CHARACTERISATION OF REFLEXIVITY BY EQUIVALENT RENORMING

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ABSTRACT. A new rotundity property of Day's norm on $c_0(\Gamma)$ is introduced. This property provides in particular a renorming characterisation of the class of all reflexive Banach spaces.

Renorming characterisation of various classes of Banach spaces is important and useful for applications. To give a few examples, the most spectacular result in this area is certainly the Enflo-Pisier characterisation of super-reflexive spaces as those admitting uniformly rotund norm [E] (or even having power type modulus of uniform convexity [P]). A beautiful characterisation of spaces admitting uniformly Gâteaux smooth (UG) norm as those for which (B_{X^*}, w^*) is a uniform Eberlein compact was obtained in [FGZ] (see also a subsequent paper [FGHZ]).

Restricting to separable Banach spaces allows more results (not valid in the general case), for example Fréchet smooth or weakly uniformly rotund (WUR) renorming characterises spaces with a separable dual (Asplund spaces). A good source of other results and references on the subject is G. Godefroy's article in [JL], or [DGZ].

In the present note we are interested in renorming characterisation of reflexivity. Let us give a brief account of the known facts. The fundamental LUR renorming of the WCG spaces ([T]) together with the standard duality gives the well-known fact that X is reflexive if and only if there is an equivalent norm such that the dual norm on X^* is Fréchet differentiable. On the other hand, Milman in [M] introduced the notions of 2-rotund (2R) and weakly 2-rotund (W2R) (see below for these definitions). He states (without proof) that separable reflexive spaces are precisely the W2R renormable and asks whether reflexive spaces with LUR norm (this condition is redundant due to [T]) are 2R renormable. The last problem was settled positively for separable spaces by Odell and Schlumprecht [OS], but the general case remains open.

The main result of this note (Theorem 3) implies a characterisation of reflexive spaces as those admitting a W2R renorming. We also give examples showing that LUR renorming of a reflexive space is not necessarily W2R and vice versa.

First let us fix some notation. For a finite set A we denote the number of elements of A by $|A|$. Given $x = (x(\gamma))_{\gamma\in\Gamma} \in c_0(\Gamma)$ and $A \subset \Gamma$, $x\restriction_A$ denotes the vector defined as $x\restriction_A(\gamma) = x(\gamma)$ for $\gamma \in A$ and $x\restriction_A(\gamma) = 0$ for $\gamma \in \Gamma \setminus A$.

Definition. We say that a norm $\|\cdot\|$ on a Banach space X is 2-rotund (2R) (resp. weakly 2-rotund (W2R)) if for every $\{x_n\} \subset B_X$ such that

$$\lim_{m,n\to\infty} \|x_m + x_n\| = 2$$

there is an $x \in X$ such that $\lim_{n\to\infty} x_n = x$ in the norm (resp. weak) topology of X.

It is well-known (see e.g. [DGZ, II.6.2]) that the definition can be equivalently restated as

Fact 1. *A norm $\|\cdot\|$ on a Banach space X is 2R (resp. W2R) if and only if for every $\{x_n\} \subset X$ such that*

$$\lim_{m,n\to\infty} 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 = 0 \tag{1}$$

there is an $x \in X$ such that $\lim_{n\to\infty} x_n = x$ in the norm (resp. weak) topology of X.

This formulation is more convenient to use because it is homogeneous.

Using Šmulyan's criterion we immediately obtain

Fact 2. *If a norm $\|\cdot\|$ on a Banach space X is 2R then its dual norm $\|\cdot\|^*$ is Fréchet differentiable with all its derivatives contained in $X \subset X^{**}$. If a norm $\|\cdot\|$ on a Banach space X is W2R then its dual norm $\|\cdot\|^*$ is Gâteaux differentiable with all its derivatives contained in $X \subset X^{**}$.*

Note however, that the norm dual to a W2R norm need not to be Fréchet differentiable as Example 6 will show.

