SMOOTH APPROXIMATIONS WITHOUT CRITICAL POINTS

PETR HÁJEK AND MICHAL JOHANIS

ABSTRACT. We show that in any separable Banach space containing c_0 which admits a C^k-smooth bump, every continuous function can be approximated by a C^k-smooth function whose range of derivative is of the first category. Moreover, the approximation can be constructed in such a way that its derivative avoids a prescribed K_σ set (in particular the approximation can have no critical points). On the other hand, in a Banach space with the RNP, the range of the derivative of every smooth bounded bump contains a set residual in some neighbourhood of 0.

In the last few years there has been a growing interest in the general problem: Given a (separable) Banach space X and a C^k-smooth function $f: X \to \mathbb{R}$, what can be said about the set $f'(X) \subset X^*$. Early results in this area were obtained by Azagra and Deville in [AD], where they construct a C^1-smooth bump function f, such that $f'(X) = X^*$, on every Banach space X admitting a C^1-smooth Lipschitz bump function. This surprising result contrasts James' characterisation of reflexive spaces as those for which $\|S_X\|' = S_{X^*}$ whenever $\|\cdot\|$ is an equivalent C^1 renorming of X. Also, by [H], C^1 smoothness cannot be in general replaced by C^2 smoothness. Subsequently, the possible shape of $f'(X)$ has been investigated e.g. in [ADJ], [AFJ], [AJ], [AJ2], [BFKL], [BFL], [FKK] and [G].

Recently, Azagra and Cepedello in [AC] proved that every continuous function on ℓ_2 can be uniformly approximated by a C^∞-smooth function without critical points (i.e. the points where $f' = 0$). Their proof is rather technical and does not seem to generalise to other spaces. In our note we give a simpler proof of a stronger statement for every separable Asplund space X (i.e. Banach space with a separable dual, cf. [DGZ]) containing a copy of c_0. We show that for any fixed K_σ set $N \subset X^*$, the set of smooth functions $\{f;\ f'(X) \cap N = \emptyset\}$ is dense among the continuous functions with uniform topology. However, due to (probably folklore) Fact 3, our method cannot be used for spaces with the Radon-Nikodým Property (RNP), in particular ℓ_2 or any reflexive space. This leaves open the natural conjecture that in every infinite-dimensional separable Asplund space the set of smooth functions without critical points is dense among all continuous functions. Let us recall that all these spaces admit a C^1 bump without a critical point ([AJ]).

First let us fix some notation. Let X be a Banach space. We denote by $B_r = \{\, x \in X;\ \|x\| \le r \,\}$, $U_r = \{\, x \in X;\ \|x\| < r \,\}$ and $S_r = \{\, x \in X;\ \|x\| = r \,\}$ the closed ball, the open ball and the sphere respectively. Sometimes we will write B_r^X to distinguish the space in which we take the ball. We say that a subset of a topological space belongs to the K_σ class if it can be written as a countable union of compact sets. If Y is a subspace of X and $L \in X^*$, by $L\restriction_Y$ we denote the restriction of L to Y (thus $L\restriction_Y \in Y^*$). For a set $N \subset X^*$, we write $N\restriction_Y = \{L\restriction_Y;\ L \in N\}$. We say a function $f: X \to \mathbb{R}$ is Gâteaux differentiable at $x \in X$ if there is $L \in X^*$ such that $\lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big) = L(h)$ for every $h \in X$. If moreover this limit is uniform for $h \in S_X$, we say that f is Fréchet differentiable at x. This L is then called the Gâteaux (Fréchet) derivative of f at x and is denoted by $L = f'(x)$. In this paper, all derivatives are Fréchet unless stated otherwise. If $X = Z \oplus Y$, $x = (z, y)$ and $f: X \to \mathbb{R}$, we use the notation $\frac{\partial f}{\partial Z}(x) = f_y'(z)$, where $f_y: Z \to \mathbb{R}$, $f_y(z) = f(z, y)$. A bump function (or a bump for short) is a non-constant function $f: X \to \mathbb{R}$ with bounded and non-empty support.

