpha} \in C^k(V_{n\alpha}, Y)$ such that

$$\|f(x) - g_{n\alpha}(x)\| \leq \min\left\{\frac{(\eta-1)CL}{2^n L_{n\alpha}}, \frac{\varepsilon(x_{n\alpha})}{3}\right\} < \varepsilon(x) \quad \text{for each } x \in U_{n\alpha}. \tag{5}$$

(The second inequality follows from (4).) Define the mapping $\tilde{g}_{n\alpha} : \Omega \to Y$ by $\tilde{g}_{n\alpha}(x) = g_{n\alpha}(x)$ for $x \in V_{n\alpha}$, $\tilde{g}_{n\alpha}(x) = 0$ otherwise.

Finally, we define the mapping $g : \Omega \to Y$ by

$$g(x) = \sum_{n\in\mathbb{N},\alpha\in\Lambda} \tilde{g}_{n\alpha}(x)\psi_{n\alpha}(x).$$

Since $\operatorname{supp}\psi_{n\alpha} \subset U_{n\alpha}$, $g_{n\alpha} \in C^k(V_{n\alpha}, Y)$, and the sum is locally finite, the mapping g is well defined and moreover $g \in C^k(\Omega, Y)$.

Choose $x \in \Omega$ and let us compute how far $g(x)$ is from $f(x)$:

$$\|f(x) - g(x)\| = \left\|\sum_{n\in\mathbb{N},\alpha\in\Lambda} \big(f(x) - \tilde{g}_{n\alpha}(x)\big)\psi_{n\alpha}(x)\right\| \leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} \|f(x) - g_{n\alpha}(x)\|\,\psi_{n\alpha}(x) < \varepsilon(x) \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} \psi_{n\alpha}(x) = \varepsilon(x),$$

where the last inequality follows from (5).

To estimate the derivative of g at some fixed $x \in \Omega$, notice that by the discreteness of $\{\operatorname{supp}\psi_{n\alpha}\}_{\alpha\in\Lambda}$, for each $n \in \mathbb{N}$ there is at most one $\alpha \in \Lambda$ such that $\psi'_{n\alpha}(x) \neq 0$. Put $M = \{n \in \mathbb{N};\ \exists\alpha \in \Lambda : \psi'_{n\alpha}(x) \neq 0\}$. Then there is a mapping $\beta : M \to \Lambda$ such that for each $n \in M$, $\psi'_{n\alpha}(x) = 0$ whenever $\alpha \neq \beta(n)$ and moreover $x \in U_{n\beta(n)}$. (Notice that if $\psi'_{n\alpha}(x) \neq 0$ then necessarily $x \in U_{n\alpha}$.) Further, since $\sum \psi_{n\alpha} = 1$, it follows that $\sum \psi'_{n\alpha} = 0$. Hence,

$$\begin{aligned}
\|g'(x)\| &= \left\|\sum_{n\in\mathbb{N},\alpha\in\Lambda} (\tilde{g}_{n\alpha}\psi_{n\alpha})'(x)\right\| = \left\|\sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} (\tilde{g}_{n\alpha}\psi_{n\alpha})'(x)\right\| = \left\|\sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} \psi_{n\alpha}(x)g'_{n\alpha}(x) + \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} g_{n\alpha}(x)\psi'_{n\alpha}(x)\right\| \\
&= \left\|\sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} \psi_{n\alpha}(x)g'_{n\alpha}(x) + \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} \big(g_{n\alpha}(x) - f(x)\big)\psi'_{n\alpha}(x)\right\| \\
&\leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} \left\|g'_{n\alpha}(x)\right\|\psi_{n\alpha}(x) + \sum_{n\in M} \left\|g_{n\beta(n)}(x) - f(x)\right\| \left\|\psi'_{n\beta(n)}(x)\right\| \\
&\leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\ x\in U_{n\alpha}}} CL\psi_{n\alpha}(x) + \sum_{n\in M} \left\|g_{n\beta(n)}(x) - f(x)\right\| L_{n\beta(n)} \leq CL + \sum_{n\in M} \frac{(\eta-1)CL}{2^n L_{n\beta(n)}} L_{n\beta(n)} \leq \eta CL,
\end{aligned}$$

where the last but one inequality follows from (5).

