UNIFORMLY GÂTEAUX SMOOTH APPROXIMATIONS ON $c_0(\Gamma)$

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ABSTRACT. Every Lipschitz mapping from $c_0(\Gamma)$ into a Banach space Y can be uniformly approximated by Lipschitz mappings that are simultaneously uniformly Gâteaux smooth and C^∞-Fréchet smooth.

The main result of this note is a construction of uniform approximations of Lipschitz mappings from $c_0(\Gamma)$ into a Banach space Y, by means of Lipschitz mappings that are also uniformly Gâteaux (UG) smooth and C^∞-Fréchet differentiable. We also construct an equivalent UG and C^∞-Fréchet smooth renorming of $c_0(\Gamma)$. Finally, we construct an example of a convex, even, Lipschitz and UG-smooth separating function, such that the Minkowski functional of its sub-level set is not UG-smooth. The first two results answer problems posed in [FMZ]. An example of the implicit function method violating UG smoothness was lacking in the literature. Its existence is not surprising to the specialists, as the known constructions of UG renormings always take a detour around this otherwise standard method of obtaining smooth renormings.

Let us recall that all Banach spaces admitting a UG-smooth bump function are in particular weak Asplund spaces by a fundamental result of Preiss, Phelps and Namioka in [PPN] (see also [F] or [BL]). However, the additional uniformity of the derivatives leads to a considerably stronger theory, with several characterisations of Banach spaces admitting a UG bump function (or a renorming). In particular, a Banach space admits an equivalent UG renorming if and only if its dual ball is a uniform Eberlein compact [FGZ]. For basic properties of UG smoothness we refer to [DGZ] and [F–Z]. To shed some light on the significance of our results, let us briefly summarise some of the more recent results concerning UG smoothness, defined below (for simplicity we assume that the domain is a whole Banach space X).

It is shown in [LV] that a continuous UG-smooth real function on a Banach space X is locally Lipschitz. Moreover, if the function is uniformly continuous (or bounded), then it is globally Lipschitz. Thus some uniformity (Lipschitz) condition is in some sense also necessary for a mapping to be UG approximable. Tang [T] has shown that the existence of a UG bump function on a Banach space implies the existence of an equivalent UG renorming (analogous statement for Gâteaux smooth bumps is false, see [H]), and used this fact to show that every convex function on such a space is uniformly approximable by convex and UG-smooth functions on bounded sets. The more general problem of approximating all Lipschitz functions seems to be still open. One of the difficulties is that the standard approach to constructing smooth approximations by using smooth partitions of unity appears to be failing (loss of uniformity). In this regard let us mention that in the stronger uniformly Fréchet case it was shown by John, Toruńczyk and Zizler [JTZ] that the UF-smooth partitions of unity always exist provided the space has a UF bump function. However, the existence of UF approximations of Lipschitz functions seems to be open.

In the separable setting, Fabian and Zizler [FZ] were able to combine the best Fréchet smoothness of the space in question together with the UG condition (recall that every separable Banach space has a UG renorming). Namely, if a separable Banach space admits a C^k-Fréchet smooth norm, then it admits also a norm which is simultaneously C^k-Fréchet smooth and UG. The techniques used in their paper are strongly separable in nature, which leads to the natural question of what happens in the general case. This is the source of the questions asked in [FMZ], resolved in our note. Let us now proceed with the preliminaries to our results.

For an arbitrary set A, we denote its cardinality by $|A|$. For $n \in \mathbb{N}$, λ_n denotes the Lebesgue measure on $\mathbb{R}^n$.

For a metric space (X, ρ), we denote $B(x,r) = \{y \in X;\ \rho(x,y) \le r\}$ and $U(x,r) = \{y \in X;\ \rho(x,y) < r\}$ the closed and open ball in X centred at $x \in X$ with radius $r \ge 0$. Let $A \subset X$. A neighbourhood $U \subset X$ of A is called an *r-uniform neighbourhood* if there is $r > 0$ such that $\bigcup_{x\in A} U(x,r) \subset U$. A neighbourhood is called a *uniform neighbourhood* if it is r-uniform for some $r > 0$.

For $F \subset \Gamma$ we denote the associated projection in $c_0(\Gamma)$ by P_F, i.e. $P_F x = \sum_{\gamma\in F} e^*_\gamma(x) e_\gamma$ where $x \in c_0(\Gamma)$. By $c_{00}(\Gamma)$ we denote the linear subspace of $c_0(\Gamma)$ consisting of finitely supported vectors. The canonical supremum norm on $c_0(\Gamma)$ will be denoted by $\|\cdot\|$.

Let X and Y be normed linear spaces, $\Omega \subset X$ be open and $f\colon \Omega \to Y$. We will denote the directional derivative of f at $x \in \Omega$ in the direction $h \in X$ by $D_h f(x) = \lim_{t\to 0} \frac{1}{t}(f(x+th) - f(x))$. If f is Gâteaux differentiable for all $x \in \Omega$ (i.e. $D_h f(x)$ exists for all $h \in X$ and $h \mapsto D_h f(x)$ is a bounded linear operator) and moreover for all fixed $h \in X$ the limit defining $D_h f(x)$ is uniform in $x \in \Omega$ we say that f is uniformly Gâteaux differentiable (UG) on Ω.

