Functional analysis 1

Functional analysis 1

- Topological vector spaces

Functional analysis 1

- Topological vector spaces
- Theory of distributions

Functional analysis 1

- Topological vector spaces
- Theory of distributions
- Bochner integral

Functional analysis 1

- Topological vector spaces
- Theory of distributions
- Bochner integral
- Compact convex sets

I. Topological vector spaces

1. Elementary properties

Definition 1

Let X be a vector space over $\mathbb{K}$ and τ a topology on X. If the operations of addition and scalar multiplication are continuous as mappings $+\colon X \times X \to X$ and $\cdot\colon \mathbb{K} \times X \to X$, then the pair (X, τ) is called a topological vector space.

Definition 1

Let X be a vector space over $\mathbb{K}$ and τ a topology on X. If the operations of addition and scalar multiplication are continuous as mappings $+ \colon X \times X \to X$ and $\cdot \colon \mathbb{K} \times X \to X$, then the pair (X, τ) is called a topological vector space.

The system of all neighbourhoods of a point $x \in X$ is denoted by $\tau(x)$.

Fact 2

Let X be a vector space and ρ a translation-invariant pseudometric na X. Then

(a) *the operation of addition is continuous as a mapping* $+\colon (X,\rho)\times(X,\rho)\to(X,\rho)$ *(it is even* 2*-Lipschitz);*

Fact 2

Let X be a vector space and ρ a translation-invariant pseudometric na X. Then

(a) *the operation of addition is continuous as a mapping* $+\colon (X,\rho)\times(X,\rho)\to(X,\rho)$ *(it is even 2-Lipschitz);*

(b) $\rho(nx,0)\le n\rho(x,0)$ *for every $x\in X$ and $n\in\mathbb{N}$.*

Proposition 3

Let X be a topological vector space over $\mathbb{K}$.

(a) *If $a \in X$ and $\lambda \in \mathbb{K} \setminus \{0\}$, then the operations $x \mapsto x + a$ and $x \mapsto \lambda x$ are homeomorphisms of X onto X.*

Proposition 3

Let X be a topological vector space over $\mathbb{K}$.

(a) *If $a \in X$ and $\lambda \in \mathbb{K} \setminus \{0\}$, then the operations $x \mapsto x + a$ and $x \mapsto \lambda x$ are homeomorphisms of X onto X.*

(b) *$\tau(x) = x + \tau(0)$ for every $x \in X$.*

Proposition 3

Let X be a topological vector space over $\mathbb{K}$.

(a) *If $a \in X$ and $\lambda \in \mathbb{K} \setminus \{0\}$, then the operations $x \mapsto x + a$ and $x \mapsto \lambda x$ are homeomorphisms of X onto X.*

(b) *$\tau(x) = x + \tau(0)$ for every $x \in X$.*

(c) *If $U \in \tau(0)$, then there exists an open $V \in \tau(0)$ such that $V + V \subset U$.*

Definition 4

Let X be a vector space over $\mathbb{K}$ and $A \subset X$. The set A is called

- absorbing, if for each $x \in X$ there exists a $\lambda_x > 0$ such that $tx \in A$ for every $t \in [0, \lambda_x]$;

Definition 4

Let X be a vector space over $\mathbb{K}$ and $A \subset X$. The set A is called

- absorbing, if for each $x \in X$ there exists a $\lambda_x > 0$ such that $tx \in A$ for every $t \in [0, \lambda_x]$;
- balanced, if $\alpha A \subset A$ for every $\alpha \in \mathbb{K}$, $|\alpha| \leq 1$.

Definition 4

Let X be a vector space over $\mathbb{K}$ and $A \subset X$. The set A is called

- absorbing, if for each $x \in X$ there exists a $\lambda_x > 0$ such that $tx \in A$ for every $t \in [0, \lambda_x]$;
- balanced, if $\alpha A \subset A$ for every $\alpha \in \mathbb{K}$, $|\alpha| \leq 1$.

Proposition 5

Let X be a topological vector space.

(a) *Every $U \in \tau(0)$ is absorbing.*

Definition 4

Let X be a vector space over $\mathbb{K}$ and $A \subset X$. The set A is called

- absorbing, if for each $x \in X$ there exists a $\lambda_x > 0$ such that $tx \in A$ for every $t \in [0, \lambda_x]$;
- balanced, if $\alpha A \subset A$ for every $\alpha \in \mathbb{K}$, $|\alpha| \leq 1$.

Proposition 5

Let X be a topological vector space.

(a) *Every $U \in \tau(0)$ is absorbing.*

(b) *$\tau(0)$ has a basis consisting of open balanced sets.*

Theorem 6 (John von Neumann (1935))

Let X be a vector space and $\mathcal{U}$ a system of subsets of X containing 0, which is a basis of a filter (i.e. it is non-empty and for each $U_1, U_2 \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that $U \subset U_1 \cap U_2$). Assume that $\mathcal{U}$ has the following properties:

(i) *For each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V + V \subset U$.*

(ii) *Each set in $\mathcal{U}$ is absorbing.*

(iii) *Each set in $\mathcal{U}$ is balanced.*

Then there is a unique topology τ on X such that (X, τ) is a topological vector space and $\mathcal{U}$ is a basis of neighbourhoods of 0.

Theorem 7

Let X be a topological vector space.

(a) *Let $K \subset X$ be compact and $C \subset X$ closed and disjoint from K. Then there exists an open balanced $V \in \tau(0)$ such that $(K + V) \cap (C + V) = \emptyset$.*

Theorem 7

Let X be a topological vector space.

(a) *Let $K \subset X$ be compact and $C \subset X$ closed and disjoint from K. Then there exists an open balanced $V \in \tau(0)$ such that $(K + V) \cap (C + V) = \emptyset$.*

(b) *X is regular (i.e. a point and a closed set can be separated by open sets).*

Theorem 7

Let X be a topological vector space.

(a) *Let $K \subset X$ be compact and $C \subset X$ closed and disjoint from K. Then there exists an open balanced $V \in \tau(0)$ such that $(K + V) \cap (C + V) = \emptyset$.*

(b) *X is regular (i.e. a point and a closed set can be separated by open sets).*

(c) *The following statements are equivalent:*

   (i) *X is Hausdorff.*

   (ii) *X is T_1 (i.e. points are closed sets).*

   (iii) *$\{0\}$ is a closed set.*

   (iv) *$\{0\} = \bigcap\{U;\ U \in \tau(0)\}$.*

Proposition 8

Let X be a topological vector space.

(a) *If $G \subset X$ is open and $A \subset X$ arbitrary, then $A + G$ is open.*

Proposition 8

Let X be a topological vector space.

(a) *If $G \subset X$ is open and $A \subset X$ arbitrary, then $A + G$ is open.*

(b) *If $F \subset X$ is closed and $K \subset X$ compact, then $F + K$ is closed.*

Proposition 8

Let X be a topological vector space.

(a) *If $G \subset X$ is open and $A \subset X$ arbitrary, then $A + G$ is open.*

(b) *If $F \subset X$ is closed and $K \subset X$ compact, then $F + K$ is closed.*

(c) *If $K, L \subset X$ are compact, then $K + L$ is also compact.*

Proposition 9

Let X be a topological vector space over $\mathbb{K}$ and $A, B \subset X$. Then the following hold:

(a) $\bar{A} = \bigcap\{A + U;\ U \in \tau(0)\}$.

Proposition 9

Let X be a topological vector space over $\mathbb{K}$ and $A, B \subset X$. Then the following hold:

(a) $\bar{A} = \bigcap\{A + U;\ U \in \tau(0)\}$.

(b) $\bar{A} + \bar{B} \subset \overline{A + B}$ *and* $\operatorname{Int} A + \operatorname{Int} B \subset \operatorname{Int}(A + B)$.

Proposition 9

Let X be a topological vector space over $\mathbb{K}$ and $A, B \subset X$. Then the following hold:

(a) $\bar{A} = \bigcap\{A + U;\ U \in \tau(0)\}$.

(b) $\bar{A} + \bar{B} \subset \overline{A + B}$ *and* $\operatorname{Int} A + \operatorname{Int} B \subset \operatorname{Int}(A + B)$.

(c) $\lambda\bar{A} = \overline{\lambda A}$ *and* $\lambda \operatorname{Int} A = \operatorname{Int}(\lambda A)$ *for any* $\lambda \in \mathbb{K} \setminus \{0\}$.

Proposition 9

Let X be a topological vector space over $\mathbb{K}$ and $A, B \subset X$. Then the following hold:

(a) $\bar{A} = \bigcap\{A + U;\ U \in \tau(0)\}$.

(b) $\bar{A} + \bar{B} \subset \overline{A + B}$ *and* $\operatorname{Int} A + \operatorname{Int} B \subset \operatorname{Int}(A + B)$.

(c) $\lambda\bar{A} = \overline{\lambda A}$ *and* $\lambda \operatorname{Int} A = \operatorname{Int}(\lambda A)$ *for any* $\lambda \in \mathbb{K} \setminus \{0\}$.

(d) *If Y is a subspace of X, then $\overline{Y}$ is also a subspace of X.*

Proposition 9

Let X be a topological vector space over $\mathbb{K}$ and $A, B \subset X$. Then the following hold:

(a) $\bar{A} = \bigcap\{A + U;\ U \in \tau(0)\}$.

(b) $\bar{A} + \bar{B} \subset \overline{A + B}$ *and* $\operatorname{Int} A + \operatorname{Int} B \subset \operatorname{Int}(A + B)$.

(c) $\lambda \bar{A} = \overline{\lambda A}$ *and* $\lambda \operatorname{Int} A = \operatorname{Int}(\lambda A)$ *for any* $\lambda \in \mathbb{K} \setminus \{0\}$.