To finish the proof we show that g is ηCL-Lipschitz on the set Ω. Without loss of generality we may assume that $\varepsilon(x) \leq (\eta C - 1)L \operatorname{dist}(x, X \setminus \Omega)$ for every $x \in \Omega$. Now fix $x, y \in \Omega$. If the line segment l with end points x and y lies in Ω, then the standard argument yields that $\|g(x) - g(y)\| \leq \eta CL \|x - y\|$. Otherwise there is $z \in l \cap (X \setminus \Omega)$. Then

$$\begin{aligned}
\|g(x) - g(y)\| &\leq \|g(x) - f(x)\| + \|f(x) - f(y)\| + \|f(y) - g(y)\| < \varepsilon(x) + L\|x - y\| + \varepsilon(y) \\
&\leq (\eta C - 1)L\|x - z\| + L\|x - y\| + (\eta C - 1)L\|y - z\| = \eta CL\|x - y\|.
\end{aligned}$$

∎

Combining Theorem 15 with Theorem I and Corollary 9 we obtain the following corollary.

Corollary 16. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Y be a Banach space. Suppose further that one of the following conditions is satisfied:*

- *X is a Banach space with an unconditional Schauder basis, or*
- *at least one of the spaces X or Y is equal to $B_0(V)$ for some topological space V, or*
- *at least one of the spaces X or Y is equal to $C_u(P)$ for some metric space P.*

Then there is a constant $C \in \mathbb{R}$ such that for any open $\Omega \subset X$, any L-Lipschitz mapping $f : \Omega \to Y$, and any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ there is a CL-Lipschitz mapping $g \in C^k(\Omega, Y)$ for which $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.

Proof. It suffices to notice that under our assumptions the hypothesis of Theorem 15 is satisfied. Indeed, since B_X is a 2-Lipschitz retract of X, every L-Lipschitz mapping defined on B_X can be extended to a $2L$-Lipschitz mapping defined on X. Thus we may apply either Theorem I or Corollary 9.

□

Further, Theorem 15 together with Theorem D gives us the next corollary.

Corollary 17. *Let X be a Hilbert space and $\Omega \subset X$ be an open subset. Then for any L-Lipschitz function $f : \Omega \to \mathbb{R}$, any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$, and any $\eta > 1$ there is an ηL-Lipschitz function $g \in C^1(\Omega)$ such that $|f(x) - g(x)| < \varepsilon(x)$ for all $x \in \Omega$.*

6. Approximation of C^1-smooth mappings

In this section we extend the result of Moulis (Theorem C) about the relation of Lipschitz approximation and the approximation of mappings together with its derivatives to non-separable case.

To refrain from repeating the same argument over and over again in various contexts, we prove the following proposition, whose statement is necessarily more technically involved. One of the main ideas is based on the same argument as the proof of Theorem 15.

Proposition 18. *Let X, Y be normed linear spaces, $k \in \mathbb{N} \cup \{\infty\}$, and $\Omega \subset X$ be open. Suppose that for any open covering $\mathcal{U}$ of Ω there is a Lipschitz C^k-smooth σ-discrete partition of unity on Ω subordinated to $\mathcal{U}$. Suppose further that $\{Y_\gamma\}_{\gamma\in\Gamma}$ is a collection of closed subspaces of Y such that for each $\gamma \in \Gamma$ there is a constant $C_\gamma \in \mathbb{R}$ such that for any L-Lipschitz mapping $f \in C^1(2U_X, Y_\gamma)$ and any $\varepsilon > 0$ there is a $C_\gamma L$-Lipschitz mapping $g \in C^k(U_X, Y)$ satisfying $\sup_{x\in U_X} \|f(x) - g(x)\| \le \varepsilon$. Let $f \in C^1(\Omega, Y)$ be such that it is locally a mapping into some Y_γ, $\gamma \in \Gamma$. Then for any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ there is $g \in C^k(\Omega, Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