Theorem 1. *Let X be a separable Banach space that contains c_0 and admits a C^k bump, $k \in \mathbb{N} \cup \{\infty\}$. Let $f \in C(X)$ and $\varepsilon > 0$. Then there is a function $g \in C^k(X)$ such that $g'(X)$ is of the first category in X^* and $\|f - g\| < \varepsilon$.*

In the proof we will use the notions of partition of unity and of functions which locally depend on finitely many coordinates. A collection $\{\psi_\gamma;\ \gamma \in \Gamma\}$ of real valued functions on X is called a (locally finite) partition of unity on X if for every $x \in X$ there is a neighbourhood of x which meets only finite number of $\operatorname{supp} \psi_\gamma$, $\gamma \in \Gamma$ and $\sum_\Gamma \psi_\gamma(x) = 1$ for each $x \in X$. If $\mathcal{U} = \{U_\gamma;\ \gamma \in \Gamma\}$ is an open covering of X, the partition of unity $\{\psi_\gamma;\ \gamma \in \Gamma\}$ is said to be subordinated to $\mathcal{U}$ if $\operatorname{supp} \psi_\gamma \subset U_\gamma$ for every $\gamma \in \Gamma$. Recall that an open covering $\mathcal{U}$ of X is called locally finite if for each $x \in X$ there is a neighbourhood of x that meets only finitely many members of $\mathcal{U}$. An open covering $\mathcal{V} = \{V_\alpha;\ \alpha \in \Lambda\}$ is a refinement of an open covering $\mathcal{U} = \{U_\gamma;\ \gamma \in \Gamma\}$ if for each $\alpha \in \Lambda$ there is a $\gamma \in \Gamma$ such that $V_\alpha \subset U_\gamma$. For more information about smooth partitions of unity and approximation we refer e.g. to [DGZ, VIII.3].

We say that $f: X \to E$ (where E is a Banach space) locally depends on finitely many coordinates if for each $x \in X$ there are a neighbourhood U of x, $n \in \mathbb{N}$, a finite collection of functionals $x_1^*, \dots, x_n^* \in X^*$ and a mapping $g: \mathbb{R}^n \to E$ such that $f(y) = g\big(x_1^*(y), \dots, x_n^*(y)\big)$ for $y \in U$. Note that the canonical supremum norm $\|\cdot\|_\infty$ on c_0 locally depends on finitely many coordinates on $c_0 \setminus \{0\}$. Indeed, given $0 \ne x = (x_i) \in c_0$, let $M \subset \mathbb{N}$ satisfy $|x_n| = \|x\|_\infty$ if and only if $n \in M$. Clearly, M is a finite set and $\|\cdot\|_\infty$ depends only on coordinates $\{x_i\}_{i\in M}$ in the $\frac{1}{2}\big(\|x\|_\infty - \sup\{|x_i|,\ i \in \mathbb{N} \setminus M\}\big)$ neighbourhood of x. It is

Date: July 2002.
2000 *Mathematics Subject Classification.* 46B20, 46G05.
Key words and phrases. derivatives, approximation, critical points.
Supported by grants GAUK 277/2001, GAČR 201-01-1198, AV 101-90-03, A1019205.

shown in [DGZ, VIII.3] how using compositions, shifts and other operations starting from $\|\cdot\|_\infty$ we can generate a dense subset of $C(c_0)$ consisting of C^∞-smooth functions locally depending on finitely many coordinates.

By [Sob] we know that c_0 is complemented in every separable overspace. Hence in the situation of Theorem 1, $X = c_0 \oplus Y$, where Y is a separable Banach space that admits a C^k bump. The following lemma will provide us with partition of unity convenient for our purpose.

Lemma 2. *Let $X = c_0 \oplus Y$, such that Y is a separable Banach space that admits a C^k bump, $k \in \mathbb{N} \cup \{\infty\}$, and $\mathcal{U}$ be a countable open covering of X. Then there is a C^k-smooth partition of unity $\{\psi_n\}$ subordinated to $\mathcal{U}$ such that for each n, $\frac{\partial \psi_n}{\partial c_0}(X)$ is contained in a K_σ set in ℓ_1.*

Proof. Denote by S_0 the set of functions in $C^\infty(c_0)$ which locally depend on finitely many coordinates, and further denote $\mathcal{B}_0 = \{f^{-1}(0,+\infty);\ f \in S_0,\ 0 \le f \le 1\}$ and $\mathcal{B}_k = \{f^{-1}(0,+\infty);\ f \in C^k(Y),\ 0 \le f \le 1\}$. Let $\mathcal{V}$ be a countable refinement of $\mathcal{U}$ of the form $\mathcal{V} = \{U_n \times V_n;\ U_n \in \mathcal{B}_0, V_n \in \mathcal{B}_k\}$. Such refinement exists, as $\mathcal{B}_0$ and $\mathcal{B}_k$ form bases of topologies in the respective spaces (see e.g. [DGZ, VIII.3]) and it can be made countable because X is separable. Now we need to construct locally finite refinement of $\mathcal{V}$ along with the partition of unity subordinated to this refinement.