(d) *If Y is a subspace of X, then $\overline{Y}$ is also a subspace of X.*

(e) *If A is convex, then $\bar{A}$ and $\operatorname{Int} A$ are convex. Moreover, if $\operatorname{Int} A$ is non-empty, then $\bar{A} = \overline{\operatorname{Int} A}$.*

Proposition 9

Let X be a topological vector space over $\mathbb{K}$ and $A, B \subset X$. Then the following hold:

(a) $\bar{A} = \bigcap\{A + U;\ U \in \tau(0)\}$.

(b) $\bar{A} + \bar{B} \subset \overline{A + B}$ *and* $\operatorname{Int} A + \operatorname{Int} B \subset \operatorname{Int}(A + B)$.

(c) $\lambda\bar{A} = \overline{\lambda A}$ *and* $\lambda \operatorname{Int} A = \operatorname{Int}(\lambda A)$ *for any* $\lambda \in \mathbb{K} \setminus \{0\}$.

(d) *If Y is a subspace of X, then $\overline{Y}$ is also a subspace of X.*

(e) *If A is convex, then $\bar{A}$ and $\operatorname{Int} A$ are convex. Moreover, if $\operatorname{Int} A$ is non-empty, then $\bar{A} = \overline{\operatorname{Int} A}$.*

(f) *If A is balanced, then $\bar{A}$ is balanced.*

P3, P8(a)

Proposition 9

Let X be a topological vector space over $\mathbb{K}$ and $A, B \subset X$. Then the following hold:

(a) $\bar{A} = \bigcap\{A + U;\ U \in \tau(0)\}$.

(b) $\bar{A} + \bar{B} \subset \overline{A + B}$ *and* $\operatorname{Int} A + \operatorname{Int} B \subset \operatorname{Int}(A + B)$.

(c) $\lambda \bar{A} = \overline{\lambda A}$ *and* $\lambda \operatorname{Int} A = \operatorname{Int}(\lambda A)$ *for any* $\lambda \in \mathbb{K} \setminus \{0\}$.

(d) *If Y is a subspace of X, then $\overline{Y}$ is also a subspace of X.*

(e) *If A is convex, then $\bar{A}$ and $\operatorname{Int} A$ are convex. Moreover, if $\operatorname{Int} A$ is non-empty, then $\bar{A} = \overline{\operatorname{Int} A}$.*

(f) *If A is balanced, then $\bar{A}$ is balanced.*

(g) *If A is balanced and $0 \in \operatorname{Int} A$, then $\operatorname{Int} A$ is balanced.*

Theorem 10

Let X be a topological vector space, $Y \subset X$ a closed subspace and $Z \subset X$ a finite-dimensional subspace. Then $Y + Z$ is closed.

Theorem 10

Let X be a topological vector space, $Y \subset X$ a closed subspace and $Z \subset X$ a finite-dimensional subspace. Then $Y + Z$ is closed.

Corollary 11

Let X be a Hausdorff topological vector space. Every finite-dimensional subspace of X is closed in X.

2. Bounded sets, metrisability

2. Bounded sets, metrisability

Definition 12

Let X be a topological vector space and $A \subset X$. The set A is called bounded if for every $U \in \tau(0)$ there exists a $t > 0$ such that $A \subset tU$.

2. Bounded sets, metrisability

Definition 12

Let X be a topological vector space and $A \subset X$. The set A is called bounded if for every $U \in \tau(0)$ there exists a $t > 0$ such that $A \subset tU$.

Proposition 13

Let X be a topological vector space over $\mathbb{K}$ and $A \subset X$. Then the following statements are equivalent:

(i) *The set A is bounded.*

2. Bounded sets, metrisability

Definition 12

Let X be a topological vector space and $A \subset X$. The set A is called bounded if for every $U \in \tau(0)$ there exists a $t > 0$ such that $A \subset tU$.

Proposition 13

Let X be a topological vector space over $\mathbb{K}$ and $A \subset X$. Then the following statements are equivalent:

(i) *The set A is bounded.*

(ii) $\gamma_n x_n \to 0$ *for every sequence* $\{x_n\} \subset A$ *and every sequence* $\{\gamma_n\} \subset \mathbb{K}$, $\gamma_n \to 0$.

2. Bounded sets, metrisability

Definition 12

Let X be a topological vector space and $A \subset X$. The set A is called bounded if for every $U \in \tau(0)$ there exists a $t > 0$ such that $A \subset tU$.

Proposition 13

Let X be a topological vector space over $\mathbb{K}$ and $A \subset X$. Then the following statements are equivalent:

(i) *The set A is bounded.*

(ii) $\gamma_n x_n \to 0$ *for every sequence* $\{x_n\} \subset A$ *and every sequence* $\{\gamma_n\} \subset \mathbb{K}$, $\gamma_n \to 0$.

(iii) $\frac{1}{n} x_n \to 0$ *for every sequence* $\{x_n\} \subset A$.

Proposition 14

Let X be a topological vector space and let $A, B \subset X$ be bounded. Then the following hold:

(a) *The sets $A \cup B$ and $A + B$ are bounded.*

Proposition 14

Let X be a topological vector space and let $A, B \subset X$ be bounded. Then the following hold:

(a) *The sets $A \cup B$ and $A + B$ are bounded.*

(b) *The set λA is bounded for any $\lambda \in \mathbb{K}$.*

Proposition 14

Let X be a topological vector space and let $A, B \subset X$ be bounded. Then the following hold:

(a) *The sets $A \cup B$ and $A + B$ are bounded.*

(b) *The set λA is bounded for any $\lambda \in \mathbb{K}$.*

(c) *The set $\bar{A}$ is bounded.*

Lemma 15

Let X be a topological vector space over $\mathbb{K}$, $V \in \tau(0)$ and $\{\delta_n\} \subset \mathbb{K} \setminus \{0\}$, $\delta_n \to 0$. Then $\{\delta_n V;\ n \in \mathbb{N}\}$ is a basis of neighbourhoods of 0 if and only if V is bounded.

Lemma 15

Let X be a topological vector space over $\mathbb{K}$, $V \in \tau(0)$ and $\{\delta_n\} \subset \mathbb{K} \setminus \{0\}$, $\delta_n \to 0$. Then $\{\delta_n V;\ n \in \mathbb{N}\}$ is a basis of neighbourhoods of 0 if and only if V is bounded.

Theorem 16

Let X be a topological vector space. Then the following statements are equivalent:

(i) *X has a countable basis of neighbourhoods of 0.*

Lemma 15

Let X be a topological vector space over $\mathbb{K}$, $V \in \tau(0)$ and $\{\delta_n\} \subset \mathbb{K} \setminus \{0\}$, $\delta_n \to 0$. Then $\{\delta_n V;\ n \in \mathbb{N}\}$ is a basis of neighbourhoods of 0 if and only if V is bounded.

Theorem 16

Let X be a topological vector space. Then the following statements are equivalent:

(i) *X has a countable basis of neighbourhoods of 0.*

(ii) *X is pseudometrisable.*

Lemma 15

Let X be a topological vector space over $\mathbb{K}$, $V \in \tau(0)$ and $\{\delta_n\} \subset \mathbb{K} \setminus \{0\}$, $\delta_n \to 0$. Then $\{\delta_n V;\ n \in \mathbb{N}\}$ is a basis of neighbourhoods of 0 if and only if V is bounded.

Theorem 16

Let X be a topological vector space. Then the following statements are equivalent:

(i) *X has a countable basis of neighbourhoods of 0.*

(ii) *X is pseudometrisable.*

(iii) *X is pseudometrisable by a translation-invariant pseudometric.*

Lemma 15

Let X be a topological vector space over $\mathbb{K}$, $V \in \tau(0)$ *and* $\{\delta_n\} \subset \mathbb{K} \setminus \{0\}$, $\delta_n \to 0$. *Then* $\{\delta_n V;\ n \in \mathbb{N}\}$ *is a basis of neighbourhoods of* 0 *if and only if V is bounded.*

Theorem 16

Let X be a topological vector space. Then the following statements are equivalent:

(i) *X has a countable basis of neighbourhoods of* 0.

(ii) *X is pseudometrisable.*

(iii) *X is pseudometrisable by a translation-invariant pseudometric.*

If X is Hausdorff, that the prefix pseudo- above can be omitted.

Definition 17

Let X be a topological vector space. We say that X is locally bounded if every $x \in X$ has a bounded neighbourhood.

Definition 17

Let X be a topological vector space. We say that X is locally bounded if every $x \in X$ has a bounded neighbourhood.

Theorem 18

Let X be a locally bounded topological vector space. Then X is pseudometrisable.

3. Total boundedness and compactness

3. Total boundedness and compactness

Definition 19

Let X be a topological vector space and $A \subset X$. The set A is called totally bounded if for every $U \in \tau(0)$ there exists a finite $F \subset A$ such that $A \subset F + U$.

Proposition 20

Let X be a topological vector space and $A, B \subset X$. Then the following hold:

(a) *A is totally bounded if and only if for every $U \in \tau(0)$ there exists a finite $F \subset X$ such that $A \subset F + U$.*

Proposition 20

Let X be a topological vector space and $A, B \subset X$. Then the following hold:

(a) *A is totally bounded if and only if for every $U \in \tau(0)$ there exists a finite $F \subset X$ such that $A \subset F + U$.*

(b) *If A is totally bounded and $B \subset A$, then B is also totally bounded.*

Proposition 20

Let X be a topological vector space and $A, B \subset X$. Then the following hold:

(a) *A is totally bounded if and only if for every $U \in \tau(0)$ there exists a finite $F \subset X$ such that $A \subset F + U$.*

(b) *If A is totally bounded and $B \subset A$, then B is also totally bounded.*

(c) *If A, B are totally bounded, then also $A \cup B$ and $A + B$ are totally bounded.*

Proposition 20

Let X be a topological vector space and $A, B \subset X$. Then the following hold:

(a) *A is totally bounded if and only if for every $U \in \tau(0)$ there exists a finite $F \subset X$ such that $A \subset F + U$.*

(b) *If A is totally bounded and $B \subset A$, then B is also totally bounded.*

(c) *If A, B are totally bounded, then also $A \cup B$ and $A + B$ are totally bounded.*

(d) *If A is totally bounded and $\lambda \in \mathbb{K}$, then also λA is totally bounded.*

Proposition 20

Let X be a topological vector space and $A, B \subset X$. Then the following hold:

(a) *A is totally bounded if and only if for every $U \in \tau(0)$ there exists a finite $F \subset X$ such that $A \subset F + U$.*

(b) *If A is totally bounded and $B \subset A$, then B is also totally bounded.*

(c) *If A, B are totally bounded, then also $A \cup B$ and $A + B$ are totally bounded.*

(d) *If A is totally bounded and $\lambda \in \mathbb{K}$, then also λA is totally bounded.*

(e) *If A is totally bounded, then also $\bar{A}$ is totally bounded.*

Proposition 21

Let X be a topological vector space. Compact subsets of X are totally bounded

Proposition 21

Let X be a topological vector space. Compact subsets of X are totally bounded and totally bounded subsets of X are bounded.