Recall that for an arbitrary set Γ, Day's norm on $c_0(\Gamma)$ is defined by

$$\|x\| = \sup\left\{\left(\sum_{k=1}^{n} 4^{-k} x^2(\gamma_k)\right)^{1/2} ;\ (\gamma_1, \dots, \gamma_n)\right\},$$

where the supremum is taken over all $n \in \mathbb{N}$ and all ordered n-tuples $(\gamma_1, \dots, \gamma_n)$ of distinct elements of Γ. Recall further that a norm $\|\cdot\|$ on a Banach space X is called locally uniformly rotund (LUR) if $\lim_{n\to\infty} \|x_n - x\| = 0$ whenever $x_n, x \in X$ are such that $\lim_{n\to\infty} 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0$. It is well-known that Day's norm is LUR ([R], cf. [DGZ, II.7.3]).

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Theorem 3. *Let Γ be an arbitrary set and $\|\cdot\|$ be Day's norm on $c_0(\Gamma)$. Let $\{x_n\} \subset c_0(\Gamma)$ satisfy* (1). *Then $\{x_n\}$ has a weak cluster point x if and only if $\lim_{n\to\infty} x_n = x$ (in the norm topology).*

Proof. Every weak cluster point of $\{x_n\}$ is a weak limit of some subsequence of $\{x_n\}$. (Indeed, $\overline{\{x_n\}}^w \subset c_0(\bigcup \operatorname{supp} x_n)$ which has separable dual, and as $\overline{\{x_n\}}^w$ is bounded (see below), it is metrisable.)

Notice further the following easy observation: If a norm $\|\cdot\|$ on a Banach space X is LUR and $\{x_n\} \subset X$ satisfies (1) and has a cluster point x (in the norm topology) then already $x_n \to x$ (in the norm).

Since obviously any subsequence of $\{x_n\}$ also satisfies (1), by the facts mentioned above we may assume that $x_n \to x$ weakly and we have to find a subsequence of $\{x_n\}$ norm converging to x.

Let $\|\cdot\|_\infty$ denote the canonical norm on $c_0(\Gamma)$. Let $\{\alpha_k^n\}$ be the support of x_n enumerated so that $\left|x_n(\alpha_1^n)\right| \geq \left|x_n(\alpha_2^n)\right| \geq \cdots$ and $\{\beta_k^{m,n}\}$ be the support of $(x_m + x_n)$ enumerated so that $\left|(x_m + x_n)(\beta_1^{m,n})\right| \geq \left|(x_m + x_n)(\beta_2^{m,n})\right| \geq \cdots$. Note that we may and do assume that $\beta_k^{m,n} = \beta_k^{n,m}$, $k \in \mathbb{N}$.

From the definition of Day's norm

$$\|x_n\|^2 = \sum_k 4^{-k} x_n^2(\alpha_k^n) \geq \sum_k 4^{-k} x_n^2(\gamma_k) \tag{2}$$

for any sequence $\{\gamma_k\} \subset \Gamma$. Hence

$$\begin{aligned} 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 &= 2\sum 4^{-k} x_m^2(\alpha_k^m) + 2\sum 4^{-k} x_n^2(\alpha_k^n) - \sum 4^{-k}(x_m + x_n)^2(\beta_k^{m,n}) \\ &\geq 2\sum 4^{-k} x_m^2(\beta_k^{m,n}) + 2\sum 4^{-k} x_n^2(\beta_k^{m,n}) - \sum 4^{-k}(x_m + x_n)^2(\beta_k^{m,n}) \\ &= \sum 4^{-k}\big(x_m(\beta_k^{m,n}) - x_n(\beta_k^{m,n})\big)^2 \geq 0. \end{aligned} \tag{3}$$

As $2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \geq \big(\|x_m\| - \|x_n\|\big)^2 \geq 0$, (1) implies that $\{\|x_n\|\}$ is Cauchy and hence $\{\|x_n\|_\infty\}$ is bounded. Therefore by passing to a suitable subsequence we may assume that there is $z \in \ell_\infty$ such that $\left|x_n(\alpha_k^n)\right| \to z(k)$, $k \in \mathbb{N}$. Notice that $z(1) \geq z(2) \geq \cdots \geq 0$. The vector z represents the asymptotic "shape" of the vectors x_n.