For $n \in \mathbb{N}$, let $u_n \in S_0$, $0 \le u_n \le 1$, be such that $U_n = u_n^{-1}(0,+\infty)$ and similarly $v_n \in C^k(Y)$, $0 \le v_n \le 1$, be such that $V_n = v_n^{-1}(0,+\infty)$. Let $g_n \in C^\infty(\mathbb{R})$ be such that $g_n = 0$ on $[1/n,+\infty)$, $g_n = 1$ on $(-\infty,0]$ and $0 < g_n < 1$ on $(0,1/n)$. Denote the coordinates of $x \in X$ as $x = (z,y)$, $z \in c_0$, $y \in Y$.

Put $W_1 = U_1 \times V_1$ and $\varphi_1(x) = u_1(z)v_1(y)$. Then $W_1 = \varphi_1^{-1}(0,+\infty)$ and $\frac{\partial \varphi_1}{\partial c_0}(x) = u_1'(z)v_1(y)$. As u_1' locally depends only on finitely many coordinates, for every $z \in c_0$ there is a neighbourhood N_z of z in c_0 such that $u_1'(N_z)$ is relatively compact in ℓ_1 (it is a continuous image of a finite-dimensional bounded set). Since c_0 is separable, $u_1'(c_0)$ is contained in a K_σ subset of ℓ_1. We can see that $\frac{\partial \varphi_1}{\partial c_0}(X)$ is contained in a K_σ set, because it is a subset of a continuous image of a product of two K_σ sets (one of them being the set that contains $u_1'(c_0)$, the other one $\mathbb{R}$).

We continue by induction. For $n > 1$, put

$$W_n = (U_n \times V_n) \cap \bigcap_{i<n} \varphi_i^{-1}(-\infty, 1/n) \quad \text{and}$$
$$\varphi_n(x) = u_n(z)v_n(y) \prod_{i<n} g_n\bigl(\varphi_i(x)\bigr).$$

Clearly, $W_n = \varphi_n^{-1}(0,+\infty)$. Further, by the Leibniz rule,

$$\frac{\partial \varphi_n}{\partial c_0}(x) = u_n'(z)v_n(y)\prod_{i<n} g_n\bigl(\varphi_i(x)\bigr) + \sum_{j<n}\left(\frac{\partial \varphi_j}{\partial c_0}(x) g_n'\bigl(\varphi_j(x)\bigr)u_n(z)v_n(y)\prod_{\substack{i<n\\ i\neq j}} g_n\bigl(\varphi_i(x)\bigr)\right),$$

the summands are all of the form $a(x)b(x)$, where $a\colon X \to \mathbb{R}$ and $b\colon X \to \ell_1$ with $b(X)$ contained in a K_σ set (for u_n' it is by the same reason as for u_1' and for $\frac{\partial \varphi_j}{\partial c_0}$ it follows from the induction) and so $\frac{\partial \varphi_n}{\partial c_0}(X)$ is also contained in a K_σ set. (It is again a subset of a continuous image of products of K_σ sets.)

For each $x \in X$, there is an $n(x) \in \mathbb{N}$ such that $x \in U_{n(x)} \times V_{n(x)}$ and $x \notin U_i \times V_i$ for $i < n(x)$. Then $x \notin W_i$ for $i < n(x)$ and so $x \in W_{n(x)}$. Therefore $\{W_n\}$ is an open covering of X. Moreover, it is a locally finite covering of X. Indeed, given $x \in X$, put $W = \varphi_{n(x)}^{-1}\bigl(\varphi_{n(x)}(x)/2, +\infty\bigr)$. Then W is a neighbourhood of x and if $m > \max\bigl\{2/\varphi_{n(x)}(x), n(x)\bigr\}$, then $W \cap W_m = \emptyset$. To see this, assume that $w \in W \cap W_m$. According to the definition of W_m, we have that $\varphi_{n(x)}(w) < 1/m$. Because $w \in W$, $\varphi_{n(x)}(w) > \varphi_{n(x)}(x)/2$, which contradicts the choice of m.