Proposition 21

Let X be a topological vector space. Compact subsets of X are totally bounded and totally bounded subsets of X are bounded.

Fact 22

Let (X, τ) be a topological vector space pseudometrisable by a translation-invariant pseudometric ρ. Then $A \subset X$ is τ-totally bounded if and only if it is ρ-totally bounded.

Definition 23

Let (X, τ_X), (Y, τ_Y) be topological vector spaces and $f\colon X \to Y$. We say that f is uniformly continuous if for every $V \in \tau_Y(0)$ there exists a $U \in \tau_X(0)$ such that $f(x) \in f(y) + V$ whenever $x, y \in X$ are such that $x \in y + U$.

Definition 23

Let (X, τ_X), (Y, τ_Y) be topological vector spaces and $f\colon X \to Y$. We say that f is uniformly continuous if for every $V \in \tau_Y(0)$ there exists a $U \in \tau_X(0)$ such that $f(x) \in f(y) + V$ whenever $x, y \in X$ are such that $x \in y + U$.

Proposition 24

Let (X, τ_X), (Y, τ_Y) be topological vector spaces and let $f\colon X \to Y$ be uniformly continuous. If $A \subset X$ is totally bounded, then $f(A)$ is also totally bounded.

4. Linear mappings

4. Linear mappings

A linear image of a balanced set is again a balanced set, a pre-image of a balanced set under a linear mapping is again a balanced set.

Theorem 25

Let X and Y be topological vector spaces and $T\colon X \to Y$ a linear mapping. Consider the following statements:

(i) *T is bounded on some neighbourhood of 0.*

(ii) *T is continuous at 0.*

(iii) *T is continuous.*

(iv) *T is uniformly continuous.*

(v) *T is sequentially continuous.*

(vi) *$T(A)$ is bounded for every bounded $A \subset X$.*

(vii) *The set $\{T(x_n);\ n \in \mathbb{N}\}$ is bounded whenever $\{x_n\} \subset X$, $x_n \to 0$.*

Then (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi)$\Leftrightarrow$(vii).

Theorem 25

Let X and Y be topological vector spaces and $T\colon X \to Y$ a linear mapping. Consider the following statements:

(i) *T is bounded on some neighbourhood of 0.*

(ii) *T is continuous at 0.*

(iii) *T is continuous.*

(iv) *T is uniformly continuous.*

(v) *T is sequentially continuous.*

(vi) *$T(A)$ is bounded for every bounded $A \subset X$.*

(vii) *The set $\{T(x_n);\ n \in \mathbb{N}\}$ is bounded whenever $\{x_n\} \subset X$, $x_n \to 0$.*

Then (i)⇒(ii)⇔(iii)⇔(iv)⇒(v)⇒(vi)⇔(vii). If Y is locally bounded, then (i)–(iv) are equivalent.

Theorem 25

Let X and Y be topological vector spaces and $T\colon X \to Y$ a linear mapping. Consider the following statements:

(i) *T is bounded on some neighbourhood of 0.*

(ii) *T is continuous at 0.*

(iii) *T is continuous.*

(iv) *T is uniformly continuous.*

(v) *T is sequentially continuous.*

(vi) *$T(A)$ is bounded for every bounded $A \subset X$.*

(vii) *The set $\{T(x_n);\ n \in \mathbb{N}\}$ is bounded whenever $\{x_n\} \subset X$, $x_n \to 0$.*

Then (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi)$\Leftrightarrow$(vii). If Y is locally bounded, then (i)–(iv) are equivalent. If X is pseudometrisable, then (ii)–(vii) are equivalent.

Lemma 26

Let X be a pseudometrisable topological vector space. If $\{x_n\} \subset X$ converges to 0, then there exists a sequence $\{\gamma_n\} \subset \mathbb{N}$ such that $\gamma_n \to +\infty$ and $\gamma_n x_n \to 0$.

Theorem 27

Let X be a topological vector space over $\mathbb{K}$ and let $f\colon X \to \mathbb{K}$ be a non-zero linear form. Then the following statements are equivalent:

(i) *f is continuous.*

(ii) *$\operatorname{Ker} f$ is closed.*

(iii) *$\overline{\operatorname{Ker} f} \neq X$.*

As usual, linear forms will be also called (linear) functionals.

As usual, linear forms will be also called (linear) functionals.

Definition 28

Let X be a topological vector space. The symbol $X^{\#}$ denotes the space of all linear forms (functionals) on X and it is called the algebraic dual.

As usual, linear forms will be also called (linear) functionals.

Definition 28

Let X be a topological vector space. The symbol $X^{\#}$ denotes the space of all linear forms (functionals) on X and it is called the algebraic dual. The symbol X^{*} denotes the subspace of $X^{\#}$ consisting of linear functionals that are continuous on X and it it is called the topological dual (or just the dual).

Definition 29

Let X and Y be topological vector spaces and $T\colon X \to Y$ a linear mapping. We say that T is an isomorphism of X onto Y (or just an isomorphism) if T is a homeomorphism of X onto Y;

Definition 29

Let X and Y be topological vector spaces and $T\colon X \to Y$ a linear mapping. We say that T is an isomorphism of X onto Y (or just an isomorphism) if T is a homeomorphism of X onto Y; we say that T is an isomorphism of X into Y (or just an isomorphism into) if T is an isomorphism of X onto $\operatorname{Rng} T$.

Fact 30

Let X be a vector space, Y a topological vector space, and $\{T_\gamma\}_{\gamma\in\Gamma}$ a net of linear mappings from X into Y. If $T\colon X \to Y$ is a pointwise limit of the net $\{T_\gamma\}$, then T is linear.

5. Finite-dimensional spaces

5. Finite-dimensional spaces

Theorem 31

Let X be a topological vector space over $\mathbb{K}$. Then the following statements are equivalent:

(i) *X is Hausdorff and $\dim X < \infty$.*

5. Finite-dimensional spaces

Theorem 31

Let X be a topological vector space over $\mathbb{K}$. Then the following statements are equivalent:

(i) *X is Hausdorff and $\dim X < \infty$.*

(ii) *The exists an $n \in \mathbb{N}$ such that X is isomorphic to $(\mathbb{K}^n, \|\cdot\|_2)$.*

5. Finite-dimensional spaces

Theorem 31

Let X be a topological vector space over $\mathbb{K}$. Then the following statements are equivalent:

(i) *X is Hausdorff and $\dim X < \infty$.*

(ii) *The exists an $n \in \mathbb{N}$ such that X is isomorphic to $(\mathbb{K}^n, \|\cdot\|_2)$.*

(iii) *X is Hausdorff and has a totally bounded neighbourhood of 0.*

Theorem 31

Let X be a topological vector space over $\mathbb{K}$. Then the following statements are equivalent:

(i) *X is Hausdorff and $\dim X < \infty$.*

(ii) *The exists an $n \in \mathbb{N}$ such that X is isomorphic to $(\mathbb{K}^n, \|\cdot\|_2)$.*

(iii) *X is Hausdorff and has a totally bounded neighbourhood of 0.*

(iv) *X is pseudometrisable and every linear mapping from X into some topological vector space is continuous.*

Theorem 31

Let X be a topological vector space over $\mathbb{K}$. Then the following statements are equivalent:

(i) *X is Hausdorff and $\dim X < \infty$.*

(ii) *The exists an $n \in \mathbb{N}$ such that X is isomorphic to $(\mathbb{K}^n, \|\cdot\|_2)$.*

(iii) *X is Hausdorff and has a totally bounded neighbourhood of 0.*

(iv) *X is pseudometrisable and every linear mapping from X into some topological vector space is continuous.*

(v) *X is pseudometrisable and every linear form on X is continuous.*

Corollary 32

Let X be a finite-dimensional vector space. Then there exists only one Hausdorff vector topology on X.

6. Locally convex spaces

Fact 33

Let X be a vector space over $\mathbb{K}$, $A, B \subset X$ convex, and $\alpha \in \mathbb{K}$. Then the sets αA and $A + B$ are convex.

Definition 34

Let X be a vector space over $\mathbb{K}$. A set $A \subset X$ is called absolutely convex if it is convex and balanced.

Definition 34

Let X be a vector space over $\mathbb{K}$. A set $A \subset X$ is called absolutely convex if it is convex and balanced.

Fact 35

Let X be a vector space over $\mathbb{K}$ and $A \subset X$. Then the following hold:

(a) *If A is balanced, then $\operatorname{conv} A$ is balanced and so absolutely convex.*

Definition 34

Let X be a vector space over $\mathbb{K}$. A set $A \subset X$ is called absolutely convex if it is convex and balanced.

Fact 35

Let X be a vector space over $\mathbb{K}$ and $A \subset X$. Then the following hold:

(a) *If A is balanced, then $\operatorname{conv} A$ is balanced and so absolutely convex.*

(b) *A is absolutely convex if and only if $\alpha x + \beta y \in A$ for every $x, y \in A$ and $\alpha, \beta \in \mathbb{K}$, $|\alpha| + |\beta| \leq 1$.*

Fact 36

Let X be a vector space and p a seminorm on X. Then the following hold:

(a) $|p(x) - p(y)| \leq p(x - y)$ *for every* $x, y \in X$.