We claim that $z \in c_0$. If this is not the case then there is a $C > 0$ such that $z(k) > C$ for $k \in \mathbb{N}$. Then there is a finite $A \subset \Gamma$ such that $\|x\restriction_{\Gamma\setminus A}\|_\infty < \frac{C}{8}$. By (3) and (1) there is $m_0 \in \mathbb{N}$ such that

$$\sum_k 4^{-k}\big(x_m(\beta_k^{m,n}) - x_n(\beta_k^{m,n})\big)^2 < 4^{-|A|-1}\frac{C^2}{16} \quad \text{for } m, n > m_0. \tag{4}$$

Since $\left|x_n\big(\alpha_{|A|+1}^n\big)\right| \to z(|A|+1) > C$, there is $n_1 > m_0$ such that $\left|x_{n_1}\big(\alpha_{|A|+1}^{n_1}\big)\right| > C$. Thus we can choose $\gamma \in \Gamma \setminus A$ for which $\left|x_{n_1}(\gamma)\right| > C$. Next we find a finite $B \subset \Gamma$ such that

$$\|x_{n_1}\restriction_{\Gamma\setminus B}\|_\infty < \frac{C}{8}. \tag{5}$$

This implies that $\gamma \in B \setminus A$. Using the weak convergence we choose $n_2 > m_0$ such that $\left\|(x_{n_2} - x)\restriction_B\right\|_\infty < \frac{C}{8}$. Therefore we have

$$\|x_{n_2}\restriction_{B\setminus A}\|_\infty < \frac{C}{4} \tag{6}$$

and so $\left|x_{n_2}(\gamma)\right| < \frac{C}{4}$. Further,

$$\left|x_{n_1}(\gamma) + x_{n_2}(\gamma)\right| > \frac{3}{4}C. \tag{7}$$

We find the smallest $k_0 \in \mathbb{N}$ for which $\beta_{k_0}^{n_1,n_2} \notin A$. It follows that $k_0 \leq |A| + 1$ and

$$\left|(x_{n_1} + x_{n_2})\left(\beta_{k_0}^{n_1,n_2}\right)\right| \geq \left|(x_{n_1} + x_{n_2})(\gamma)\right|. \tag{8}$$

Now either $\beta_{k_0}^{n_1,n_2} \in B \setminus A$ and we can use (8), (7) and (6) to obtain

$$\begin{aligned} \left|x_{n_1}(\beta_{k_0}^{n_1,n_2}) - x_{n_2}(\beta_{k_0}^{n_1,n_2})\right| &\geq \left|x_{n_1}(\beta_{k_0}^{n_1,n_2}) + x_{n_2}(\beta_{k_0}^{n_1,n_2})\right| - 2\left|x_{n_2}(\beta_{k_0}^{n_1,n_2})\right| \\ &\geq \left|x_{n_1}(\gamma) + x_{n_2}(\gamma)\right| - 2\left|x_{n_2}(\beta_{k_0}^{n_1,n_2})\right| \geq \frac{3}{4}C - \frac{1}{2}C \geq \frac{C}{4}, \end{aligned}$$

or $\beta_{k_0}^{n_1,n_2} \in \Gamma \setminus (B \cup A)$ and we use (8), (7) and (5) instead to get the same conclusion. Finally

$$\sum_k 4^{-k}\big(x_{n_1}(\beta_k^{n_1,n_2}) - x_{n_2}(\beta_k^{n_1,n_2})\big)^2 \geq 4^{-k_0}\big(x_{n_1}(\beta_{k_0}^{n_1,n_2}) - x_{n_2}(\beta_{k_0}^{n_1,n_2})\big)^2 \geq 4^{-|A|-1}\frac{C^2}{16}$$

which contradicts (4).