To build a partition of unity from the collection $\{\varphi_n\}$, define $\psi_n = \varphi_n / \sum_i \varphi_i$ and notice that since the sum is locally finite, the image of $\frac{\partial \psi_n}{\partial c_0}$ is still contained in some K_σ set.

The partition of unity $\{\psi_n\}$ is subordinated to $\{W_n\}$ which is a refinement of $\mathcal{U}$. To finish the proof we simply add the appropriate functions from the collection $\{\psi_n\}$ to make the partition of unity subordinated to $\mathcal{U}$. □

Proof of Theorem 1. We construct the function g by a standard procedure using the partition of unity supplied by Lemma 2: Let $\mathcal{I}$ be a countable open covering of $\mathbb{R}$ by intervals with the length ε. Then $\mathcal{U} = f^{-1}(\mathcal{I}) = \{f^{-1}(I);\ I \in \mathcal{I}\}$ is a countable open covering of X. Let $\{\psi_n\}$ be a partition of unity from Lemma 2 subordinated to $\mathcal{U}$. For each $n \in \mathbb{N}$ such that ψ_n is not identically zero, we choose $x_n \in X$ such that $\psi_n(x_n) \neq 0$. It follows that if $x \in X$ and $n \in \mathbb{N}$ are such that $\psi_n(x) \neq 0$, then $f(x)$ and $f(x_n)$ both lie in some $I \in \mathcal{I}$ and therefore $|f(x) - f(x_n)| < \varepsilon$. Define

$$g(x) = \sum_{n=1}^{\infty} f(x_n)\psi_n(x).$$

The sum is locally finite. It follows that $g \in C^k(X)$ and we can see that $\frac{\partial g}{\partial c_0}(X)$ is contained in a K_σ subset of ℓ_1. Because $g'(X) \subset \left(\frac{\partial g}{\partial c_0}(X) \times Y^*\right)$, it is a subset of an F_σ set of the first category in X^*. Moreover, for $x \in X$ we have

$$\left|f(x) - g(x)\right| = \left|\sum_{n=1}^{\infty} f(x_n)\psi_n(x) - f(x)\sum_{n=1}^{\infty}\psi_n(x)\right| \leq \sum_{\substack{n=1\\ \psi_n(x)\neq 0}}^{\infty} \left|f(x_n) - f(x)\right|\psi_n(x) < \sum_{\substack{n=1\\ \psi_n(x)\neq 0}}^{\infty} \varepsilon\psi_n(x) = \varepsilon.$$

□

On the other hand, in spaces with the RNP the range of the derivative of non-trivial smooth function is always large: (Recall that e.g. reflexive spaces have the RNP.)

Fact 3. *Let X be a Banach space with the RNP, $b\colon X \to \mathbb{R}$ be a lower semicontinuous Gâteaux differentiable bounded below bump function with $\operatorname{supp} b \subset B_R$ and $y \in B_R$ such that $b(y) < 0$. Then $b'(U_R)$ contains a residual subset of $U_r^{X^*}$, where $r = \frac{-b(y)}{R+\|y\|}$.*

The proof of Fact 3 relies on

Stegall's variational principle [Ste]. *Let X be a Banach space with the RNP, E be a non-empty closed bounded subset of X. Let $\varphi\colon E \to \mathbb{R}$ be a bounded below lower semicontinuous function. Then the set of $x^* \in X^*$ such that the function $\varphi - x^*$ attains its minimum at one point in E is residual in X^*.*

Proof of Fact 3. We apply Stegall's variational principle on $b\colon B_R \to \mathbb{R}$. This gives us a set A residual in X^*, such that $b - x^*$ attains its minimum on B_R at one point for all $x^* \in A$. Pick any $x^* \in A \cap U_r^{X^*}$. Then $b - x^*$ attains its minimum at some unique point $x \in U_R$ and thus $b'(x) = x^*$.

□

By utilising the fact that the partition of unity in Lemma 2 has the partial derivatives contained in a K_σ set a little bit more, we can perturb the approximating function in such a way that its derivative avoids a K_σ set.