Fact 36

Let X be a vector space and p a seminorm on X. Then the following hold:

(a) $|p(x) - p(y)| \leq p(x - y)$ *for every* $x, y \in X$.

(b) *The set* $Z = p^{-1}(0)$ *is a subspace of* X. *If* $x, y \in X$ *are such that* $x - y \in Z$, *then* $p(x) = p(y)$.

Fact 36

Let X be a vector space and p a seminorm on X. Then the following hold:

(a) $|p(x) - p(y)| \leq p(x - y)$ *for every* $x, y \in X$.

(b) *The set* $Z = p^{-1}(0)$ *is a subspace of* X. *If* $x, y \in X$ *are such that* $x - y \in Z$, *then* $p(x) = p(y)$.

(c) *The sets* $\{x \in X;\ p(x) < c\}$ *and* $\{x \in X;\ p(x) \leq c\}$ *are absolutely convex for every* $c \in [0, +\infty)$.

Definition 37

Let X be a vector space and $f\colon X \to \mathbb{R}$. We say that f is positively homogeneous if $f(tx) = tf(x)$ for every $t \geq 0$.

Definition 37

Let X be a vector space and $f\colon X \to \mathbb{R}$. We say that f is positively homogeneous if $f(tx) = tf(x)$ for every $t \geq 0$.

Fact 38

Let X be a vector space and f a positively homogeneous function on X. Denote $F_c = \{x \in X;\ f(x) \leq c\}$ and $G_c = \{x \in X;\ f(x) < c\}$ for $c \in \mathbb{R}$. For every $c > 0$ the sets F_c and G_c are absorbing and moreover $F_c = cF_1$, $G_c = cG_1$.

Definition 39

Let X be a vector space and let $A \subset X$ be absorbing. The Minkowski functional of the set A is a function $\mu_A \colon X \to [0, +\infty)$ defined by

$$\mu_A(x) = \inf\{\lambda > 0;\ x \in \lambda A\}.$$

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

(b) *μ_A is positively homogeneous.*

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

(b) *μ_A is positively homogeneous.*

(c) *If A is convex, then μ_A is a non-negative sub-linear functional.*

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

(b) *μ_A is positively homogeneous.*

(c) *If A is convex, then μ_A is a non-negative sub-linear functional.*

(d) *If A is absolutely convex, then μ_A is a seminorm.*

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

(b) *μ_A is positively homogeneous.*

(c) *If A is convex, then μ_A is a non-negative sub-linear functional.*

(d) *If A is absolutely convex, then μ_A is a seminorm.*

(e) *$A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.*

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

(b) *μ_A is positively homogeneous.*

(c) *If A is convex, then μ_A is a non-negative sub-linear functional.*

(d) *If A is absolutely convex, then μ_A is a seminorm.*

(e) $A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.

(f) *If A is balanced or convex, then*
$\{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

(b) *μ_A is positively homogeneous.*

(c) *If A is convex, then μ_A is a non-negative sub-linear functional.*

(d) *If A is absolutely convex, then μ_A is a seminorm.*

(e) *$A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.*

(f) *If A is balanced or convex, then $\{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.*

(g) *Let $p\colon X \to [0, +\infty)$ be positively homogeneous and $B \subset X$.*
- *If B is absorbing and $B \subset \{x \in X;\ p(x) \leq 1\}$, then $\mu_B \geq p$.*

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

(b) *μ_A is positively homogeneous.*

(c) *If A is convex, then μ_A is a non-negative sub-linear functional.*

(d) *If A is absolutely convex, then μ_A is a seminorm.*

(e) *$A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.*

(f) *If A is balanced or convex, then $\{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.*

(g) *Let $p\colon X \to [0, +\infty)$ be positively homogeneous and $B \subset X$.*
- *If B is absorbing and $B \subset \{x \in X;\ p(x) \leq 1\}$, then $\mu_B \geq p$.*
- *If $\{x \in X;\ p(x) < 1\} \subset B$, then $\mu_B \leq p$.*

Theorem 40

Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:

(a) *If $B \supset A$, then $\mu_B \leq \mu_A$.*

(b) *μ_A is positively homogeneous.*

(c) *If A is convex, then μ_A is a non-negative sub-linear functional.*

(d) *If A is absolutely convex, then μ_A is a seminorm.*

(e) $A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.

(f) *If A is balanced or convex, then* $\{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.

(g) *Let $p\colon X \to [0, +\infty)$ be positively homogeneous and $B \subset X$.*
- *If B is absorbing and $B \subset \{x \in X;\ p(x) \leq 1\}$, then $\mu_B \geq p$.*
- *If $\{x \in X;\ p(x) < 1\} \subset B$, then $\mu_B \leq p$.*

So, if $\{x \in X;\ p(x) < 1\} \subset B \subset \{x \in X;\ p(x) \leq 1\}$, then $\mu_B = p$.

Proposition 41

Let X be a topological vector space and let $A \subset X$ be absorbing. Then $\operatorname{Int} A \subset \{x \in X;\ \mu_A(x) < 1\}$.

Proposition 41

Let X be a topological vector space and let $A \subset X$ be absorbing. Then $\operatorname{Int} A \subset \{x \in X;\ \mu_A(x) < 1\}$*. If moreover A is balanced or convex, then*

$$\operatorname{Int} A \subset \{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\} \subset \overline{A}.$$

Lemma 42

Let X be a topological vector space and p a sub-linear functional on X. Then p is uniformly continuous if and only if it is bounded above on some neighbourhood of 0.

Lemma 42

Let X be a topological vector space and p a sub-linear functional on X. Then p is uniformly continuous if and only if it is bounded above on some neighbourhood of 0.

Corollary 43

Let X be a topological vector space and $A \subset X$ an absorbing convex set. Then μ_A is continuous if and only if A is a neighbourhood of 0. In this case

$$\operatorname{Int} A = \{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\} = \bar{A}.$$

Theorem 44

Let X be a topological vector space. Then $X^* \neq \{0\}$ if and only if there is a convex neighbourhood of 0 in X that is different from X.

Definition 45

- We say that a topological vector space is locally convex if it has a basis of neighbourhoods of 0 consisting of convex sets.

Definition 45

- We say that a topological vector space is **locally convex** if it has a basis of neighbourhoods of 0 consisting of convex sets.
- A locally convex space whose topology is induced by a complete translation-invariant metric is called a **Fréchet space**.

Definition 45

- We say that a topological vector space is locally convex if it has a basis of neighbourhoods of 0 consisting of convex sets.
- A locally convex space whose topology is induced by a complete translation-invariant metric is called a Fréchet space.
- We say that a topological vector space is normable if its topology is generated by a norm.

Proposition 46

Let X be a topological vector space. If $U \in \tau(0)$ is convex, then there exists an open absolutely convex $V \in \tau(0)$ such that $V \subset U$.

Proposition 46

Let X be a topological vector space. If $U \in \tau(0)$ is convex, then there exists an open absolutely convex $V \in \tau(0)$ such that $V \subset U$.

Corollary 47

In a locally convex space $\tau(0)$ has a basis consisting of open absolutely convex absorbing sets and also a basis consisting of closed absolutely convex absorbing sets.

Let X be a vector space, $p_1, \dots, p_n$ seminorms on X, and $\varepsilon > 0$. We denote

$$U_{p_1,\dots,p_n,\varepsilon} = \{x \in X;\ p_1(x) < \varepsilon, \dots, p_n(x) < \varepsilon\}.$$

Theorem 48

Let X be a vector space and $\mathcal{P}$ a system of seminorms on X. Then there is a locally convex topology τ on X such that the system $\mathcal{S} = \{U_{p,\varepsilon};\ p \in \mathcal{P}, \varepsilon > 0\}$ is a sub-basis of neighbourhoods of 0 and the system $\mathcal{U} = \{U_{p_1,\dots,p_n,\varepsilon};\ n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0\}$ is a basis of neighbourhoods of 0.

Theorem 48

Let X be a vector space and $\mathcal{P}$ a system of seminorms on X. Then there is a locally convex topology τ on X such that the system $\mathcal{S} = \{U_{p,\varepsilon};\ p \in \mathcal{P}, \varepsilon > 0\}$ is a sub-basis of neighbourhoods of 0 and the system $\mathcal{U} = \{U_{p_1,\dots,p_n,\varepsilon};\ n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0\}$ is a basis of neighbourhoods of 0. The topology τ has the following properties:

(a) *Every seminorm $p \in \mathcal{P}$ is τ-continuous.*

Theorem 48

Let X be a vector space and $\mathcal{P}$ a system of seminorms on X. Then there is a locally convex topology τ on X such that the system $\mathcal{S} = \{U_{p,\varepsilon};\ p \in \mathcal{P}, \varepsilon > 0\}$ is a sub-basis of neighbourhoods of 0 and the system $\mathcal{U} = \{U_{p_1,\dots,p_n,\varepsilon};\ n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0\}$ is a basis of neighbourhoods of 0. The topology τ has the following properties:

(a) *Every seminorm $p \in \mathcal{P}$ is τ-continuous.*
(b) *A set $A \subset X$ is τ-bounded if and only if $p(A)$ is bounded for each $p \in \mathcal{P}$.*

Theorem 48

Let X be a vector space and $\mathcal{P}$ a system of seminorms on X. Then there is a locally convex topology τ on X such that the system $\mathcal{S} = \{U_{p,\varepsilon};\ p \in \mathcal{P}, \varepsilon > 0\}$ is a sub-basis of neighbourhoods of 0 and the system $\mathcal{U} = \{U_{p_1,\dots,p_n,\varepsilon};\ n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0\}$ is a basis of neighbourhoods of 0. The topology τ has the following properties:

(a) *Every seminorm $p \in \mathcal{P}$ is τ-continuous.*
(b) *A set $A \subset X$ is τ-bounded if and only if $p(A)$ is bounded for each $p \in \mathcal{P}$.*
(c) *A net $\{x_\gamma\}_{\gamma \in \Gamma} \subset X$ converges to $x \in X$ in τ if and only if $p(x_\gamma - x) \to 0$ for each $p \in \mathcal{P}$.*

Theorem 48

Let X be a vector space and $\mathcal{P}$ a system of seminorms on X. Then there is a locally convex topology τ on X such that the system $\mathcal{S} = \{U_{p,\varepsilon};\ p \in \mathcal{P}, \varepsilon > 0\}$ is a sub-basis of neighbourhoods of 0 and the system $\mathcal{U} = \{U_{p_1,\dots,p_n,\varepsilon};\ n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0\}$ is a basis of neighbourhoods of 0. The topology τ has the following properties:

(a) *Every seminorm $p \in \mathcal{P}$ is τ-continuous.*

(b) *A set $A \subset X$ is τ-bounded if and only if $p(A)$ is bounded for each $p \in \mathcal{P}$.*

(c) *A net $\{x_\gamma\}_{\gamma \in \Gamma} \subset X$ converges to $x \in X$ in τ if and only if $p(x_\gamma - x) \to 0$ for each $p \in \mathcal{P}$.*

The topology τ will be called a topology generated by the system of seminorms $\mathcal{P}$.