Now we stabilise the supports of the vectors x_n. By (3),

$$\begin{aligned}0 &\le 2\sum 4^{-k}x_m^2(\alpha_k^m) + 2\sum 4^{-k}x_n^2(\alpha_k^n) - \sum 4^{-k}(x_m+x_n)^2(\beta_k^{m,n}) \\ &\quad - \Big(2\sum 4^{-k}x_m^2(\beta_k^{m,n}) + 2\sum 4^{-k}x_n^2(\beta_k^{m,n}) - \sum 4^{-k}(x_m+x_n)^2(\beta_k^{m,n})\Big) \\ &\le 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m+x_n\|^2,\end{aligned}$$

which together with (2) and (1) gives

$$\lim_{m,n\to\infty}\Big(\sum 4^{-k}x_n^2(\alpha_k^n) - \sum 4^{-k}x_n^2(\beta_k^{m,n})\Big) = 0. \tag{9}$$

But, for every $j \in \mathbb{N}$

$$\begin{aligned}\sum_{k=1}^{\infty} 4^{-k}x_n^2(\alpha_k^n) - \sum_{k=1}^{\infty} 4^{-k}x_n^2(\beta_k^{m,n}) &= \sum_{k=1}^{\infty}\Big(4^{-k} - 4^{-(k+1)}\Big)\left(\sum_{i=1}^{k} x_n^2(\alpha_i^n) - \sum_{i=1}^{k} x_n^2(\beta_i^{m,n})\right) \\ &\ge \Big(4^{-j} - 4^{-(j+1)}\Big)\left(x_n^2(\alpha_j^n) - x_n^2(\alpha_{j+1}^n)\right),\end{aligned} \tag{10}$$

unless $\{\alpha_i^n;\ 1 \le i \le j\} = \{\beta_i^{m,n};\ 1 \le i \le j\}$.

Indeed, if $\{\alpha_i^n;\ 1 \le i \le j\} \ne \{\beta_i^{m,n};\ 1 \le i \le j\}$, then $x_n^2(\alpha_1^n) + x_n^2(\alpha_2^n) + \dots + x_n^2(\alpha_{j-1}^n) + x_n^2(\alpha_{j+1}^n) \ge \sum_{i=1}^{j} x_n^2(\beta_i^{m,n})$.

If $z(1) = 0$ then easily $\|x_n\|_\infty \le |x_n(\alpha_1^n)| \to z(1) = 0$. Otherwise choose $0 < \varepsilon \le z(1)$. As $z \in c_0$ we can find $k_1 \in \mathbb{N}$ such that $z(k_1+1) < \varepsilon$ and $z(k_1) \ge \varepsilon$. Put $\delta = \frac{1}{3}\big(z(k_1) - z(k_1+1)\big)$. There is $n_3 \in \mathbb{N}$ such that $\big||x_n(\alpha_k^n)| - z(k)\big| < \min\{\delta, \varepsilon\}$ for $n > n_3$ and $1 \le k \le k_1+1$ and thus $|x_n(\alpha_{k_1}^n)| - |x_n(\alpha_{k_1+1}^n)| > \delta$ for $n > n_3$. By putting this fact together with (10) and (9) we obtain $m_1 > n_3$ such that $\{\alpha_k^n;\ 1 \le k \le k_1\} = \{\beta_k^{m,n};\ 1 \le k \le k_1\}$ for $m, n > m_1$. Because we have $\{\alpha_k^m;\ 1 \le k \le k_1\} = \{\beta_k^{n,m};\ 1 \le k \le k_1\} = \{\beta_k^{m,n};\ 1 \le k \le k_1\} = \{\alpha_k^n;\ 1 \le k \le k_1\}$ for $m, n > m_1$, the sets $\{\alpha_k^n;\ 1 \le k \le k_1\}$ are equal for $n > m_1$ and we denote this set by E.

