ON UNIFORMLY GÂTEAUX SMOOTH NORMS AND NORMAL STRUCTURE

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ABSTRACT. It is shown that every separable Banach space admits an equivalent norm that is uniformly Gâteaux smooth and yet lacks asymptotic normal structure.

A Banach space is said to have the fixed point property (FPP), if for every non-empty bounded closed convex $C \subset X$ and every non-expansive self-mapping $T\colon C \to C$ there is a fixed point of T in C. A Banach space is said to have the weak fixed point property (w-FPP) if for every non-empty weakly compact convex $C \subset X$ there is a fixed point for every non-expansive $T\colon C \to C$. Clearly, a Banach space has w-FPP if it has FPP. The space c_0 has w-FPP but does not have FPP, see [M]. These two notions obviously coincide in reflexive spaces.

The classical results in metric fixed point theory state that a Banach space has w-FPP if its norm is uniformly Fréchet differentiable ([K]), or uniformly rotund ([B]). In fact, instead of uniformly rotund, it is sufficient to assume that the norm is only uniformly rotund in every direction (URED), [Z]. It is a natural question, whether the uniform Fréchet differentiability can be weakened to uniform Gâteaux differentiability (UG), since the notion of UG is dual (in a sense) to URED. (In fact, UG is dual to weak* uniform rotundity, which is a stronger notion than URED.)

We note that in a non-separable case, a theorem of [DLT] states that for any uncountable set Γ, the non-separable space $c_0(\Gamma)$ does not have FPP under any equivalent renorming. But it is well known that for any set Γ, $c_0(\Gamma)$ has an equivalent renorming that is simultaneously locally uniformly rotund, Fréchet differentiable and UG, see e.g. [DGZ, II.7.8]. Thus even norms with rather good geometrical properties do not assure FPP.

In our note we show that the usual proofs of "UF, UR or URED implies w-FPP" cannot be adapted, since they prove the w-FPP by showing that UF, UR or URED implies that the norm has a normal structure. We show that, in contrast to that, if the norm of a Banach space is UG, it does not necessarily have a normal structure. Even more, every separable Banach space can be equivalently renormed to have a uniformly Gâteaux smooth norm that lacks asymptotic normal structure. This notion was defined by J. B. Baillon and R. Schöneberg in [BS] as a weakening of the normal structure, which is still sufficient for w-FPP.

The norm $\|\cdot\|$ on a Banach space X is said to have *asymptotic normal structure* if for every closed convex bounded set $C \subset X$ with $\operatorname{diam} C > 0$ and every sequence $\{x_n\} \subset C$ satisfying $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ there exists $x \in C$ such that

$$\liminf_{n\to\infty} \|x_n - x\| < \operatorname{diam}_{\|\cdot\|} C.$$

The norm is called *uniformly Gâteaux smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + th\| - \|x\|}{t} = \|\cdot\|'(x, h)$$

is uniform in $x \in S_X$ for each $h \in S_X$, where S_X is the unit sphere of X. It follows, that the derivative of the norm at $x \in X \setminus \{0\}$, i.e. $h \mapsto \|\cdot\|'(x, h)$, is an element of X^*.

Recall that a Markushevich basis of a Banach space X is a biorthogonal system $\{e_n; f_n\} \subset X \times X^*$ such that $\overline{\operatorname{span}}\{e_n\} = X$ and $\{f_n\}$ separates the points of X (i.e. for any $x \neq y \in X$ there is $n \in \mathbb{N}$ such that $f_n(x) \neq f_n(y)$).

Theorem 1. *Let X be a separable Banach space. Then there exists an equivalent uniformly Gâteaux smooth norm lacking asymptotic normal structure.*

Proof. First, we will define a norm that lacks asymptotic normal structure. It will be done similarly as in [MS]. Let $\{e_n; f_n\}$ be a Markushevich basis of $(X, \|\cdot\|)$ such that $\|e_n\| = 1$ and $\|f_n\| \leq 20$ for all $n \in \mathbb{N}$ (see e.g. [LT, 1.f.4]). We put

$$C = \{x \in X;\ \|x\| \leq 2,\ 0 \leq f_n(x) \leq 1 \text{ for all } n \in \mathbb{N}\}.$$

This is a closed convex bounded set, $0 \in C$ and $\{e_n\} \subset C$. For an arbitrary $\beta \geq \operatorname{diam}_{\|\cdot\|} C$, we define a new norm

$$\|x\|_\beta = \max\Big\{\|x\|,\ \beta \sup_{n\in\mathbb{N}} |f_n(x)|\Big\},$$

which is obviously an equivalent norm on X.