Theorem 48

Let X be a vector space and $\mathcal{P}$ a system of seminorms on X. Then there is a locally convex topology τ on X such that the system $\mathcal{S} = \{U_{p,\varepsilon};\ p \in \mathcal{P}, \varepsilon > 0\}$ is a sub-basis of neighbourhoods of 0 and the system $\mathcal{U} = \{U_{p_1,\dots,p_n,\varepsilon};\ n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0\}$ is a basis of neighbourhoods of 0. The topology τ has the following properties:

(a) *Every seminorm $p \in \mathcal{P}$ is τ-continuous.*
(b) *A set $A \subset X$ is τ-bounded if and only if $p(A)$ is bounded for each $p \in \mathcal{P}$.*
(c) *A net $\{x_\gamma\}_{\gamma\in\Gamma} \subset X$ converges to $x \in X$ in τ if and only if $p(x_\gamma - x) \to 0$ for each $p \in \mathcal{P}$.*

The topology τ will be called a topology generated by the system of seminorms $\mathcal{P}$.
On the other hand, if (X, τ) is a locally convex space and $\mathcal{V}$ is a sub-basis of neighbourhoods of 0 consisting of absolutely convex sets, then τ is generated by the system of seminorms $\{\mu_V;\ V \in \mathcal{V}\}$.

Proposition 49

Let (X, τ) be a locally convex space. The following statements are equivalent:

(i) *X is Hausdorff.*

Proposition 49

Let (X, τ) be a locally convex space. The following statements are equivalent:

(i) *X is Hausdorff.*

(ii) *Every system of seminorms $\mathcal{P}$ generating τ has the following property:*
For each $x \in X \setminus \{0\}$ there is a $p \in \mathcal{P}$ such that $p(x) > 0$.

Proposition 49

Let (X, τ) be a locally convex space. The following statements are equivalent:

(i) *X is Hausdorff.*

(ii) *Every system of seminorms $\mathscr{P}$ generating τ has the following property:*
For each $x \in X \setminus \{0\}$ there is a $p \in \mathscr{P}$ such that $p(x) > 0$.

(iii) *There exists a system of seminorms $\mathscr{P}$ generating τ with the property from statement (ii).*

Lemma 50

Let (X, τ) *be a locally convex space generated by a countable system of seminorms* $\{p_n\}$*. Then*

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{p_n(x - y), 1\}$$

is a translation-invariant pseudometric on X *generating* τ*.*

Theorem 51 (A. N. Kolmogorov (1934))

Let (X, τ) *be a topological vector space. Then* X *is seminormable (resp. normable, if* X *is Hausdorff) if and only if it has a bounded convex neighbourhood of* 0*.*

Proposition 52

Let X be a locally convex space and $A \subset X$. Then the following hold:

(a) *If A is bounded, then also the set $\operatorname{conv} A$ is bounded.*

Proposition 52

Let X be a locally convex space and $A \subset X$. Then the following hold:

(a) *If A is bounded, then also the set $\operatorname{conv} A$ is bounded.*

(b) *If A is totally bounded, then also the set $\operatorname{conv} A$ is totally bounded.*

7. Separation theorems

Lemma 53

Let X be a topological vector space and $f \in X^* \setminus \{0\}$. Then f is an open mapping.

Theorem 54

Let X be a topological vector space and let $A, B \subset X$ be disjoint convex sets. Then the following hold:

(a) *If A is open, then there exist $f \in X^*$ such that* $\operatorname{Re} f(x) < \inf_B \operatorname{Re} f$ *for every $x \in A$.*

Theorem 54

Let X be a topological vector space and let $A, B \subset X$ be disjoint convex sets. Then the following hold:

(a) *If A is open, then there exist $f \in X^*$ such that $\operatorname{Re} f(x) < \inf_B \operatorname{Re} f$ for every $x \in A$.*

(b) *If X is locally convex, A closed and B compact, then there exists $f \in X^*$ such that $\sup_A \operatorname{Re} f < \inf_B \operatorname{Re} f$. If A is moreover absolutely convex, then even $\sup_A |f| < \inf_B \operatorname{Re} f$.*

Corollary 55

Let X be a locally convex space. Then the following hold:

(a) *If X is Hausdorff, then X^* separates the points of X.*

Corollary 55

Let X be a locally convex space. Then the following hold:

(a) *If X is Hausdorff, then X^* separates the points of X.*

(b) *If Y is a closed subspace of X and $x \in X \setminus Y$, then there exists $f \in X^*$ such that $f\restriction_Y = 0$ and $f(x) = 1$.*

Corollary 55

Let X be a locally convex space. Then the following hold:

(a) *If X is Hausdorff, then X^* separates the points of X.*

(b) *If Y is a closed subspace of X and $x \in X \setminus Y$, then there exists $f \in X^*$ such that $f\restriction_Y = 0$ and $f(x) = 1$.*

(c) *If Y is a subspace of X and $f \in Y^*$, then there exists $F \in X^*$ such that $F\restriction_Y = f$.*

8. Weak topologies and polars

8. Weak topologies and polars

Weak topologies

Lemma 56

Let X be a vector space and let $f, f_1, \ldots, f_n$ be linear forms on X. Then $f \in \operatorname{span}\{f_1, \ldots, f_n\}$ if and only if $\bigcap_{j=1}^{n} \operatorname{Ker} f_j \subset \operatorname{Ker} f$.

Lemma 56

Let X be a vector space and let $f, f_1, \ldots, f_n$ be linear forms on X. Then $f \in \operatorname{span}\{f_1, \ldots, f_n\}$ if and only if $\bigcap_{j=1}^{n} \operatorname{Ker} f_j \subset \operatorname{Ker} f$.

Fact 57

Let X, Y, Z be vector spaces and $L\colon X \to Y$ and $S\colon X \to Z$ linear mappings. Then there exists a linear mapping $T\colon Z \to Y$ such that $L = T \circ S$ if and only if $\operatorname{Ker} S \subset \operatorname{Ker} L$.

Definition 58

Let X be a vector space and let $M \subset X^{\#}$ be non-empty. The symbol $\sigma(X, M)$ denotes the locally convex topology on X generated by the system of seminorms $\{|f|;\ f \in M\}$.

Definition 58

Let X be a vector space and let $M \subset X^{\#}$ be non-empty. The symbol $\sigma(X, M)$ denotes the locally convex topology on X generated by the system of seminorms $\{|f|;\ f \in M\}$.

Proposition 59

Let X be a vector space and let $M, N \subset X^{\#}$ be non-empty. Then $\sigma(X, M) = \sigma(X, N)$ if and only if $\operatorname{span} M = \operatorname{span} N$. In particular, $\sigma(X, M) = \sigma(X, \operatorname{span} M)$.

Definition 58

Let X be a vector space and let $M \subset X^\#$ be non-empty. The symbol $\sigma(X, M)$ denotes the locally convex topology on X generated by the system of seminorms $\{|f|;\ f \in M\}$.

Proposition 59

Let X be a vector space and let $M, N \subset X^\#$ be non-empty. Then $\sigma(X, M) = \sigma(X, N)$ if and only if $\operatorname{span} M = \operatorname{span} N$. In particular, $\sigma(X, M) = \sigma(X, \operatorname{span} M)$.

Proposition 60

Let X be a vector space and let $M \subset X^\#$ be non-empty. Then the topology $\sigma(X, M)$ is Hausdorff if and only if M separates the points of X.

Theorem 61

Let X be a vector space and let $M \subset X^{\#}$ be non-empty. Then $(X, \sigma(X, M))^{*} = \operatorname{span} M$.

Definition 62

Let X be a topological vector space.

- The topology $w = \sigma(X, X^*)$ is called the weak topology on X.

Definition 62

Let X be a topological vector space.

- The topology $w = \sigma(X, X^*)$ is called the weak topology on X.
- The topology $w^* = \sigma(X^*, \varepsilon(X))$ is called the weak star topology on X^*.

Definition 62

Let X be a topological vector space.

- The topology $w = \sigma(X, X^*)$ is called the weak topology on X.
- The topology $w^* = \sigma(X^*, \varepsilon(X))$ is called the weak star topology on X^*.

Corollary 63

Let (X, τ) be a topological vector space. Then the following hold:

(a) $w \subset \tau$ *and* $(X, w)^* = X^*$.

Definition 62

Let X be a topological vector space.

- The topology $w = \sigma(X, X^*)$ is called the weak topology on X.
- The topology $w^* = \sigma(X^*, \varepsilon(X))$ is called the weak star topology on X^*.

Corollary 63

Let (X, τ) be a topological vector space. Then the following hold:

(a) $w \subset \tau$ *and* $(X, w)^* = X^*$.