The definitions of E, n_3, and k_1 in the previous paragraph give $\|x_n\restriction_{\Gamma\setminus E}\|_\infty \le |x_n(\alpha_{k_1+1}^n)| < z(k_1+1)+\varepsilon < 2\varepsilon$ for $n > m_1$. This together with the weak convergence implies $\|x\restriction_{\Gamma\setminus E}\|_\infty < 2\varepsilon$ and so $\|(x-x_n)\restriction_{\Gamma\setminus E}\|_\infty \le \|x\restriction_{\Gamma\setminus E}\|_\infty + \|x_n\restriction_{\Gamma\setminus E}\|_\infty < 4\varepsilon$ for $n > m_1$. Finally, using the weak convergence again, pick $n_0 > m_1$ such that $\|(x-x_n)\restriction_E\|_\infty < \varepsilon$ for $n > n_0$. Then $\|x - x_n\|_\infty \le \max\{\|(x-x_n)\restriction_E\|_\infty, \|(x-x_n)\restriction_{\Gamma\setminus E}\|_\infty\} < 4\varepsilon$ for $n > n_0$.

□

Corollary 4. *Let $(X, |\cdot|)$ be a Banach space such that there is a one-to-one bounded linear operator $T\colon X \to c_0(\Gamma)$ for some Γ. Then there is an equivalent norm $\|\cdot\|$ on X such that any sequence $\{x_n\} \in X$ satisfying* (1) *has at most one weak cluster point.*

Proof. Define a norm on X by $\|x\|^2 = |x|^2 + \|Tx\|_D^2$, where $\|\cdot\|_D$ is Day's norm on $c_0(\Gamma)$. Clearly it is an equivalent norm on X. Let $\{x_n\}$ satisfy (1). Similarly as above it follows that $\lim_{m,n\to\infty} 2\|Tx_m\|_D^2 + 2\|Tx_n\|_D^2 - \|Tx_m + Tx_n\|_D^2 = 0$ and so $\{Tx_n\}$ satisfies (1) in the norm $\|\cdot\|_D$. We can apply Theorem 3 to the sequence $\{Tx_n\}$ and since T is w–w-continuous and one-to-one, $\{x_n\}$ cannot have more than one weak cluster point.

□

Corollary 5. *Let X be a Banach space. Then X is reflexive if and only if it admits an equivalent W2R norm.*

Proof. The "if" part follows easily from James' theorem: Let $\|\cdot\|$ be a W2R norm on X. Fix any $f \in X^* \setminus \{0\}$. Choose x_n in B_X such that $f(x_n) \to \|f\|$. Then $0 \le 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \le 4 - \|f\|^{-2} f(x_m+x_n)^2 \to 0$. Thus there is $x \in X$ such that $x_n \to x$ weakly, hence $f(x) = \lim f(x_n) = \|f\|$ and by James' theorem X is reflexive.

Alternatively (as non-separable James' theorem is rather hard) we can use Fact 2 to see that for any $F \in X^{**}$ that attains its norm we have $F \in X$ and then by the Bishop-Phelps theorem X is reflexive.

The "only if" part:

It is very easy to construct an equivalent W2R norm on a separable reflexive X. Let $|\cdot|$ be the original norm on X, $\{f_k\}$ be a countable subset of B_{X^*} that distinguishes points of X. Define a new norm by

$$\|x\|^2 = |x|^2 + \sum_{k=1}^{\infty} 2^{-k} f_k(x)^2.$$

Clearly it is an equivalent norm on X. Observe that since X is reflexive, to show that $\|\cdot\|$ is W2R it only suffices to show that for any sequence $\{x_n\}$ satisfying (1) and such that $x_{2n} \to x \in X$ weakly and $x_{2n+1} \to y \in X$ weakly, we have $x = y$. Indeed, as $2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \ge \big(\|x_m\| - \|x_n\|\big)^2 \ge 0$, any sequence $\{x_n\}$ satisfying (1) is bounded and hence relatively weakly compact and so we only need to show that it has only one weak cluster point.