We denote by $C^\infty(X,Y)$ the space of all C^∞-Fréchet smooth mappings from X into Y.

We are now ready to formulate the main results of our note.

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Theorem 1. *Let Γ be an arbitrary set, Y be a Banach space, $f\colon c_0(\Gamma) \to Y$ be an L-Lipschitz mapping, and let $\varepsilon > 0$. Then there is an L-Lipschitz mapping $g \in C^\infty(c_0(\Gamma), Y)$ which is uniformly Gâteaux differentiable and such that it satisfies $\sup_{c_0(\Gamma)} \|f(x) - g(x)\| \le \varepsilon$.*

Theorem 2. *For any set Γ, $c_0(\Gamma)$ admits an equivalent norm which is simultaneously C^∞-smooth and uniformly Gâteaux differentiable.*

Theorem 1 is an immediate consequence of the following theorem.

Theorem 3. *Let Γ be an arbitrary set, Y be a Banach space, $M \subset c_0(\Gamma)$, $U \subset c_0(\Gamma)$ be a uniform neighbourhood of M, $f\colon U \to Y$ be a uniformly continuous mapping with modulus of continuity ω and let $\varepsilon > 0$. Then there is a mapping $g \in C^\infty(c_0(\Gamma), Y)$ which uniformly locally depends on finitely many coordinates such that $\sup_M \|f(x) - g(x)\| \le \varepsilon$ and g is uniformly continuous on M with modulus of continuity dominated by ω. If f is moreover L-Lipschitz, then g is L-Lipschitz on M and uniformly Gâteaux differentiable on* $\operatorname{Int} M$.

We will prove the theorems by using a few lemmata.

Lemma 4. *Let X, Y be normed linear spaces, $\Omega \subset X$ be open and $f\colon \Omega \to Y$ be a Gâteaux differentiable mapping. If for each $h \in X$ the mapping $x \mapsto D_h f(x)$ is uniformly continuous on Ω, then f is uniformly Gâteaux differentiable on Ω provided that Ω is convex; otherwise f is uniformly Gâteaux differentiable on any open $V \subset \Omega$ satisfying* $\operatorname{dist}(V, X \setminus \Omega) > 0$. *Conversely, if f is uniformly Gâteaux differentiable and uniformly continuous on Ω, then for each $h \in X$ the mapping $x \mapsto D_h f(x)$ is uniformly continuous on any $A \subset \Omega$ satisfying* $\operatorname{dist}(A, X \setminus \Omega) > 0$.

Proof. Choose $h \in S_X$ and $\varepsilon > 0$, and find $\delta > 0$ such that $\|D_h f(x + th) - D_h f(x)\| < \varepsilon$ for all $x \in \Omega$ and $t \in (-\delta, \delta)$ such that $x + th \in \Omega$. If Ω is convex we set $V = \Omega$ and $\eta = \delta$, otherwise we let $\eta = \min\{\delta, \operatorname{dist}(V, X \setminus \Omega)\}$. Fix $x \in V$ and define a mapping $g\colon I \to Y$ by $g(t) = f(x + th) - tD_h f(x)$, where $I = \{t \in (-\eta, \eta);\ x + th \in \Omega\}$. Notice that I is an open interval containing 0 and $g'(t) = D_h f(x + th) - D_h f(x)$ for $t \in I$. By the assumption, $\|g'(t)\| \le \varepsilon$ for $t \in I$, hence g is ε-Lipschitz on I, and so $\left\|\frac{1}{t}(f(x + th) - f(x)) - D_h f(x)\right\| = \left\|\frac{1}{t}(g(t) - g(0))\right\| \le \varepsilon$ for all $t \in I$.

To prove the converse statement, choose $h \in X$, $h \ne 0$, a subset $A \subset \Omega$ for which $\operatorname{dist}(A, X \setminus \Omega) > 0$, and $\varepsilon > 0$. Find $0 < \eta < \operatorname{dist}(A, X \setminus \Omega)/\|h\|$ such that $\left\|\big(f(x + \eta h) - f(x)\big)/\eta - D_h f(x)\right\| < \frac{\varepsilon}{4}$ for any $x \in A$. Let $\delta > 0$ be such that $\|f(x) - f(y)\| < \eta\frac{\varepsilon}{4}$ whenever $x, y \in A$ are such that $\|x - y\| < \delta$. Then, for such x, y, we have

$$\|D_h f(x) - D_h f(y)\| < \frac{\varepsilon}{2} + \frac{1}{\eta}\|f(x + \eta h) - f(x) - f(y + \eta h) + f(y)\| < \varepsilon.$$

□

Lemma 5. *Let X and Y be normed linear spaces, H be a dense subset of X, $\Omega \subset X$ be open and $f\colon \Omega \to Y$ be a Gâteaux differentiable Lipschitz mapping such that for each $h \in H$ the mapping $x \mapsto D_h f(x)$ is uniformly continuous on Ω. Then the mapping $x \mapsto D_h f(x)$ is uniformly continuous on Ω for every $h \in X$.*