(b) $(X^*, w^*)^* = \varepsilon(X)$.

Definition 62

Let X be a topological vector space.

- The topology $w = \sigma(X, X^*)$ is called the weak topology on X.
- The topology $w^* = \sigma(X^*, \varepsilon(X))$ is called the weak star topology on X^*.

Corollary 63

Let (X, τ) be a topological vector space. Then the following hold:

(a) $w \subset \tau$ *and* $(X, w)^* = X^*$.

(b) $(X^*, w^*)^* = \varepsilon(X)$.

Proposition 64

Let X be a Banach space. Then X is reflexive if and only if on the space $(X^*, \|\cdot\|)$ the topologies weak and w^* coincide.

Proposition 65

Let X be a topological vector space and Y a subspace of X. Denote by w_{XY} the restriction of the topology $\sigma(X, X^*)$ onto Y. Then $w_{XY} \subset \sigma(Y, Y^*)$. If X is locally convex, then $w_{XY} = \sigma(Y, Y^*)$. In other words, in a locally convex space X the original weak topology on Y coincides with the topology inherited from X.

Theorem 66

Let X be a locally convex space and let $A \subset X$ be convex. Then the following hold:

(a) $\overline{A}^{w} = \overline{A}$.

Theorem 66

Let X be a locally convex space and let $A \subset X$ be convex. Then the following hold:

(a) $\overline{A}^{w} = \overline{A}$.

(b) *A is weakly closed if and only if it is closed.*

Theorem 66

Let X be a locally convex space and let $A \subset X$ be convex. Then the following hold:

(a) $\overline{A}^w = \overline{A}$.

(b) *A is weakly closed if and only if it is closed.*

(c) *If X is pseudometrisable and $x_n \to x$ weakly, then there exist $y_n \in \operatorname{conv}\{x_j;\ j \geq n\}$ such that $y_n \to x$.*

Theorem 67 (George Whitelaw Mackey (1946))

Let X be a locally convex space and $A \subset X$. Then A is bounded if and only if it is weakly bounded.

Theorem 67 (George Whitelaw Mackey (1946))

Let X be a locally convex space and $A \subset X$. Then A is bounded if and only if it is weakly bounded.

Proposition 68

Let X be a Banach space and $A \subset X^*$. Then A is bounded if and only if it is w^*-bounded.

Theorem 69

Let X, Y be topological vector spaces and let $T\colon X \to Y$ be a continuous linear mapping. Then T is w–w continuous, i.e. it is continuous as a mapping $T\colon (X, w) \to (Y, w)$.

Proposition 70

Let X be a vector space and let $M \subset X^{\#}$ be non-empty. Then $\sigma(X, M)$ is pseudometrisable if and only if $\operatorname{span} M$ *has a countable algebraic basis.*

Proposition 70

Let X be a vector space and let $M \subset X^{\#}$ be non-empty. Then $\sigma(X, M)$ is pseudometrisable if and only if span M *has a countable algebraic basis.*

Proposition 71

Let X be an infinite-dimensional topological vector space metrisable by a complete metric. Then X does not have a countable algebraic basis.

Proposition 70

Let X be a vector space and let $M \subset X^{\#}$ be non-empty. Then $\sigma(X, M)$ is pseudometrisable if and only if span M *has a countable algebraic basis.*

Proposition 71

Let X be an infinite-dimensional topological vector space metrisable by a complete metric. Then X does not have a countable algebraic basis.

Corollary 72

(a) *Let X be a normed linear space. Then (X, w) is metrisable if and only if X is finite-dimensional. In this case the weak topology coincides with the norm topology.*

Proposition 70

Let X be a vector space and let $M \subset X^{\#}$ be non-empty. Then $\sigma(X, M)$ is pseudometrisable if and only if span M *has a countable algebraic basis.*

Proposition 71

Let X be an infinite-dimensional topological vector space metrisable by a complete metric. Then X does not have a countable algebraic basis.

Corollary 72

(a) *Let X be a normed linear space. Then (X, w) is metrisable if and only if X is finite-dimensional. In this case the weak topology coincides with the norm topology.*

(b) *Let X be a Fréchet space. Then (X^*, w^*) is metrisable if and only if X is finite-dimensional.*

Polars

Polars

Definition 73

Let X be a vector space and $A \subset X$. The absolutely convex hull of the set A is defined by

$$\operatorname{aconv} A = \bigcap \{B \supset A;\ B \subset X \text{ is absolutely convex}\}.$$

Polars

Definition 73

Let X be a vector space and $A \subset X$. The absolutely convex hull of the set A is defined by

$$\operatorname{aconv} A = \bigcap \{B \supset A;\ B \subset X \text{ is absolutely convex}\}.$$

Proposition 74

Let X be a vector space over $\mathbb{K}$ and $A \subset X$. Then

$$\operatorname{aconv} A = \left\{ \sum_{i=1}^{n} \lambda_i x_i;\ x_1, \dots, x_n \in A, \right.$$

$$\left. \lambda_1, \dots, \lambda_n \in \mathbb{K}, \sum_{i=1}^{n} |\lambda_i| \leq 1, n \in \mathbb{N} \right\}.$$

Definition 75

Let X be a topological vector space and $A \subset X$. The closed absolutely convex hull of A is defined by

$$\overline{\operatorname{aconv}}\, A = \bigcap \{B \supset A;\ B \subset X \text{ is closed absolutely convex}\}.$$

Definition 75

Let X be a topological vector space and $A \subset X$. The closed absolutely convex hull of A is defined by

$$\overline{\operatorname{aconv}} A = \bigcap \{B \supset A;\ B \subset X \text{ is closed absolutely convex}\}.$$

Proposition 76

Let X be a topological vector space and $A \subset X$. Then $\overline{\operatorname{span}} A = \overline{\operatorname{span} A}$, $\overline{\operatorname{conv}} A = \overline{\operatorname{conv} A}$, *and* $\overline{\operatorname{aconv}} A = \overline{\operatorname{aconv} A}$.

Definition 77

If X is a topological vector space and $A \subset X$, then we define the (absolute) polar of the set A by

$$A^\circ = \{f \in X^*;\ |f(x)| \leq 1 \text{ for every } x \in A\}.$$

Definition 77

If X is a topological vector space and $A \subset X$, then we define the (absolute) polar of the set A by

$$A^\circ = \{f \in X^*;\ |f(x)| \leq 1 \text{ for every } x \in A\}.$$

For the set $B \subset X^*$ we define the (absolute) prepolar by

$$B_\circ = \{x \in X;\ |f(x)| \leq 1 \text{ for every } f \in B\}.$$

Definition 77

If X is a topological vector space and $A \subset X$, then we define the (absolute) polar of the set A by

$$A^\circ = \{f \in X^*;\ |f(x)| \leq 1 \text{ for every } x \in A\}.$$

For the set $B \subset X^*$ we define the (absolute) prepolar by

$$B_\circ = \{x \in X;\ |f(x)| \leq 1 \text{ for every } f \in B\}.$$

Fact 78

Let X be a topological vector space, $A \subset X$, and $B \subset X^*$. If we consider X^* with the topology w^*, then $A^\circ = \varepsilon(A)_\circ$, $\varepsilon(B_\circ) = B^\circ$, and $(B^\circ)_\circ = (B_\circ)^\circ$.

Proposition 79

Let X be a topological vector space over $\mathbb{K}$, $A \subset X$, and $B \subset X^*$. Then the following hold:

(a) *The set A° is absolutely convex and w^*-closed. The set $B_\circ$ is absolutely convex and weakly closed.*

Proposition 79

Let X be a topological vector space over $\mathbb{K}$, $A \subset X$, and $B \subset X^*$. Then the following hold:

(a) *The set A° is absolutely convex and w^*-closed. The set $B_\circ$ is absolutely convex and weakly closed.*

(b) *If A is a subspace of X, then $A^\circ = A^\perp$. If B is a subspace of X^*, then $B_\circ = B_\perp$.*

Proposition 79

Let X be a topological vector space over $\mathbb{K}$, $A \subset X$, and $B \subset X^*$. Then the following hold:

(a) *The set A° is absolutely convex and w^*-closed. The set $B_\circ$ is absolutely convex and weakly closed.*

(b) *If A is a subspace of X, then $A^\circ = A^\perp$. If B is a subspace of X^*, then $B_\circ = B_\perp$.*

(c) *$\{0\}^\circ = X^*$, $X^\circ = \{0\}$, $\{0\}_\circ = X$, and if X^* separates the points of X, then $(X^*)_\circ = \{0\}$.*

Proposition 79

Let X be a topological vector space over $\mathbb{K}$, $A \subset X$, and $B \subset X^*$. Then the following hold:

(a) *The set A° is absolutely convex and w^*-closed. The set $B_\circ$ is absolutely convex and weakly closed.*

(b) *If A is a subspace of X, then $A^\circ = A^\perp$. If B is a subspace of X^*, then $B_\circ = B_\perp$.*

(c) $\{0\}^\circ = X^*$, $X^\circ = \{0\}$, $\{0\}_\circ = X$, *and if X^* separates the points of X, then $(X^*)_\circ = \{0\}$.*

(d) *If $\lambda \in \mathbb{K} \setminus \{0\}$, then $(\lambda A)^\circ = \frac{1}{\lambda} A^\circ$ and $(\lambda B)_\circ = \frac{1}{\lambda} B_\circ$.*

Proposition 79

Let X be a topological vector space over $\mathbb{K}$, $A \subset X$, and $B \subset X^*$. Then the following hold:

(a) *The set A° is absolutely convex and w^*-closed. The set $B_\circ$ is absolutely convex and weakly closed.*

(b) *If A is a subspace of X, then $A^\circ = A^\perp$. If B is a subspace of X^*, then $B_\circ = B_\perp$.*

(c) $\{0\}^\circ = X^*$, $X^\circ = \{0\}$, $\{0\}_\circ = X$, *and if X^* separates the points of X, then $(X^*)_\circ = \{0\}$.*

(d) *If $\lambda \in \mathbb{K} \setminus \{0\}$, then $(\lambda A)^\circ = \frac{1}{\lambda} A^\circ$ and $(\lambda B)_\circ = \frac{1}{\lambda} B_\circ$.*

(e) *If $A_\gamma \subset X$, $\gamma \in \Gamma$ is an arbitrary system, then $\left(\bigcup_{\gamma \in \Gamma} A_\gamma\right)^\circ = \bigcap_{\gamma \in \Gamma} A_\gamma^\circ$. If $B_\gamma \subset X^*$, $\gamma \in \Gamma$ is an arbitrary system, then $\left(\bigcup_{\gamma \in \Gamma} B_\gamma\right)_\circ = \bigcap_{\gamma \in \Gamma} (B_\gamma)_\circ$.*

Theorem 80 (Bipolar theorem; Jean Dieudonné (1950))

Let X be a topological vector space.