Let $\{x_n\}$ satisfy (1), $x_{2n} \to x \in X$ weakly and $x_{2n+1} \to y \in X$ weakly. As

$$\begin{aligned}2\|x_{2n+1}\|^2 + 2\|x_{2n}\|^2 - \|x_{2n+1} + x_{2n}\|^2 &= 2|x_{2n+1}|^2 + 2|x_{2n}|^2 - |x_{2n+1} + x_{2n}|^2 \\ &\quad + \sum 2^{-k}\left(2f_k(x_{2n+1})^2 + 2f_k(x_{2n})^2 - f_k(x_{2n+1} + x_{2n})^2\right) \\ &\ge \big(|x_{2n+1}| - |x_{2n}|\big)^2 + \sum 2^{-k}\big(f_k(x_{2n+1}) - f_k(x_{2n})\big)^2 \ge 0\end{aligned}$$

and all the summands in the last term are non-negative, (1) implies $0 = \lim_{n\to\infty} f_k(x_{2n+1}) - f_k(x_{2n}) = f_k(y) - f_k(x)$ for any $k \in \mathbb{N}$. Therefore $x = y$.

In a general case of X non-separable we can use the renorming from Corollary 4 (The existence of the required operator is well-known, see e.g. [DGZ, VI.5].) The weak compactness of B_X implies that this renorming is W2R.

□

The following two examples show that Troyanski's construction of the LUR norm on a reflexive space is neither sufficient for nor overcome by W2R renorming. Recall that a norm $\|\cdot\|$ on a Banach space X is said to be midpoint locally uniformly rotund (MLUR) if given sequences $\{x_n\}$, $\{y_n\}$ and x in X we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$ whenever $\|x_n\| \le \|x\|$, $\|y_n\| \le \|x\|$ and $\lim_{n\to\infty} \|x_n + y_n - 2x\| = 0$. It is easy to show that if all of the points of $S_{(X,\|\cdot\|)}$ are strongly exposed then $\|\cdot\|$ is MLUR. Hence if $\|\cdot\|$ is LUR or $\|\cdot\|^*$ is Fréchet differentiable then $\|\cdot\|$ is MLUR.

Example 6. *There is an equivalent norm $\|\cdot\|$ on ℓ_2 such that it is W2R but not MLUR (and thus neither $\|\cdot\|$ is LUR nor $\|\cdot\|^*$ is Fréchet differentiable).*

Proof. Let $\|\cdot\|_2$ be the canonical norm on ℓ_2 and let us define the new norm by

$$\|x\|^2 = \bigl(\max\{\|x\|_2, 2|x_1|\}\bigr)^2 + \sum_{i=2}^{\infty} 2^{-i} x_i^2.$$

This is clearly an equivalent norm.

Let us denote the ith coordinate of a vector $x_n \in \ell_2$ by $x_n(i)$. In view of the construction of the W2R norm on a separable reflexive space in the proof of Corollary 5 it remains to show that if $\{x_n\}$ satisfies (1), $x_{2n} \to x \in \ell_2$ weakly and $x_{2n+1} \to y \in \ell_2$ weakly then $x(1) = y(1)$. By passing to a subsequence we may assume that either always $\|x_{2n} + x_{2n+1}\|_2 \ge 2|x_{2n}(1) + x_{2n+1}(1)|$ or always $2|x_{2n}(1) + x_{2n+1}(1)| \ge \|x_{2n} + x_{2n+1}\|_2$. In the first case

$$\begin{aligned}
&2\|x_{2n+1}\|^2 + 2\|x_{2n}\|^2 - \|x_{2n+1} + x_{2n}\|^2 \\
&\qquad = 2\bigl(\max\{\|x_{2n+1}\|_2, 2|x_{2n+1}(1)|\}\bigr)^2 + 2\bigl(\max\{\|x_{2n}\|_2, 2|x_{2n}(1)|\}\bigr)^2 - \|x_{2n+1} + x_{2n}\|_2^2 \\
&\qquad\quad + 2\sum_{i=2}^{\infty} 2^{-i} x_{2n+1}(i)^2 + 2\sum_{i=2}^{\infty} 2^{-i} x_{2n}(i)^2 - \sum_{i=2}^{\infty} 2^{-i}\bigl(x_{2n+1}(i) + x_{2n}(i)\bigr)^2 \\
&\qquad \ge 2\|x_{2n+1}\|_2^2 + 2\|x_{2n}\|_2^2 - \|x_{2n+1} + x_{2n}\|_2^2 \ge 0
\end{aligned}$$