Proof. Let L be a Lipschitz constant of f. Pick an arbitrary $h \in X$ and let $\varepsilon > 0$. Find $h_0 \in H$ such that $\|h - h_0\| < \frac{\varepsilon}{4L}$. By the uniform continuity of $D_{h_0} f(x)$ there is $\delta > 0$ such that $\|D_{h_0} f(x) - D_{h_0} f(y)\| < \frac{\varepsilon}{2}$ whenever $x, y \in \Omega$, $\|x - y\| < \delta$. Then

$$\|D_h f(x) - D_h f(y)\| \le \|D_{h_0} f(x) - D_{h_0} f(y)\| + \|D_{h-h_0} f(x) - D_{h-h_0} f(y)\| < \frac{\varepsilon}{2} + 2L\|h - h_0\| < \varepsilon,$$

whenever $x, y \in \Omega$, $\|x - y\| < \delta$.

□

Lemma 6. *Let X be a normed linear space, $k \in \mathbb{N} \cup \{\infty\}$, $g\colon X \to \mathbb{R}$ be a C^k-smooth, UG, Lipschitz, even and convex function that is separating (i.e. there is an $r > 0$ such that $\inf_{x \in rS_X} |g(x) - g(0)| > 0$). Then X admits an equivalent C^k-smooth UG norm.*

Proof. As shown in the Example below, UG smoothness does not, in general, survive the standard use of the implicit function theorem (Minkowski functional). To be able to use the Minkowski functional of some sub-level set of g, we need to gain more control over $g'(x)x$. To this end we introduce a transformation, the idea of which comes from [FZ]. Basically, we construct a function that is "primitive" to g in a sense, so that its derivative is g back again (more or less), hence Lipschitz. (For a more detailed exposition of the method we refer to [FZ].) So, define $f\colon X \to \mathbb{R}$ by

$$f(x) = \int_{[0,1]} g(tx)\,\mathrm{d}\lambda(t).$$

Let L be the Lipschitz constant of g. It is easy to check that f is $L/2$-Lipschitz, even and convex.

Without loss of generality we may assume that $g(0) = 0$. By the convexity of g and the fact that g is even, $g(x) \ge 0$ for $x \in X$. Since g is separating, there are $r > 0$ and $a > 0$ such that $g(x) \ge a$ for all $x \in rS_X$. Hence $g(tx) \ge a - Lr(1 - t)$ whenever $t \in [0, 1]$ and $\|x\| = r$. It follows that

$$f(x) \ge \int_{1-a/(Lr)}^{1} \big(a - Lr(1 - t)\big)\,\mathrm{d}\lambda(t) = \frac{a^2}{2Lr} = b \quad \text{for any } x \in rS_X. \tag{1}$$

Let $x \in X$. Using the compactness of the set $\{tx;\ t \in [0,1]\}$ and the continuity of g', we can find a neighbourhood U of x such that $g'(ty)$ is bounded for $y \in U$ and $t \in [0,1]$. Hence, using the theory of integration, we can check that $f \in C^1(U)$ and

$$f'(x)h = \int\limits_{[0,1]} g'(tx)(th)\,\mathrm{d}\lambda(t). \tag{2}$$

By repeating the same argument it follows that $f \in C^k(X)$.

Using Lemma 4 and (2) we can see that the function $x \mapsto f'(x)h$ is uniformly continuous on X for any $h \in X$.

Moreover, the function $x \mapsto f'(x)x$ is Lipschitz on X. Indeed, using the substitution $t(1+\tau) = s$ we get $f(x+\tau x) = \int_{[0,1]} g(tx + t\tau x)\,\mathrm{d}\lambda(t) = \frac{1}{1+\tau}\int_{[0,1+\tau]} g(sx)\,\mathrm{d}\lambda(s)$. Thus, using the continuity of g along the way,

$$\begin{aligned} f'(x)x &= \lim_{\tau\to 0}\frac{1}{\tau}\big(f(x+\tau x) - f(x)\big) = \lim_{\tau\to 0}\frac{1}{\tau}\left(\left(\frac{1}{1+\tau} - 1\right)\int\limits_0^{1+\tau} g(tx)\,\mathrm{d}\lambda(t) + \int\limits_1^{1+\tau} g(tx)\,\mathrm{d}\lambda(t)\right) \\ &= \lim_{\tau\to 0}\frac{-1}{1+\tau}\int\limits_0^{1+\tau} g(tx)\,\mathrm{d}\lambda(t) + \lim_{\tau\to 0}\frac{1}{\tau}\int\limits_1^{1+\tau} g(tx)\,\mathrm{d}\lambda(t) = g(x) - \int\limits_0^1 g(tx)\,\mathrm{d}\lambda(t) = g(x) - f(x). \end{aligned}$$

Since both f and g are L-Lipschitz, the function $x \mapsto f'(x)x$ is $2L$-Lipschitz. Clearly, $f(0) = 0$. So, the convexity of f implies

$$f'(x)x \ge f(x) \quad \text{for any } x \in X. \tag{3}$$

Using the convexity of f, the estimate (1) and the fact that $f(0) = 0$, we obtain that $f(x) > b$ whenever $\|x\| > r$. It follows that $B = \{x \in X;\ f(x) \le b\}$ is a closed absolutely convex bounded set that contains a neighbourhood of 0. The Minkowski functional ν of the set B is therefore an equivalent norm on X and $\nu(x) = 1$ if and only if $f(x) = b$.