Proof. First notice that from approximations on U_X by translating and scaling we immediately obtain approximations on any open ball in X. For each $x \in \Omega$ find $r(x) > 0$ and $\gamma(x) \in \Gamma$ such that $U(x, 4r(x)) \subset \Omega$, $f\big(U(x, 4r(x))\big) \subset Y_{\gamma(x)}$,

$$\varepsilon(y) > \frac{\varepsilon(x)}{3} \quad \text{for each } y \in U(x, 4r(x)), \text{ and} \tag{6}$$

$$\big\|f'(x) - f'(y)\big\| < \frac{\varepsilon(x)}{9C_{\gamma(x)}} \quad \text{for each } y \in U(x, 4r(x)). \tag{7}$$

By our assumption there is a σ-discrete Lipschitz C^k-smooth partition of unity on Ω subordinated to $\{U(x, r(x));\ x \in \Omega\}$. We may assume that the partition of unity is of the form $\{\psi_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, where for each $n \in \mathbb{N}$ the family $\{\operatorname{supp}\psi_{n\alpha}\}_{\alpha\in\Lambda}$ is discrete in Ω. For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ let $U_{n\alpha} = U(x_{n\alpha}, r(x_{n\alpha}))$ be such that $\operatorname{supp}\psi_{n\alpha} \subset U_{n\alpha}$. Let $L_{n\alpha}$ be the Lipschitz constant of $\psi_{n\alpha}$. Further, denote $C_{n\alpha} = C_{\gamma(x_{n\alpha})}$ and $V_{n\alpha} = U(x_{n\alpha}, 2r(x_{n\alpha}))$. Without loss of generality assume that $L_{n\alpha} \ge 1$ and $C_{n\alpha} \ge 1$.

For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ let us define the mapping $f_{n\alpha}\colon U(x_{n\alpha}, 4r(x_{n\alpha})) \to Y_{\gamma(x_{n\alpha})}$ by $f_{n\alpha}(x) = f(x) - f'(x_{n\alpha})x$. Then, by (7) and (6),

$$\big\|f'_{n\alpha}(x)\big\| = \big\|f'(x) - f'(x_{n\alpha})\big\| < \frac{\varepsilon(x_{n\alpha})}{9C_{n\alpha}} < \frac{\varepsilon(x)}{3C_{n\alpha}} \le \frac{\varepsilon(x)}{3} \quad \text{for each } x \in U(x_{n\alpha}, 4r(x_{n\alpha})). \tag{8}$$

According to our assumption, for each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ we can approximate $f_{n\alpha}$ on $V_{n\alpha}$ by $g_{n\alpha} \in C^k(V_{n\alpha}, Y)$ such that

$$\big\|g'_{n\alpha}(x)\big\| \le \frac{\varepsilon(x_{n\alpha})}{9} < \frac{\varepsilon(x)}{3} \quad \text{for each } x \in V_{n\alpha}, \tag{9}$$

$$\|f_{n\alpha}(x) - g_{n\alpha}(x)\| \le \frac{\varepsilon(x_{n\alpha})}{9 \cdot 2^n L_{n\alpha}} < \frac{\varepsilon(x)}{3 \cdot 2^n L_{n\alpha}} < \varepsilon(x) \quad \text{for each } x \in V_{n\alpha}. \tag{10}$$

(The second inequalities follow from (6).) Define the mapping $\tilde{g}_{n\alpha}\colon \Omega \to Y$ by $\tilde{g}_{n\alpha}(x) = g_{n\alpha}(x)$ for $x \in V_{n\alpha}$, $\tilde{g}_{n\alpha}(x) = 0$ otherwise. Finally, we define the mapping $g\colon \Omega \to Y$ by

$$g(x) = \sum_{n\in\mathbb{N},\alpha\in\Lambda} \big(\tilde{g}_{n\alpha}(x) + f'(x_{n\alpha})x\big)\psi_{n\alpha}(x).$$

Since $\operatorname{supp}\psi_{n\alpha} \subset U_{n\alpha}$, $g_{n\alpha} \in C^k(V_{n\alpha}, Y)$, and the sum is locally finite, the mapping g is well defined and moreover $g \in C^k(\Omega, Y)$.