and the uniform rotundity of $\|\cdot\|_2$ implies $x = y$. In the second case similarly

$$\begin{aligned}
2\|x_{2n+1}\|^2 + 2\|x_{2n}\|^2 - \|x_{2n+1} + x_{2n}\|^2 &\ge 2\cdot 4|x_{2n+1}(1)|^2 + 2\cdot 4|x_{2n}(1)|^2 - 4|x_{2n+1}(1) + x_{2n}(1)|^2 \\
&= 4\bigl(x_{2n+1}(1) - x_{2n}(1)\bigr)^2 \ge 0.
\end{aligned}$$

Thus obviously $x(1) = y(1)$ and we can conclude that $\|\cdot\|$ is W2R.

Now put $\tilde{x}_n = e_1 + e_n$, $\tilde{y}_n = e_1 - e_n$, $x_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$, $y_n = \frac{\tilde{y}_n}{\|\tilde{y}_n\|}$ and $x = \frac{e_1}{2}$. Then, $\|\tilde{x}_n\| = \|\tilde{y}_n\| = (4 + 2^{-n})^{1/2} \to 2$ and $\|x_n + y_n - 2x\| = \left\|\frac{2e_1}{\|\tilde{x}_n\|} - e_1\right\| \to 0$, but $\|x_n - y_n\| = \frac{2\|e_n\|}{\|\tilde{x}_n\|} > \frac{2}{3}$ and so $\|\cdot\|$ is not MLUR.

□

Example 7. *There is an equivalent norm $\|\cdot\|$ on ℓ_2 such that it is LUR but not W2R.*

Proof. Let us define the norm on ℓ_2 by

$$\begin{aligned}
|x|_{i,j}^2 &= \bigl(|x_1| + |x_i| + |x_j|\bigr)^2 + \frac{1}{i+j}\left(x_1^2 + x_i^2 + x_j^2\right) + \sum_{\substack{k=2\\ k\neq i,j}}^{\infty} x_k^2 \\
\|x\|^2 &= \sup_{1<i<j}\left\{|x|_{i,j}^2\right\}.
\end{aligned}$$

This is clearly an equivalent norm.

We claim that locally (away from the origin) the supremum can be taken over a finite set. To see this fix any $x \in \ell_2 \setminus \{0\}$. We have to distinguish two cases.

If we can choose $k > 1$ such that $x_k \neq 0$ then there is i_0 such that $|x_i| < \frac{|x_k|}{3}$ for $i > i_0$ and if we denote by $m(y) \in \mathbb{N}$ the largest index for which $|y_{m(y)}| = \max\{|y_i|, i > 1\}$ then clearly $m(y) \le i_0$ for any $y \in \ell_2$ such that $\|x - y\|_2 < \frac{|x_k|}{3}$. Let $\varepsilon = \frac{1}{3i_0}\frac{x_k^2}{4}\frac{1}{16\|x\|_2}$ and find j_0 such that $8\|x\|_2\left(|x_j| + \varepsilon\right) + \frac{4\|x\|_2^2}{j} < \frac{1}{3i_0}\frac{x_k^2}{4}$ for $j > j_0$. Then for any $y \in \ell_2$ such that