Theorem 4. *Let X be a separable Banach space that contains c_0 and admits a C^k bump, $k \in \mathbb{N} \cup \{\infty\}$. Let $f \in C(X)$, $\varepsilon > 0$, and $N \subset X^*$ be a K_σ set. Then there is a function $g \in C^k(X)$ such that $\|f - g\| < \varepsilon$, $g'(X)$ is of the first category in X^* and $g'(X) \cap N = \emptyset$.*

In the proof we will make use of the following lemma (we assume that $X = c_0 \oplus Y$ again):

Lemma 5. *Let X be as in Theorem 4, $L \in X^*$, $r > 0$, and $\varepsilon > 0$. Then there is a function $h \in C^k(X)$ such that $h(x) = L(x)$ for $x \in B_r$, $h(x) = 0$ for $x \notin U_{r+\varepsilon}$, $\frac{\partial h}{\partial c_0}(X)$ is contained in a K_σ set in ℓ_1, and $\|h\|_{C(X)} < \|L\|_{X^*}(r + \varepsilon)$.*

Proof. Using the partition of unity provided by Lemma 2 we construct a bump $\varphi \in C^k(X)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ for $x \in B_r$, $\varphi = 0$ outside $U_{r+\varepsilon}$ and $\frac{\partial \varphi}{\partial c_0}(X)$ is contained in a K_σ set. (Consider the open covering of X formed by $U_{r+\varepsilon}$, $X \setminus B_{r+\varepsilon}$ and a countable covering of $S_{r+\varepsilon}$ by open balls with diameter ε. Take as φ the function from the partition of unity with its support in $U_{r+\varepsilon}$.) Put $h(x) = \varphi(x)L(x)$. Then $\frac{\partial h}{\partial c_0}(x) = \varphi(x)L\restriction_{c_0} + L(x)\frac{\partial \varphi}{\partial c_0}(x)$, the image of the first summand is a subset of a line in ℓ_1 and the image of the second summand is contained in a continuous image of a product of two K_σ sets (one of them being $\mathbb{R}$), hence $\frac{\partial h}{\partial c_0}(X)$ is a subset of a K_σ set. The other assertions are evident.

□

Proof of Theorem 4. The proof of Theorem 1 gives a function $g_0 \in C^k(X)$ such that $\frac{\partial g_0}{\partial c_0}(X)$ is a subset of a K_σ set in ℓ_1 and $\|f - g_0\| < \frac{\varepsilon}{2}$.

Let $D_1 = \bigcup_{w\in N}\left(\frac{\partial g_0}{\partial c_0}(B_1) - w\restriction_{c_0}\right)$. Since N is a K_σ set and $\frac{\partial g_0}{\partial c_0}(B_1)$ is a subset of a K_σ set, D_1 is a subset of a K_σ set as well. A K_σ set in ℓ_1 has an empty interior and so the complement of D_1 in ℓ_1 contains a dense G_δ subset of ℓ_1. Let us denote this G_δ set by A_1 and let ρ_1 be a complete metric on A_1 compatible with the norm topology of ℓ_1. Let $G_1 = U^{\ell_1}_{\varepsilon/(2^3\cdot 1)}$. G_1 is an open non-empty set and A_1 is dense in ℓ_1 and thus there is $L_1 \in A_1 \cap G_1$. Extend this L_1 by the Hahn-Banach theorem to the whole of X (preserving the norm) and denote the extended functional by L_1 again. Now Lemma 5 produces a function $h_1 \in C^k(X)$ such that $h_1 = L_1$ on B_1 and $\|h_1\| < \frac{\varepsilon}{2^2}$. Finally put $g_1 = g_0 - h_1$. We claim that $(N\restriction_{c_0} + L_1) \cap \frac{\partial g_0}{\partial c_0}(B_1) = \emptyset$. Indeed, take any $w \in N$. By our choice $L_1 \in A_1$, hence $L_1 \notin D_1$, and so $L_1 + w\restriction_{c_0} \notin \frac{\partial g_0}{\partial c_0}(B_1)$. From this and the fact that $h_1'(x) = L_1$ on B_1 we have $g_1'(B_1) \cap N = \emptyset$.