Choose $x \in \Omega$ and let us compute how far $g(x)$ is from $f(x)$:

$$\|f(x) - g(x)\| = \left\| \sum_{n\in\mathbb{N},\alpha\in\Lambda} \big(f(x) - \tilde{g}_{n\alpha}(x) - f'(x_{n\alpha})x\big)\psi_{n\alpha}(x) \right\| = \left\| \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \big(f_{n\alpha}(x) - g_{n\alpha}(x)\big)\psi_{n\alpha}(x) \right\|$$
$$\leq \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \|f_{n\alpha}(x) - g_{n\alpha}(x)\|\,\psi_{n\alpha}(x) < \varepsilon(x) \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \psi_{n\alpha}(x) = \varepsilon(x),$$

where the last inequality follows from (10).

To estimate the distance between the derivatives at some fixed $x \in \Omega$, notice that by the discreteness of $\{\operatorname{supp}\psi_{n\alpha}\}_{\alpha\in\Lambda}$, for each $n \in \mathbb{N}$ there is at most one $\alpha \in \Lambda$ such that $\psi'_{n\alpha}(x) \neq 0$. Put $M = \{n \in \mathbb{N};\ \exists\alpha \in \Lambda\colon \psi'_{n\alpha}(x) \neq 0\}$. Then there is a mapping $\beta\colon M \to \Lambda$ such that for each $n \in M$, $\psi'_{n\alpha}(x) = 0$ whenever $\alpha \neq \beta(n)$, and moreover $x \in U_{n\beta(n)}$. (Notice that if $\psi'_{n\alpha}(x) \neq 0$ then necessarily $x \in U_{n\alpha}$.) Hence,

$$\|f'(x) - g'(x)\| = \|(f-g)'(x)\| = \left\| \sum_{n\in\mathbb{N},\alpha\in\Lambda} \Big(\big(f - \tilde{g}_{n\alpha} - f'(x_{n\alpha})\big)\psi_{n\alpha}\Big)'(x) \right\|$$
$$= \left\| \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \Big(\big(f - g_{n\alpha} - f'(x_{n\alpha})\big)\psi_{n\alpha}\Big)'(x) \right\| = \left\| \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \big((f_{n\alpha} - g_{n\alpha})\psi_{n\alpha}\big)'(x) \right\|$$
$$= \left\| \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \psi_{n\alpha}(x)(f_{n\alpha} - g_{n\alpha})'(x) + \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \big(f_{n\alpha}(x) - g_{n\alpha}(x)\big)\psi'_{n\alpha}(x) \right\|$$
$$\leq \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \|f'_{n\alpha}(x) - g'_{n\alpha}(x)\|\,\psi_{n\alpha}(x) + \sum_{n\in M} \|f_{n\beta(n)}(x) - g_{n\beta(n)}(x)\|\,\big\|\psi'_{n\beta(n)}(x)\big\|$$
$$\leq \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \Big(\|f'_{n\alpha}(x)\| + \|g'_{n\alpha}(x)\|\Big)\psi_{n\alpha}(x) + \sum_{n\in M} \|f_{n\beta(n)}(x) - g_{n\beta(n)}(x)\|\,L_{n\beta(n)}$$
$$< \left(\frac{\varepsilon(x)}{3} + \frac{\varepsilon(x)}{3}\right) \sum_{\substack{n\in\mathbb{N} \\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \psi_{n\alpha}(x) + \sum_{n\in M} \frac{\varepsilon(x)}{3\cdot 2^n L_{n\beta(n)}} L_{n\beta(n)} \leq \varepsilon(x),$$

where the last but one inequality follows from (8), (9) and (10).

□

Theorem 19. *Let X, Y be normed linear spaces, $k \in \mathbb{N} \cup \{\infty\}$. Consider the following statements:*

(i) There is $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f : 2U_X \to Y$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X, Y)$ such that $\sup_{x\in U_X} \|f(x) - g(x)\| \leq \varepsilon$.

(ii) For any open $\Omega \subset X$ and any open covering $\mathcal{U}$ of Ω there is a Lipschitz C^k-smooth σ-discrete partition of unity on Ω subordinated to $\mathcal{U}$. There is $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f \in C^1(2U_X, Y)$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X, Y)$ such that $\sup_{x\in U_X} \|f(x) - g(x)\| \leq \varepsilon$.