(a) *If $A \subset X$, then $(A^\circ)_\circ = \overline{\mathrm{aconv}}^{\,w} A$ ($= \overline{\mathrm{aconv}}\, A$ if X is locally convex).*

Theorem 80 (Bipolar theorem; Jean Dieudonné (1950))

Let X be a topological vector space.

(a) *If $A \subset X$, then $(A^\circ)_\circ = \overline{\mathrm{aconv}}^{w} A$ (= $\overline{\mathrm{aconv}}\, A$ if X is locally convex).*

(b) *If $B \subset X^*$, then $(B_\circ)^\circ = \overline{\mathrm{aconv}}^{w^*} B$.*

Lemma 81

Let X be a topological vector space and $A \subset X$, $B \subset X^*$. Then

(a) $A^\perp$ *is a* w^**-closed subspace of* X^**,*

(b) $B_\perp$ *is a weakly closed subspace of* X*,*

(c) $(A^\perp)_\perp = \overline{\operatorname{span}}^{w} A$ *(*$= \overline{\operatorname{span}}\, A$ *if* X *is locally convex),*

(d) $(B_\perp)^\perp = \overline{\operatorname{span}}^{w^*} B$*.*

Theorem 82

If X, Y are topological vector spaces such that Y^* separates the points of Y, and $T\colon X \to Y$ is a continuous linear mapping, then

(a) $\operatorname{Ker} T^* = (\operatorname{Rng} T)^{\perp}$,

(b) $\operatorname{Ker} T = (\operatorname{Rng} T^*)_{\perp}$,

(c) $\overline{\operatorname{Rng} T}^{w} = (\operatorname{Ker} T^*)_{\perp}$,

(d) $\overline{\operatorname{Rng} T^*}^{w^*} = (\operatorname{Ker} T)^{\perp}$.

Theorem 83 (Herman Heine Goldstine (1938))

If X is a normed linear space, then $\overline{\varepsilon(B_X)}^{w^*} = B_{X^{**}}$.

Theorem 84 (Banach-Alaoglu-Bourbaki)

Let X be a topological vector space. If U is a neighbourhood of 0 in X, then U° is a w^*-compact set.

Theorem 84 (Banach-Alaoglu-Bourbaki)

Let X be a topological vector space. If U is a neighbourhood of 0 in X, then U° is a w^*-compact set.

Corollary 85

Let X be a normed linear space. Then B_{X^*} is w^*-compact.

Proposition 86

Let X be a separable topological vector space and let $\{x_n\}_{n=1}^{\infty}$ be dense in X. If $U \subset X$ is a neighbourhood of 0, then (U°, w^*) is a topological space metrisable by the metric

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{|(f-g)(x_n)|, 1\}.$$

Fact 87

Let X be a normed linear space. Then the canonical embedding $\varepsilon\colon X \to X^{**}$ is an isomorphism of locally convex spaces (X, w) and $(\varepsilon(X), w^*)$.

Fact 87

Let X *be a normed linear space. Then the canonical embedding* $\varepsilon\colon X \to X^{**}$ *is an isomorphism of locally convex spaces* (X, w) *and* $(\varepsilon(X), w^*)$. *In particular,* ε *is a homeomorphism of topological spaces* (B_X, w) *and* $(\varepsilon(B_X), w^*)$.

Proposition 88

Let X be a normed linear space.

(a) *If X is separable and $\{x_n\}$ is dense in S_X, then (B_{X^*}, w^*) is metrisable by the metric*

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} |(f - g)(x_n)|.$$

Proposition 88

Let X be a normed linear space.

(a) *If X is separable and $\{x_n\}$ is dense in S_X, then (B_{X^*}, w^*) is metrisable by the metric*

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} |(f-g)(x_n)|.$$

(b) *If X^* is separable and $\{f_n\}$ is dense in S_{X^*}, then (B_X, w) is metrisable by the metric*

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x-y)|.$$

Theorem 89

If X is a Banach space, then X is reflexive if and only if B_X is weakly compact.

Theorem 89

If X is a Banach space, then X is reflexive if and only if B_X is weakly compact.

Theorem 90

Let X be a reflexive Banach space. Then B_X is weakly sequentially compact. That is every bounded sequence in X has a weakly convergent subsequence.

Proposition 91

Let X be a normed linear space. Then the mapping $x \mapsto \varepsilon_x \restriction_{B_{X^*}}$ is a linear isometry from X into $C((B_{X^*}, w^*))$. So every normed linear space is isometric to a subspace of $C(K)$ for some Hausdorff compact K.

II. Theory of distributions

Lemma 92

Let $\Omega \subset \mathbb{R}^d$ be open.

(a) *Let μ be a Borel complex measure on Ω. If $\int_\Omega \varphi \,\mathrm{d}\mu = 0$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $\mu = 0$.*

Lemma 92

Let $\Omega \subset \mathbb{R}^d$ be open.

(a) *Let μ be a Borel complex measure on Ω. If $\int_\Omega \varphi \, \mathrm{d}\mu = 0$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $\mu = 0$.*

(b) *Let $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$. If $\int_\Omega f\varphi \, \mathrm{d}\lambda = 0$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $f = 0$ a. e. on Ω.*

Lemma 92

Let $\Omega \subset \mathbb{R}^d$ be open.

(a) *Let μ be a Borel complex measure on Ω. If $\int_\Omega \varphi \, \mathrm{d}\mu = 0$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $\mu = 0$.*

(b) *Let $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$. If $\int_\Omega f\varphi \, \mathrm{d}\lambda = 0$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $f = 0$ a. e. on Ω.*

(c) *Let μ be a Borel complex measure on Ω and $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$. If $\int_\Omega \varphi \, \mathrm{d}\mu = \int_\Omega f\varphi \, \mathrm{d}\lambda$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $f \in L_1(\Omega, \lambda)$ and $\mu(A) = \int_A f \, \mathrm{d}\lambda$ for every Borel $A \subset \Omega$.*

Lemma 93

Let $A, U \subset \mathbb{R}^d$ *be such that* $\operatorname{dist}(A, \mathbb{R}^d \setminus U) > 0$. *Then there exists* $\varphi \in C^\infty(\mathbb{R}^d)$ *such that* $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subset U$, *and* $\varphi = 1$ *on* A.

Lemma 93

Let $A, U \subset \mathbb{R}^d$ *be such that* $\operatorname{dist}(A, \mathbb{R}^d \setminus U) > 0$. *Then there exists* $\varphi \in C^\infty(\mathbb{R}^d)$ *such that* $0 \le \varphi \le 1$, $\operatorname{supp} \varphi \subset U$, *and* $\varphi = 1$ *on* A.

Corollary 94

Let $K \subset \mathbb{R}^d$ *be compact and let* $G \subset \mathbb{R}^d$ *be open such that* $G \supset K$. *Then there exist* $U \subset G$ *open,* $U \supset K$ *and* $\varphi \in \mathcal{D}(G)$ *such that* $0 \le \varphi \le 1$ *and* $\varphi = 1$ *on* U.

1. Weak derivatives

Proposition 95

Let $(a, b) \subset \mathbb{R}$ *and* $f \in C^1((a, b))$. *Then*

$$\int_a^b f'\varphi \,\mathrm{d}\lambda = -\int_a^b f\varphi' \,\mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}((a, b))$.

Definition 96

Let $(a, b) \subset \mathbb{R}$ and $f \in L_1^{\mathrm{loc}}((a, b))$. We say that a function $g \in L_1^{\mathrm{loc}}((a, b))$ is the weak derivative of f if

$$\int_a^b g\varphi \,\mathrm{d}\lambda = -\int_a^b f\varphi' \,\mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}((a, b))$.

Definition 96

Let $(a, b) \subset \mathbb{R}$ and $f \in L_1^{\mathrm{loc}}((a, b))$. We say that a function $g \in L_1^{\mathrm{loc}}((a, b))$ is the weak derivative of f if

$$\int_a^b g\varphi \, \mathrm{d}\lambda = -\int_a^b f\varphi' \, \mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}((a, b))$. We say that a Borel complex measure μ on (a, b) is the weak derivative of f if

$$\int_a^b \varphi \, \mathrm{d}\mu = -\int_a^b f\varphi' \, \mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}((a, b))$.