Put $M = X \times (0, +\infty)$ and define $F\colon M \to \mathbb{R}$ by $F(x,y) = f(\frac{x}{y}) - b$. It is obvious that $F \in C^k(M)$. Pick any $0 \ne x \in X$. Then $(x, \nu(x)) \in M$, $F(x,\nu(x)) = 0$, and

$$\frac{\partial F}{\partial y}(x,\nu(x)) = f'\left(\frac{x}{\nu(x)}\right)\left(-\frac{x}{\nu(x)^2}\right) = -\frac{1}{\nu(x)}f'\left(\frac{x}{\nu(x)}\right)\left(\frac{x}{\nu(x)}\right).$$

From (3) it follows that $\frac{\partial F}{\partial y}(x,\nu(x)) \le -b/\nu(x)$. Therefore we can use the Implicit Function Theorem to conclude that on some neighbourhood of x, ν is a C^k-smooth function. Hence ν is a C^k-smooth norm.

Finally, we claim that the function $x \mapsto \nu'(x)h$ is uniformly continuous on $A_R = X \setminus B(0,R)$ for any $h \in X$ and any $R > 0$, which according to Lemma 4 means that the norm ν is UG.

Since $F\big(x,\nu(x)\big) = 0$ for any $0 \ne x \in X$, it follows that $\big(F(x,\nu(x))\big)' = 0$. A simple computation yields

$$\nu'(x) = \frac{f'\left(\frac{x}{\nu(x)}\right)}{f'\left(\frac{x}{\nu(x)}\right)\left(\frac{x}{\nu(x)}\right)}.$$

Fix any $R > 0$ and $h \in X$. Denote $S = \{x \in X;\ \nu(x) = 1\}$. As the mapping $\psi\colon A_R \to S$, $\psi(x) = x/\nu(x)$ is Lipschitz and $\nu'(x) = \nu'(\psi(x))$, it is enough to show that $x \mapsto \nu'(x)h$ is uniformly continuous on S. Let $\varepsilon > 0$. Find $0 < \delta < \varepsilon$ such that $|f'(x)h - f'(y)h| < \varepsilon$ whenever $\|x - y\| < \delta$. Then, for any $x, y \in S$, $\|x-y\| < \delta$ we have

$$\begin{aligned} \left|\nu'(x)h - \nu'(y)h\right| &= \left|\frac{f'(x)h}{f'(x)x} - \frac{f'(y)h}{f'(y)y}\right| \le \frac{1}{|f'(x)x|}\left|f'(x)h - f'(y)h\right| + |f'(y)h|\left|\frac{1}{f'(x)x} - \frac{1}{f'(y)y}\right| \\ &\le \frac{\varepsilon}{b} + \frac{L}{2}\|h\|\frac{|f'(x)x - f'(y)y|}{|f'(x)x|\,|f'(y)y|} \le \frac{\varepsilon}{b} + \frac{L}{2}\|h\|\frac{2L\|x-y\|}{b^2} < \varepsilon\left(\frac{1}{b} + \frac{L^2}{b^2}\|h\|\right). \end{aligned}$$

□

Let X be a topological vector space, $\Omega \subset X$ an open subset, E be an arbitrary set, $M \subset X^*$ and $g\colon \Omega \to E$. We say that g *depends only on* M on a set $U \subset \Omega$ if $g(x) = g(y)$ whenever $x, y \in U$ are such that $f(x) = f(y)$ for all $f \in M$. We say that g *depends locally on finitely many coordinates from* M (LFC-M for short) if for each $x \in \Omega$ there are a neighbourhood $U \subset \Omega$ of x and a finite subset $F \subset M$ such that g depends only on F on U. We say that g *depends locally on finitely many coordinates* (LFC for short) if it is LFC-X^*.

Let $U \in \tau(0)$ be a neighbourhood of zero in X. We say that g *depends U-uniformly locally on finitely many coordinates from* M (U-ULFC-M for short) if for each $x \in \Omega$ there is a finite subset $F \subset M$ such that g depends only on F on $(x+U) \cap \Omega$.

Lemma 7. *Let Γ be an arbitrary set, Y be any Banach space, and $\Phi\colon c_0(\Gamma) \to Y$ be a mapping that is $U(0,r)$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ for some $r > 0$. Further, let $M \subset c_0(\Gamma)$, $\Omega \subset c_0(\Gamma)$ be an open uniform neighbourhood of M, Φ be uniformly continuous on Ω with modulus of continuity ω, let $\Phi = 0$ on $c_0(\Gamma) \setminus \Omega$, and let $\varepsilon > 0$. Then there is a $U(0,\frac{r}{2})$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ mapping $\Psi \in C^\infty(c_0(\Gamma), Y)$ such that $\sup_M \|\Phi(x) - \Psi(x)\| \le \varepsilon$, Ψ is uniformly continuous on M with modulus of continuity dominated by ω, and the mapping $x \mapsto \Psi'(x)h$ is uniformly continuous on M for any $h \in c_{00}(\Gamma)$. If Φ is even, then so is Ψ. If moreover $Y = \mathbb{R}$ and Φ is convex, then so is Ψ.*