Let $D_2 = \bigcup_{w\in N}\left(\frac{\partial g_1}{\partial c_0}(B_2) - w\restriction_{c_0}\right)$, which is a subset of a K_σ set. The complement of D_2 contains a dense G_δ subset of ℓ_1. Let us denote this G_δ set by $\tilde{A}_2$. Let $A_2 = \tilde{A}_2 \cap (A_1 - L_1)$, hence A_2 is again a dense G_δ set. Let ρ_2 be the complete metric on A_2 compatible with the norm topology of ℓ_1. The set $\tilde{M}_2^1 = \{\, L \in A_1 - L_1;\ \rho_1(L_1 + L, L_1) < \frac{1}{2^2}\,\}$ is relatively open (in the norm topology of ℓ_1) and non-empty (containing at least zero) and so there is a set M_2^1 open in ℓ_1 such that $\tilde{M}_2^1 = M_2^1 \cap (A_1 - L_1)$. Let $G_2 = M_2^1 \cap U^{\ell_1}_{\varepsilon/(2^4\cdot 2)}$. G_2 is an open non-empty set and A_2 is dense in ℓ_1 and thus there is $L_2 \in A_2 \cap G_2$. Note that $L_1 + L_2 \in A_1$. Extend this L_2 by the Hahn-Banach theorem to the whole of X (preserving the norm) and denote the extended functional by L_2 again. Now Lemma 5 produces a function $h_2 \in C^k(X)$ such that $h_2 = L_2$ on B_2 and $\|h_2\| < \frac{\varepsilon}{2^3}$. Put $g_2 = g_1 - h_2$ and notice that, since $h_2'(x) = L_2$ on B_2 and $(N\restriction_{c_0} + L_2) \cap \frac{\partial g_1}{\partial c_0}(B_2) = \emptyset$ (which we show the same way as in the previous paragraph), we have $g_2'(B_2) \cap N = \emptyset$.

Now let us proceed by induction: Suppose that $g_1, \dots, g_{n-1}$ have already been defined. Let $D_n = \bigcup_{w \in N} \bigl(\frac{\partial g_{n-1}}{\partial c_0}(B_n) - w\restriction_{c_0}\bigr)$ (which is a subset of a K_σ set) and $\tilde{A}_n$ be a dense G_δ subset of ℓ_1 which is contained in the complement of the set D_n. Let $A_n = \tilde{A}_n \cap (A_{n-1} - L_{n-1})$ and ρ_n be the complete metric on A_n. For $j < n$, the sets

$$\tilde{M}_n^j = \left\{ L \in A_j - \sum_{i=j}^{n-1} L_i;\ \rho_j\left(\sum_{i=j}^{n-1} L_i + L, \sum_{i=j}^{n-1} L_i\right) < \frac{1}{2^n} \right\}$$

are relatively open (in the respective sets) and thanks to the induction hypothesis they contain at least zero. Therefore there are sets M_n^j open in ℓ_1 such that $\tilde{M}_n^j = M_n^j \cap \bigl(A_j - \sum_{i=j}^{n-1} L_i\bigr)$. Let $G_n = \bigcap_{j<n} M_n^j \cap U^{\ell_1}_{\varepsilon/(2^{n+2}\cdot n)}$. It is open and non-empty (contains at least zero) and so there is $L_n \in A_n \cap G_n$. Notice that by the induction hypothesis $\sum_{i=j}^n L_i \in A_j$. Extend again the L_n to the whole of X. From Lemma 5 we get a function $h_n \in C^k(X)$ such that $h_n = L_n$ on B_n and $\|h_n\| < \frac{\varepsilon}{2^{n+1}}$. Put $g_n = g_{n-1} - h_n$, then $g_n'(B_n) \cap N = \emptyset$.

The sequence $\{g_n\}$ is Cauchy in $C(X)$ (because $\sum_{k=m}^n \|h_k\| < \frac{\varepsilon}{2^m}$) and so we can define

$$g = \lim_{n\to\infty} g_n = g_0 - \sum_{k=1}^{\infty} h_k.$$

Notice that $\|f - g\| < \varepsilon$. Fix any $n \in \mathbb{N}$. On B_n, $g = g_{n-1} - \sum_{k=n}^\infty h_k = g_{n-1} - \sum_{k=n}^\infty L_k$ and since $\bigl\{\sum_{k=n}^j L_k\bigr\}_j$ is Cauchy in X^* (through the choice of G_k), $g \in C^1(X)$ and $\frac{\partial g}{\partial c_0}(X)$ is contained in a K_σ subset of ℓ_1 (as it holds for g_n on B_n). Therefore $g'(X)$ is of the first category in X^*.