$\|x-y\|_2 < \min\left\{\frac{|x_k|}{3}, \varepsilon\right\}$ we have

$$|y|^2_{i,j} = \|y\|_2^2 + 2\big(|y_1 y_i| + |y_1 y_j| + |y_i y_j|\big) + \frac{1}{i+j}\big(y_1^2 + y_i^2 + y_j^2\big) \le \|y\|_2^2 + 2|y_1||y_i| + 4\|y\|_2|y_j| + \frac{1}{i+j}\|y\|_2^2$$

$$\le \|y\|_2^2 + 2|y_1||y_{m(y)}| + 8\|x\|_2\big(|x_j| + \varepsilon\big) + \frac{4}{i+j}\|x\|_2^2 < \|y\|_2^2 + 2|y_1||y_{m(y)}| + \frac{1}{3i_0}\frac{x_k^2}{4} \qquad \text{for } j > j_0.$$

On the other hand,

$$\|y\|^2 \ge \|y\|_2^2 + 2|y_1||y_{m(y)}| + \frac{1}{m(y)+2m(y)}y_{m(y)}^2 \ge \|y\|_2^2 + 2|y_1||y_{m(y)}| + \frac{1}{3i_0}y_k^2 > \|y\|_2^2 + 2|y_1||y_{m(y)}| + \frac{1}{3i_0}\frac{x_k^2}{4}.$$

In the second case we have $x_i = 0$ for $i > 1$ and $x_1 \ne 0$. Let $\varepsilon = \frac{1}{5}\frac{x_1^2}{4}\frac{1}{24\|x\|_2}$ and find j_0 such that $12\|x\|_2\,\varepsilon + \frac{4\|x\|_2^2}{j} < \frac{1}{5}\frac{x_1^2}{4}$ for $j > j_0$. Then for any $y \in \ell_2$ such that $\|x-y\|_2 < \min\left\{\frac{|x_1|}{2}, \varepsilon\right\}$ we have for $j > j_0$

$$|y|^2_{i,j} \le \|y\|_2^2 + 2\|y\|_2\, 3\varepsilon + \frac{1}{i+j}\|y\|_2^2 \le \|y\|_2^2 + 12\|x\|_2\,\varepsilon + \frac{4}{i+j}\|x\|_2^2 < \|y\|_2^2 + \frac{1}{5}\frac{x_1^2}{4}.$$

On the other hand, $\|y\|^2 \ge \|y\|_2^2 + \frac{1}{5}y_1^2 > \|y\|_2^2 + \frac{1}{5}\frac{x_1^2}{4}$.

Because

$$|x|^2_{i,j} = \frac{1}{i+j}\|x\|_2^2 + \big(|x_1| + |x_i| + |x_j|\big)^2 + \left(1 - \frac{1}{i+j}\right)\sum_{\substack{k=2\\ k\ne i,j}}^{\infty} x_k^2,$$

it is clearly a LUR norm for each i, j and thus $\|\cdot\|$ is also LUR as it is locally a maximum of a finitely many LUR norms.

Now define a sequence $\{x_n\}$ by $x_{2n} = \frac{1}{2}(e_1 + e_{2n})$ and $x_{2n+1} = e_{2n+1}$. We can easily compute that $\|x_{2n}\|^2 = 1 + \frac{1}{2(2n+2)} \to 1$, $\|x_{2n+1}\|^2 = 1 + \frac{1}{2n+3} \to 1$, and

$$\|x_m + x_n\|^2 = \begin{cases} \|e_1 + \frac{e_m+e_n}{2}\|^2 = 4 + \frac{3}{2(m+n)} \to 4 & \text{for } m, n \text{ even,}\\ \|e_m + e_n\|^2 = 4 + \frac{2}{m+n} \to 4 & \text{for } m, n \text{ odd,}\\ \|\frac{e_1+e_m}{2} + e_n\|^2 = 4 + \frac{3}{2(m+n)} \to 4 & \text{for } m \text{ even, } n \text{ odd.}\end{cases}$$

But since $x_{2n} \to \frac{e_1}{2}$ weakly and $x_{2n+1} \to 0$ weakly, the norm $\|\cdot\|$ is not W2R.

□

Still there remains quite an interesting problem: Does every (non-separable) reflexive Banach space admit a 2R norm?

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