Proof. Let $\eta > 0$ be such that Ω is an η-uniform neighbourhood of M and find $0 < \delta < \min\{\eta, \frac{r}{2}\}$ such that $\omega(\delta) < \varepsilon$. Choose φ to be an even C^∞-smooth non-negative function on $\mathbb{R}$ such that $\operatorname{supp}\varphi \subset [-\delta, \delta]$ and $\int_{\mathbb{R}} \varphi = 1$. We denote $C = \int_{\mathbb{R}} |\varphi'(t)| \,\mathrm{d}\lambda$.

Let $\mathcal{F} \subset 2^\Gamma$ be a partially ordered set of non-empty finite subsets of Γ, ordered by inclusion. For any $F \in \mathcal{F}$, we define the mapping $\Psi_F : c_0(\Gamma) \to Y$ by

$$\Psi_F(x) = \int_{\mathbb{R}^{|F|}} \Phi\Big(x - \sum_{\gamma\in F} t_\gamma e_\gamma\Big) \prod_{\gamma\in F} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|F|}(t),$$

where we integrate in the Bochner sense. Notice, that the integral is well-defined, since $\Phi = 0$ on the closed set $c_0(\Gamma) \setminus \Omega$ and Φ is uniformly continuous on Ω.

The net $\{\Psi_F\}_{\mathcal{F}}$ converges on $c_0(\Gamma)$ to a mapping $\Psi : c_0(\Gamma) \to Y$. In fact, we claim that for any $x \in c_0(\Gamma)$, there is an $F \in \mathcal{F}$ such that $\Psi_F(y) = \Psi_H(y)$ for any $F \subset H \in \mathcal{F}$ and any $y \in U(x, \frac{r}{2})$. Indeed, for a fixed $x \in c_0(\Gamma)$ let $F \in \mathcal{F}$ be such that Φ depends only on $\{e^*_\gamma\}_{\gamma\in F}$ on $U(x, r)$ and $\|x - P_F x\| < \frac{r}{2}$. Choose any $y \in U(x, \frac{r}{2})$ and $H \in \mathcal{F}$, $H \supset F$. Suppose that $t_\gamma \in [-\frac{r}{2}, \frac{r}{2}]$ for all $\gamma \in H$. Then $\big\|x - \big(y - \sum_{\gamma\in H} t_\gamma e_\gamma\big)\big\| < r$ and consequently $\Phi\left(y - \sum_{\gamma\in H} t_\gamma e_\gamma\right) = \Phi\left(y - \sum_{\gamma\in F} t_\gamma e_\gamma\right)$. Thus, by Fubini's theorem,

$$\begin{aligned}\Psi_H(y) &= \int_{[-\delta,\delta]^{|H|}} \Phi\Big(y - \sum_{\gamma\in H} t_\gamma e_\gamma\Big) \prod_{\gamma\in H} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|H|}(t)\\ &= \int_{[-\delta,\delta]^{|F|}} \Phi\Big(y - \sum_{\gamma\in F} t_\gamma e_\gamma\Big) \prod_{\gamma\in F} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|F|}(t) \prod_{\gamma\in H\setminus F} \int_{[-\delta,\delta]} \varphi(t_\gamma) \,\mathrm{d}\lambda = \Psi_F(y).\end{aligned}$$

Moreover, $\|x - P_F y\| \le \|x - P_F x\| + \|P_F\| \|x - y\| < r$ and so we can easily see that $\Psi_F(y) = \Psi_F(P_F y)$. The mapping $\Psi_F\restriction_{P_F c_0(\Gamma)}$ is in fact a finite-dimensional convolution with a smooth kernel on $R^{|F|}$, and so Ψ_F is a C^∞-smooth mapping on $U(x, \frac{r}{2})$.

The mapping Ψ is therefore $U(0, \frac{r}{2})$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ and $\Psi \in C^\infty(c_0(\Gamma), Y)$, as for any $x \in c_0(\Gamma)$, $\Psi = \Psi_F \circ P_F$ on $U(x, \frac{r}{2})$ for some $F \in \mathcal{F}$.