Moreover, on B_n, $\frac{\partial g}{\partial c_0} = \frac{\partial g_{n-1}}{\partial c_0} - \sum_{k=n}^\infty L_k$ and because $\bigl\{\sum_{k=n}^j L_k\bigr\}_j$ is Cauchy in ρ_n, which is complete on A_n, we obtain $\sum_{k=n}^\infty L_k \in A_n$. Thus $g'(B_n) \cap N = \emptyset$. Finally, for the second and higher derivatives $g^{(j)} = g_{n-1}^{(j)}$ on B_n for $1 < j \le k$ and so $g \in C^k(X)$.

□

Notice that we only needed $N\restriction_{c_0}$ to be K_σ.

As the Fact 3 shows, our method of perturbation by linear functionals doesn't work in spaces with the RNP. Spaces that don't contain c_0 and have bumps with smoothness of higher order are known to be super-reflexive ([FWZ, Theorem 3.3]), hence we have the following corollary:

Corollary 6. *Let X be a separable non-super-reflexive Banach space that admits a C^k bump, $k > 1$. Let $f \in C(X)$, $\varepsilon > 0$, and $N \subset X^*$ be a K_σ set. Then there is a function $g \in C^k(X)$ such that $\|f - g\| < \varepsilon$, $g'(X)$ is of the first category in X^*, and $g'(X) \cap N = \emptyset$.*

References

[AC] D. Azagra and M. Cepedello Boiso, *Uniform approximation of continuous mappings by smooth mappings with no critical points on Hilbert manifolds*, Duke Math. J. **124** (2004), no. 1, 47–66.

[AD] D. Azagra and R. Deville, *James' theorem fails for starlike bodies*, J. Funct. Anal. **180** (2001), 328–346.

[ADJ] D. Azagra, R. Deville and M. Jiménez-Sevilla, *On the range of the derivatives of a smooth mapping between Banach spaces*, Math. Proc. Cambridge Phil. Soc. **134** (2003), no. 1, 163–185.

[AFJ] D. Azagra, M. Fabian and M. Jiménez-Sevilla, *Exact filling of figures with the derivatives of smooth mappings between Banach spaces*, Can. Math. Bull. **48** (2005), no. 4, 481–499.

[AJ] D. Azagra and M. Jiménez-Sevilla, *The failure of Rolle's Theorem in infinite dimensional Banach spaces*, J. Funct. Anal. **182** (2001), 207–226.

[AJ2] D. Azagra and M. Jiménez-Sevilla, *Geometrical and topological properties of bumps and starlike bodies in Banach spaces*, Extracta Math. **17** (2002), no. 2, 151–200.

[BFKL] J.M. Borwein, M. Fabian, I. Kortezov and P.D. Loewen, *The range of the gradient of a continuously differentiable bump*, J. Nonlinear and Convex Anal. **2** (2001), 1–19.

[BFL] J.M. Borwein, M. Fabian and P.D. Loewen, *The range of the gradient of a Lipschitz C^1-smooth bump in infinite dimensions*, Israel J. Math. **132** (2002), 239–251.

[DGZ] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics 64, Longman Scientific & Technical, 1993.

[FKK] M. Fabian, O. Kalenda and J. Kolář, *Filling analytic sets by the derivatives of C^1-smooth bumps*, Proc. Amer. Math. Soc. **133** (2005), no. 1, 295–303.

[FWZ] M. Fabian, J.H.M. Whitfield and V. Zizler, *Norms with locally Lipschitzian derivatives*, Israel J. Math. **44** (1983), no. 3, 262–276.

[G] T. Gaspari, *On the range of the derivative of a real-valued function with bounded support*, Studia Math. **153** (2002), no. 1, 81–99.

[H] P. Hájek, *Smooth functions on c_0*, Israel Journal of Mathematics **104** (1998), no. 3, 17–27.

[Sob] A. Sobczyk, *Projection of the space m on its subspace c_0*, Bull. Amer. Math. Soc. **47** (1941), 938–947.

[Ste] C. Stegall, *Optimization of functions on certain subsets of Banach spaces*, Math. Ann. **236** (1978), 171–176.

Mathematical Institute, Czech Academy of Science, Žitná 25, 115 67 Praha 1, Czech Republic
E-mail address: hajek@math.cas.cz

Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: johanis@karlin.mff.cuni.cz