Fact 2. *For all $n \in \mathbb{N}$, $\|e_n\|_\beta = \beta$ and $\|f_n\|_\beta^* = 1/\beta$.*

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Proof of the Fact 2.
$$\|e_n\|_\beta = \max\Big\{\|e_n\|, \beta \sup_{k\in\mathbb{N}} |f_k(e_n)|\Big\} = \max\{1,\beta\} = \beta.$$
Regarding f_n, we have
$$\|f_n\|_\beta^* \geq f_n\Big(\frac{e_n}{\beta}\Big) = \frac{1}{\beta},$$
and, on the other hand,
$$\|f_n\|_\beta^* = \sup\left\{f_n\left(\frac{\sum_{k=1}^N a_k e_k}{\left\|\sum_{k=1}^N a_k e_k\right\|_\beta}\right);\ N \geq n, a_1,\dots,a_N \in \mathbb{R}\right\} = \sup_{a_n\neq 0} \frac{a_n}{\left\|\sum_{k=1}^N a_k e_k\right\|_\beta} \leq \frac{a_n}{\beta a_n} = \frac{1}{\beta},$$
where the inequality holds because, by the definition, $\left\|\sum_{k=1}^N a_k e_k\right\|_\beta \geq \beta f_n\left(\sum_{k=1}^N a_k e_k\right) = \beta a_n$.

□

Fact 3. $\operatorname{diam}_{\|\cdot\|_\beta} C = \beta$.

Proof of the Fact 3. First, $\operatorname{diam}_{\|\cdot\|_\beta} C \geq \|e_1 - 0\|_\beta = \beta$. On the other hand, if $x, y \in C$, then (as $f_n(x), f_n(y) \in [0,1]$) $|f_n(x-y)| \leq 1$ and thus
$$\|x-y\|_\beta = \max\Big\{\|x-y\|, \beta \sup_{n\in\mathbb{N}} |f_n(x-y)|\Big\} \leq \beta.$$

□

Now we define a norm $|||\cdot|||_\beta^*$ on X^* by a formula
$$\left(|||f|||_\beta^*\right)^2 = \left(\|f\|_\beta^*\right)^2 + \sum_{n=1}^\infty \frac{1}{2^n} f^2(e_n).$$
By a standard convexity argument (see [DGZ, Fact II.2.3]), the norm $|||\cdot|||_\beta^*$ is W*UR. Since $|||\cdot|||_\beta^*$ is weak*-lsc, it is a dual norm. Let $|||\cdot|||_\beta$ be the norm on X that is predual to $|||\cdot|||_\beta^*$. By a standard duality argument (see [DGZ, Thm. II.6.7]), the norm $|||\cdot|||_\beta$ is uniformly Gâteaux smooth.

Fact 4. *a)* $\lim_{n\to\infty} |||f_n|||_\beta^* = 1/\beta$,
b) $\lim_{n\to\infty} |||e_n|||_\beta = \beta$,
c) $\operatorname{diam}_{|||\cdot|||_\beta} C = \beta$.

Proof of the Fact 4. a) Follows directly from Fact 2.

b) Since $|||f|||_\beta^* \geq \|f\|_\beta^*$ for all $f \in X^*$, we have $|||x|||_\beta \leq \|x\|_\beta$ for all $x \in X$ and thus $|||e_n|||_\beta \leq \beta$. On the other hand
$$\liminf_{n\to\infty} |||e_n|||_\beta \geq \liminf_{n\to\infty} \frac{f_n(e_n)}{|||f_n|||_\beta^*} = \beta.$$
c) As above, we get $\operatorname{diam}_{|||\cdot|||_\beta} C \leq \operatorname{diam}_{\|\cdot\|_\beta} C = \beta$. On the other hand,
$$\operatorname{diam}_{|||\cdot|||_\beta} C \geq |||e_n|||_\beta \to \beta.$$

□

Now we are ready to prove that $|||\cdot|||_\beta$ does not have asymptotic normal structure. Indeed, we define the sequence $\{x_n\} \subset C$ by
$$x_n = \begin{cases} (1 - j2^{-2k})e_k + e_{k+1} & \text{for } n = 2^{2k} + j,\ j = 1,\dots,2^{2k}, \\ e_{k+1} + j2^{-2k-1}e_{k+2} & \text{for } n = 2^{2k+1} + j,\ j = 1,\dots,2^{2k+1}. \end{cases}$$
Clearly, $x_n \in C$ and
$$\lim_{n\to\infty} |||x_n - x_{n+1}|||_\beta = 0$$
Choose $x \in C$. For any $\varepsilon > 0$ let $N \in \mathbb{N}$ and $y = \sum_{l=1}^N a_l e_l$ be such that $|||x-y|||_\beta < \varepsilon$. Then, for all $k > N$ and all $n = 2^{2k+i} + j$, $j = 1,\dots,2^{2k+i}$, $i = 0,1$,
$$|||x - x_n|||_\beta > |||y - x_n|||_\beta - \varepsilon \geq \frac{f_{k+1}(y - x_n)}{|||f_{k+1}|||_\beta^*} - \varepsilon = \frac{1}{|||f_{k+1}|||_\beta^*} - \varepsilon.$$
Thus,
$$\beta \geq \liminf_{n\to\infty} |||x - x_n|||_\beta \geq \beta - \varepsilon,$$
and consequently $\lim_{n\to\infty} |||x - x_n|||_\beta = \beta = \operatorname{diam}_{|||\cdot|||_\beta} C$.

□

Remark. Notice that in the proof we could take $C = \overline{\operatorname{conv}}\{0, e_n, e_n + e_{n+1};\ n \in \mathbb{N}\}$. If the basis $\{e_n\}$ is weakly null, then by Krein's theorem C is weakly compact and hence we have an example of a weakly compact convex set without asymptotic normal structure.

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