To show that $\sup_M \|\Phi(x) - \Psi(x)\| \le \varepsilon$ choose any $x \in M \subset \Omega$. Let $F \in \mathcal{F}$ be such that $\Psi(x) = \Psi_F(x)$. Notice that $\big\|x - \big(x - \sum_{\gamma\in F} t_\gamma e_\gamma\big)\big\| = \big\|\sum_{\gamma\in F} t_\gamma e_\gamma\big\| \le \delta < \eta$ whenever $t_\gamma \in [-\delta, \delta]$ for all $\gamma \in F$. Hence $x - \sum_{\gamma\in F} t_\gamma e_\gamma \in \Omega$ and

$$\begin{aligned}\|\Phi(x) - \Psi(x)\| = \|\Phi(x) - \Psi_F(x)\| &= \bigg\| \int_{\mathbb{R}^{|F|}} \Phi(x) \prod_{\gamma\in F} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|F|}(t) - \int_{\mathbb{R}^{|F|}} \Phi\Big(x - \sum_{\gamma\in F} t_\gamma e_\gamma\Big) \prod_{\gamma\in F} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|F|}(t) \bigg\|\\ &\le \int_{[-\delta,\delta]^{|F|}} \Big\|\Phi(x) - \Phi\Big(x - \sum_{\gamma\in F} t_\gamma e_\gamma\Big)\Big\| \prod_{\gamma\in F} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|F|}(t) \le \int_{[-\delta,\delta]^{|F|}} \omega(\delta) \prod_{\gamma\in F} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|F|}(t) = \omega(\delta) < \varepsilon.\end{aligned}$$

To see that the modulus of continuity of Ψ on M is dominated by ω, choose $x, y \in M$ and find $F, H \in \mathcal{F}$ such that $\Psi(x) = \Psi_F(x)$ and $\Psi(y) = \Psi_H(y)$. Then for $K = F \cup H$ we have $\Psi(x) = \Psi_K(x)$ and $\Psi(y) = \Psi_K(y)$. As $x - \sum_{\gamma\in K} t_\gamma e_\gamma \in \Omega$ and $y - \sum_{\gamma\in K} t_\gamma e_\gamma \in \Omega$ whenever $t_\gamma \in (-\eta, \eta)$ for all $\gamma \in K$,

$$\|\Psi(x) - \Psi(y)\| = \|\Psi_K(x) - \Psi_K(y)\| \le \int_{[-\delta,\delta]^{|K|}} \Big\|\Phi\Big(x - \sum_{\gamma\in K} t_\gamma e_\gamma\Big) - \Phi\Big(y - \sum_{\gamma\in K} t_\gamma e_\gamma\Big)\Big\| \prod_{\gamma\in K} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|K|}(t) \le \omega\big(\|x - y\|\big).$$

Similarly we can check that Ψ is even if Φ is even and Ψ is convex under the additional assumptions that $Y = \mathbb{R}$ and Φ is convex.

We finish the proof by showing that the directional derivatives of Ψ in the directions of $c_{00}(\Gamma)$ are uniformly continuous on M. So first, choose any $\alpha \in \Gamma$. For $x, y \in M$ find $F, H \in \mathcal{F}$ such that $\Psi(x) = \Psi_F(x)$ on $U(x, \frac{r}{2})$ and $\Psi(y) = \Psi_H(y)$ on $U(y, \frac{r}{2})$. Put $K = F \cup H \cup \{\alpha\}$. It is standard to show that

$$\Psi'_K(x) e_\alpha = \int_{\mathbb{R}^{|K|}} \Phi\Big(x - \sum_{\gamma\in K} t_\gamma e_\gamma\Big) \varphi'(t_\alpha) \prod_{\gamma\in K\setminus\{\alpha\}} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|K|}(t).$$

Hence, similarly as above,

$$\begin{aligned}\big\|\Psi'(x)e_\alpha - \Psi'(y)e_\alpha\big\| &\le \int_{\mathbb{R}^{|K|}} \Big\|\Phi\Big(x - \sum_{\gamma\in K} t_\gamma e_\gamma\Big) - \Phi\Big(y - \sum_{\gamma\in K} t_\gamma e_\gamma\Big)\Big\| \big|\varphi'(t_\alpha)\big| \prod_{\gamma\in K\setminus\{\alpha\}} \varphi(t_\gamma) \,\mathrm{d}\lambda_{|K|}(t)\\ &\le \omega\big(\|x - y\|\big) \int_{\mathbb{R}} \big|\varphi'(t)\big| \,\mathrm{d}\lambda = C\omega\big(\|x - y\|\big).\end{aligned} \tag{4}$$

Next, choose any $h \in c_{00}(\Gamma)$ and $x, y \in M$. It follows from (4) that

$$\big\|\Psi'(x)h - \Psi'(y)h\big\| \le \sum_{\gamma\in\operatorname{supp} h} \big\|\Psi'(x)(e^*_\gamma(h)e_\gamma) - \Psi'(y)(e^*_\gamma(h)e_\gamma)\big\| \le C\omega\big(\|x - y\|\big) \sum_{\gamma\in\operatorname{supp} h} \big|e^*_\gamma(h)\big| = C\,\|h\|_{\ell_1}\,\omega\big(\|x - y\|\big).$$

□

Proof of Theorem 3. Without loss of generality we may assume that U is open. Let $r > 0$ be such that U is an r-uniform open neighbourhood of M and find $0 < \eta \le \frac{r}{2}$ such that $\omega(\eta) < \frac{\varepsilon}{2}$. Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = \max\{0, t-\eta\} + \min\{0, t+\eta\}$. Then φ is 1-Lipschitz and $|\varphi(t) - t| \le \eta$ for all $t \in \mathbb{R}$.