(iii) For any open $\Omega \subset X$, any mapping $f \in C^1(\Omega, Y)$, and any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ there is $g \in C^k(\Omega, Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.

(iv) For any open $\Omega \subset X$, any L-Lipschitz mapping $f \in C^1(\Omega, Y)$, any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$, and any $\eta > 1$ there is an ηL-Lipschitz mapping $g \in C^k(\Omega, Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.

Then (i)⇒(ii)⇒(iii)⇒(iv).

Proof. (i)⇒(ii) follows from Corollary 14, (ii)⇒(iii) follows from Proposition 18 (consider the collection of subspaces of Y consisting only of the space Y itself), and for (iii)⇒(iv) see the end of the proof of Theorem 15.

□

Corollary 20. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Y be a Banach space. Suppose further that one of the following conditions is satisfied:*

- *at least one of the spaces X or Y is equal to $B_0(V)$ for some topological space V, or*
- *at least one of the spaces X or Y is equal to $C_u(P)$ for some metric space P, or*
- *X is a Banach space with an unconditional Schauder basis, or*
- *Y is a Banach space with an unconditional Schauder basis and with a separable dual.*

Then for any open $\Omega \subset X$, any mapping $f \in C^1(\Omega, Y)$, and any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ there is $g \in C^k(\Omega, Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.

Proof. Suppose that one of the first three conditions is satisfied. Then our corollary follows from Theorem 19. It suffices to notice that under our assumptions the statement (i) of Theorem 19 holds. Indeed, since B_X is a 2-Lipschitz retract of X, every L-Lipschitz mapping defined on B_X can be extended to a $2L$-Lipschitz mapping defined on X. Thus we may apply either Corollary 9 or Theorem I.

It remains to prove the case that Y has an unconditional Schauder basis $\{e_i\}$ and has a separable dual (which means that Y admits a C^1-smooth Lipschitz bump function). We will show that statement (ii) in Theorem 19 is satisfied, which will prove our claim. Since X is separable, it is not overly difficult to construct the required partitions of unity directly. Or, we may use Theorem G together with Lemma 13.

To prove the second assertion in statement (ii) of Theorem 19 let K be the constant from Lemma 5 used on the space Y. Put $C = 2K$. Let $f \in C^1(U_X, Y)$ be L-Lipschitz and $\varepsilon > 0$. Denote $Y_n = \operatorname{span}\{e_i\}_{i=1}^n$. By Lemma 5 there is a K-Lipschitz mapping $\psi \in C^1(Y, Y)$ which locally maps into some Y_n and such that $\|y - \psi(y)\| < \varepsilon/2$ for every $y \in Y$. Put $h = \psi \circ f$. Then $h \in C^1(U_X, Y)$ is a KL-Lipschitz mapping which locally maps into some Y_n and such that $\sup_{x \in U_X} \|f(x) - h(x)\| \leq \varepsilon/2$. Since the spaces Y_n, $n \in \mathbb{N}$, are finite-dimensional, by Corollary 9 there are constants C_n such that any M-Lipschitz mapping from U_X into Y_n can be approximated by $C_n M$-Lipschitz C^k-smooth mapping. Therefore we can use Proposition 18 to find a CL-Lipschitz mapping $g \in C^k(U_X, Y)$ such that $\sup_{x \in U_X} \|g(x) - h(x)\| \leq \varepsilon/2$. As $\sup_{x \in U_X} \|f(x) - g(x)\| \leq \varepsilon$, we have just shown that the statement (ii) in Theorem 19 holds.

□

We remind that in Corollary 20 the case when X has an unconditional Schauder basis was (basically) proven already by Moulis (Theorem A).

Finally, combining Theorem H and Theorem 19 we obtain the following corollary.

Corollary 21. *Let Γ be an arbitrary set, Y be a Banach space, $\Omega \subset c_0(\Gamma)$ open, $f \in C^1(\Omega, Y)$, and $\varepsilon\colon \Omega \to (0, +\infty)$ a continuous function. Then there is $g \in C^\infty(\Omega, Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

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