Further, define a mapping $\phi\colon c_0(\Gamma) \to c_0(\Gamma)$ by $\phi(x) = \sum_{\gamma\in\Gamma} \varphi\big(e^*_\gamma(x)\big)e_\gamma$. (Notice that in fact ϕ maps into $c_{00}(\Gamma)$.) Then ϕ is 1-Lipschitz and $\|\phi(x) - x\| \le \eta$ for all $x \in c_0(\Gamma)$. Moreover, we claim that ϕ is $U(0, \frac{\eta}{2})$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$.

Indeed, fix $x \in c_0(\Gamma)$ and find $F \subset \Gamma$, $|F| < \infty$ such that $\|x - P_F x\| < \frac{\eta}{2}$. Then for any $y \in U(x, \frac{\eta}{2})$ we have $\|y - P_F y\| < \eta$. This means that if $y, z \in U(x, \frac{\eta}{2})$ are such that $e^*_\gamma(y) = e^*_\gamma(z)$ for all $\gamma \in F$, then $\varphi(e^*_\gamma(y)) = 0 = \varphi(e^*_\gamma(z))$ for all $\gamma \in \Gamma \setminus F$ and of course $\varphi(e^*_\gamma(y)) = \varphi(e^*_\gamma(z))$ for all $\gamma \in F$. Hence $\phi(y) = \phi(z)$, and so ϕ depends only on $\{e^*_\gamma\}_{\gamma\in F}$ on $U(x, \frac{\eta}{2})$.

We extend f to the whole of $c_0(\Gamma)$ by $f(x) = 0$ for $x \in c_0(\Gamma) \setminus U$ and put $\Phi = f \circ \phi$. Clearly, the mapping $\Phi\colon c_0(\Gamma) \to Y$ is $U(0, \frac{\eta}{2})$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$. Put $\Omega = \phi^{-1}(U)$. Then Ω is an $\frac{r}{2}$-uniform open neighbourhood of M in $c_0(\Gamma)$. Indeed, choose $x \in M$ and $y \in c_0(\Gamma)$ such that $\|x - y\| < \frac{r}{2}$. Then $\|\phi(y) - x\| \le \|\phi(y) - y\| + \|y - x\| < \eta + \frac{r}{2} \le r$, which means that $\phi(y) \in U$ and so $y \in \Omega$.

Moreover, Φ is uniformly continuous on Ω with modulus of continuity dominated by ω. To see this, choose any $x, y \in \Omega$. Then $\phi(x), \phi(y) \in U$ and hence $\|\Phi(x) - \Phi(y)\| \le \omega\big(\|\phi(x) - \phi(y)\|\big) \le \omega\big(\|x - y\|\big)$.

Finally, $\sup_\Omega \|f(x) - \Phi(x)\| \le \sup_\Omega \omega\big(\|x - \phi(x)\|\big) \le \omega(\eta) < \frac{\varepsilon}{2}$, and Lemma 7 together with Lemma 5 and Lemma 4 finishes the proof.

□

Proof of Theorem 2. Define a function $\Phi\colon c_0(\Gamma) \to \mathbb{R}$ by $\Phi(x) = \max\{0, \|x\| - 1\}$. Then Φ is a 1-Lipschitz convex even function which is $U(0, \frac{1}{2})$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$. (Notice that $\Phi = \|\cdot\| \circ \phi$ as in the proof of Theorem 3 for $\eta = 1$.)

Let $g \in C^\infty(c_0(\Gamma))$ be a 1-Lipschitz convex even function with uniformly continuous directional derivatives produced by Lemma 7 combined with Lemma 5, such that $|g(x) - \Phi(x)| \le 1$ for all $x \in c_0(\Gamma)$. Then g is separating, as $g(0) \le 1$ and $g(x) \ge 2$ on $4S_X$. The function g is also UG by Lemma 4, and so we can finish by using Lemma 6.

□

The technique used in the above proof can be used to strengthen the main result in [FHZ] on the existence of C^∞-Fréchet smooth approximations of strongly lattice norms on $c_0(\Gamma)$, by placing the additional UG smoothness requirement. We prefer to omit the details of the proof.

Example. We will sketch a construction of a UG, Lipschitz, even and convex function on c_0 that is separating, but the Minkowski functional of its sub-level set is an equivalent norm on c_0 that is not UG-smooth. The existence of such examples was since long suspected by specialists (e.g. [FZ], [T]), but no explicit construction seems to be available in the literature.

For any $n \in \mathbb{N}$, let $f_n : c_0 \to \mathbb{R}$ and $g_n : c_0 \to \mathbb{R}$ be defined as

$$f_n = e^*_1 + e^*_{2n} - e^*_{2n+1},$$
$$g_n = e^*_1 + \left(2 - \frac{1}{2n}\right) e^*_{2n} + e^*_{2n+1} - 1.$$

Let $\tilde{f}_n = f_n - e^*_{2n}/4$, $\tilde{g}_n = g_n - e^*_{2n}/4$, and further $g(x) = \sup_n\{\tilde{f}_n(x), \tilde{g}_n(x), \|x\|/4\}$ and $f(x) = g(x) + g(-x)$. It is easy to see that $f(0) = 0$, f is 8-Lipschitz, even, convex and separating. Further, for all $n \in \mathbb{N}$ and $w \in c_0$, let

$$x_n = e_{2n}, \qquad y_n = e_{2n} + \frac{1}{2n} e_{2n+1},$$
$$U_n(w) = \left\{x \in c_0;\ |e^*_{2n}(x - w)| < \frac{1}{16n}, \left|e^*_{2n+1}(x - w)\right| < \frac{1}{16n}, \left|e^*_j(x - w)\right| < \frac{1}{16} \text{ for } j \in \mathbb{N} \setminus \{2n, 2n+1\}\right\}.$$

We can check that $g(x) = \tilde{f}_n(x)$ on $U_n(x_n)$, $g(x) = \tilde{g}_n(x)$ on $U_n(y_n)$, and $g(-x) = \|x\|/4 = e^*_{2n}(x)/4$ on both $U_n(x_n)$ and $U_n(y_n)$. Hence, $f = f_n$ on $U_n(x_n)$ and $f = g_n$ on $U_n(y_n)$.

Now for each $n \in \mathbb{N}$ let φ_n be an even C^∞-smooth non-negative function on $\mathbb{R}$ such that $\int_{\mathbb{R}} \varphi_n = 1$. Moreover, choose these functions so that $\operatorname{supp}\varphi_1 \subset [-\frac{1}{32}, \frac{1}{32}]$, $\operatorname{supp}\varphi_{2n} \subset [-\frac{1}{32n}, \frac{1}{32n}]$ and $\operatorname{supp}\varphi_{2n+1} \subset [-\frac{1}{32n}, \frac{1}{32n}]$ for $n \in \mathbb{N}$. Similarly as in [FZ] or [J] we can show that the function

$$F(x) = \lim_{n\to\infty} \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j e_j\right) \prod_{j=1}^n \varphi_j(t_j)\, \mathrm{d}\lambda_n(t)$$

is well defined for all $x \in c_0$ and that it is Lipschitz, even, convex, separating and UG.

Furthermore, notice that for each $n \in \mathbb{N}$, f is affine on both $U_n(x_n)$ and $U_n(y_n)$. Since the convolution of an affine function with an even kernel is the same affine function again, there are neighbourhoods of x_n and y_n such that $F = f_n$ (or $F = g_n$ respectively) on those neighbourhoods.

Let ν be the Minkowski functional of the set $B = \{x \in c_0;\ F(x) \leq 1\}$. Then $\nu(x_n) = 1 = \nu(y_n)$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \|x_n - y_n\| = 0$, but

$$\nu'(x_n)e_1 = \frac{F'(x_n)e_1}{F'(x_n)x_n} = \frac{f_n'(x_n)e_1}{f_n'(x_n)x_n} = \frac{1}{1} = 1 \qquad \text{and} \qquad \nu'(y_n)e_1 = \frac{F'(y_n)e_1}{F'(y_n)y_n} = \frac{g_n'(y_n)e_1}{g_n'(y_n)y_n} = \frac{1}{2}.$$

References

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, vol. 1, Colloq. Publ. AMS **48**, 2000.

[DGZ] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics 64, Longman Scientific & Technical, 1993.

[F] M. Fabian, *Gâteaux differentiability of convex functions and topology – weak Asplund spaces*, John Wiley & Sons, Interscience, 1997.

[FGZ] M. Fabian, G. Godefroy and V. Zizler, *The structure of uniformly Gâteaux smooth Banach spaces*, Israel J. Math. **124** (2001), 243–252.

[F–Z] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant and V. Zizler, *Functional analysis and infinite dimensional geometry*, CMS books in Mathematics, Springer-Verlag, 2001.

[FHZ] M. Fabian, P. Hájek and V. Zizler, *A note on lattice renormings*, Comment. Math. Univ. Carolinae **38** (1997), no. 2, 263–272.

[FMZ] M. Fabian, V. Montesinos and V. Zizler, *Smoothness in Banach spaces. Selected problems*, Rev. R. Acad. Cien. Serie A Mat. **100** (2006), 101–125.

[FZ] M. Fabian and V. Zizler, *On uniformly Gâteaux smooth $C^{(n)}$-smooth norms on separable Banach spaces*, Czechoslovak Math. J. **49** (1999), 657–672.

[H] R. Haydon, *Trees in renorming theory*, Proc. London Math. Soc. **78** (1999), 549–584.

[J] M. Johanis, *Approximation of Lipschitz mappings*, Serdica Math. J. **29** (2003), no. 2, 141–148.

[JTZ] K. John, H. Toruńczyk and V. Zizler, *Uniformly smooth partitions of unity on superreflexive Banach spaces*, Studia Math. **70** (1981), 129–137.

[LV] D.P. McLaughlin and J.D. Vanderwerff, *Higher order Gâteaux smooth bump functions on Banach spaces*, Bull. Austral. Math. Soc. **51** (1995), 291–300.

[PPN] D. Preiss, R.R. Phelps and I. Namioka, *Smooth Banach spaces, weak Asplund spaces and monotone usco mappings*, Israel J. Math. **72** (1990), 257–279.

[T] W.K. Tang, *Uniformly differentiable bump functions*, Arch. Math. **68** (1997), 55–59.

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