Theorem 97

The weak derivative of a function $f \in L_1^{\mathrm{loc}}((a,b))$ *is uniquely determined.*

Theorem 97

The weak derivative of a function $f \in L_1^{\mathrm{loc}}((a,b))$ *is uniquely determined. More precisely, if* $g_1, g_2 \in L_1^{\mathrm{loc}}((a,b))$ *are weak derivatives of* f*, then* $g_1 = g_2$ *almost everywhere.*

Theorem 97

The weak derivative of a function $f \in L_1^{\mathrm{loc}}((a,b))$ *is uniquely determined. More precisely, if* $g_1, g_2 \in L_1^{\mathrm{loc}}((a,b))$ *are weak derivatives of* f*, then* $g_1 = g_2$ *almost everywhere. If Borel complex measures* μ_1, μ_2 *on* (a,b) *are weak derivatives of* f*, then* $\mu_1 = \mu_2$*.*

Theorem 97

The weak derivative of a function $f \in L_1^{\mathrm{loc}}((a,b))$ *is uniquely determined. More precisely, if* $g_1, g_2 \in L_1^{\mathrm{loc}}((a,b))$ *are weak derivatives of* f*, then* $g_1 = g_2$ *almost everywhere. If Borel complex measures* μ_1, μ_2 *on* (a,b) *are weak derivatives of* f*, then* $\mu_1 = \mu_2$*. If* $g \in L_1^{\mathrm{loc}}((a,b))$ *and a Borel complex measure* μ *on* (a,b) *are weak derivatives of* f*, then* $g \in L_1((a,b))$ *and* $\mu(A) = \int_A g \,\mathrm{d}\lambda$ *for every Borel* $A \subset (a,b)$*.*

Proposition 98

Let $(a, b) \subset \mathbb{R}$ *and* $f \in L_1^{\mathrm{loc}}((a, b))$. *Then the weak derivative of* f *is zero if and only if* f *is a. e. constant (i.e. there exists* $c \in \mathbb{K}$ *such that* $f = c$ *a. e. on* (a, b)*).*

Theorem 99

Let $(a, b) \subset \mathbb{R}$ *and* $f \in L_1^{\mathrm{loc}}((a, b))$*.*

(a) *Let* $a, b \in \mathbb{R}$*. If* f *is absolutely continuous on* $[a, b]$*, then it has a finite derivative a. e.,* $f' \in L_1((a, b))$*, and* f' *is the weak derivative of* f*.*

Theorem 99

Let $(a, b) \subset \mathbb{R}$ *and* $f \in L_1^{\mathrm{loc}}((a, b))$.

(a) *Let* $a, b \in \mathbb{R}$. *If* f *is absolutely continuous on* $[a, b]$, *then it has a finite derivative a. e.,* $f' \in L_1((a, b))$, *and* f' *is the weak derivative of* f. *On the other hand, if* f *has a weak derivative* $g \in L_1((a, b))$, *then there exists a function* f_0 *absolutely continuous on* $[a, b]$ *such that* $f = f_0$ *a. e. Then* $g = f_0'$ *a. e.*

Theorem 99

Let $(a, b) \subset \mathbb{R}$ and $f \in L_1^{\mathrm{loc}}((a, b))$.

(a) *Let $a, b \in \mathbb{R}$. If f is absolutely continuous on $[a, b]$, then it has a finite derivative a. e., $f' \in L_1((a, b))$, and f' is the weak derivative of f. On the other hand, if f has a weak derivative $g \in L_1((a, b))$, then there exists a function f_0 absolutely continuous on $[a, b]$ such that $f = f_0$ a. e. Then $g = f_0'$ a. e.*

(b) *The function f has a weak derivative in $L_1^{\mathrm{loc}}((a, b))$ if and only if there exists a function f_0 locally absolutely continuous on (a, b) such that $f = f_0$ a. e.*

Theorem 99

Let $(a, b) \subset \mathbb{R}$ and $f \in L_1^{\text{loc}}((a, b))$.

(a) *Let $a, b \in \mathbb{R}$. If f is absolutely continuous on $[a, b]$, then it has a finite derivative a. e., $f' \in L_1((a, b))$, and f' is the weak derivative of f. On the other hand, if f has a weak derivative $g \in L_1((a, b))$, then there exists a function f_0 absolutely continuous on $[a, b]$ such that $f = f_0$ a. e. Then $g = f_0'$ a. e.*

(b) *The function f has a weak derivative in $L_1^{\text{loc}}((a, b))$ if and only if there exists a function f_0 locally absolutely continuous on (a, b) such that $f = f_0$ a. e.*

(c) *Let $a, b \in \mathbb{R}$. The function f has a weak derivative equal to a Borel complex measure μ on $[a, b]$ if and only if there exists a left continuous function f_0 of bounded variation on $[a, b]$ such that $f = f_0$ a. e. In this case $\mu([a, x)) = f_0(x) - f_0(a)$ for every $x \in [a, b]$.*

2. The space of test functions and distributions

2. The space of test functions and distributions

Definition 100

For $N \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we define

$$\|\varphi\|_N = \max_{|\alpha| \leq N} \|D^\alpha \varphi\|_\infty.$$

2. The space of test functions and distributions

Definition 100

For $N \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we define

$$\|\varphi\|_N = \max_{|\alpha| \leq N} \|D^\alpha \varphi\|_\infty.$$

The sequence of norms $\{\|\cdot\|_N\}_{N=0}^\infty$ on $\mathcal{D}(\mathbb{R}^d)$ generates a Hausdorff locally convex topology τ_ρ metrisable by a translation invariant metric

$$\rho(\varphi, \psi) = \sum_{N=0}^{\infty} \frac{1}{2^N} \min\{\|\varphi - \psi\|_N, 1\}$$

for $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$.

Theorem 101

The metric ρ has the following properties:

(a) *Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. The following statements are equivalent:*

  (i) $\varphi_n \to \varphi$ *in the metric* ρ.

  (ii) $\|\varphi_n - \varphi\|_N \to 0$ *for each* $N \in \mathbb{N}_0$.

  (iii) $D^\alpha \varphi_n \to D^\alpha \varphi$ *uniformly on* $\mathbb{R}^d$ *for each multi-index* α *of length* d.

Theorem 101

The metric ρ has the following properties:

(a) *Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. The following statements are equivalent:*

   (i) $\varphi_n \to \varphi$ *in the metric* ρ.

   (ii) $\|\varphi_n - \varphi\|_N \to 0$ *for each* $N \in \mathbb{N}_0$.

   (iii) $D^\alpha \varphi_n \to D^\alpha \varphi$ *uniformly on* $\mathbb{R}^d$ *for each multi-index* α *of length* d.

(b) *If α is a multi-index of length d, then the mapping $\varphi \mapsto D^\alpha \varphi$ is continuous as a mapping from $(\mathcal{D}(\mathbb{R}^d), \rho)$ to $(\mathcal{D}(\mathbb{R}^d), \rho)$.*

Theorem 101

The metric ρ has the following properties:

(a) *Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. The following statements are equivalent:*

   (i) $\varphi_n \to \varphi$ *in the metric* ρ.

   (ii) $\|\varphi_n - \varphi\|_N \to 0$ *for each* $N \in \mathbb{N}_0$.

   (iii) $D^\alpha \varphi_n \to D^\alpha \varphi$ *uniformly on* $\mathbb{R}^d$ *for each multi-index* α *of length* d.

(b) *If α is a multi-index of length d, then the mapping $\varphi \mapsto D^\alpha \varphi$ is continuous as a mapping from $(\mathcal{D}(\mathbb{R}^d), \rho)$ to $(\mathcal{D}(\mathbb{R}^d), \rho)$.*

(c) *$(\mathcal{D}(K), \rho)$ is a complete metric space for every compact $K \subset \mathbb{R}^d$.*

Theorem 102

Let $\Omega \subset \mathbb{R}^d$ be open. Set

$$\mathcal{U} = \{U \subset \mathcal{D}(\Omega);\ U \text{ absolutely convex,}\\ U \cap \mathcal{D}(K) \in \tau_K(0) \text{ for every compact } K \subset \Omega\}.$$

Then $\mathcal{U}$ is a basis of neighbourhoods of 0 for a Hausdorff locally convex topology τ on $\mathcal{D}(\Omega)$ which has the following properties:

Theorem 102

Let $\Omega \subset \mathbb{R}^d$ *be open. Set*

$$\mathcal{U} = \{U \subset \mathcal{D}(\Omega);\ U \text{ absolutely convex}, \\ U \cap \mathcal{D}(K) \in \tau_K(0) \text{ for every compact } K \subset \Omega\}.$$

Then $\mathcal{U}$ *is a basis of neighbourhoods of* 0 *for a Hausdorff locally convex topology* τ *on* $\mathcal{D}(\Omega)$ *which has the following properties:*

(a) $\tau_\rho \restriction_{\mathcal{D}(\Omega)} \subset \tau$.

Theorem 102

Let $\Omega \subset \mathbb{R}^d$ be open. Set

$$\mathcal{U} = \{U \subset \mathcal{D}(\Omega);\ U \text{ absolutely convex}, \\ U \cap \mathcal{D}(K) \in \tau_K(0) \text{ for every compact } K \subset \Omega\}.$$

Then $\mathcal{U}$ is a basis of neighbourhoods of 0 for a Hausdorff locally convex topology τ on $\mathcal{D}(\Omega)$ which has the following properties:

(a) $\tau_\rho \restriction_{\mathcal{D}(\Omega)} \subset \tau$.

(b) *$\mathcal{D}(K)$ is a closed subspace of $(\mathcal{D}(\Omega), \tau)$ and $\tau \restriction_{\mathcal{D}(K)} = \tau_K$ for every compact $K \subset \Omega$.*

Theorem 102

Let $\Omega \subset \mathbb{R}^d$ be open. Set

$$\mathcal{U} = \{U \subset \mathcal{D}(\Omega);\ U \text{ absolutely convex}, \\ U \cap \mathcal{D}(K) \in \tau_K(0) \text{ for every compact } K \subset \Omega\}.$$

Then $\mathcal{U}$ is a basis of neighbourhoods of 0 for a Hausdorff locally convex topology τ on $\mathcal{D}(\Omega)$ which has the following properties:

(a) $\tau_\rho \restriction_{\mathcal{D}(\Omega)} \subset \tau$.

(b) *$\mathcal{D}(K)$ is a closed subspace of $(\mathcal{D}(\Omega), \tau)$ and $\tau \restriction_{\mathcal{D}(K)} = \tau_K$ for every compact $K \subset \Omega$.*

(c) *If $A \subset (\mathcal{D}(\Omega), \tau)$ is bounded, then there exists a compact $K \subset \Omega$ such that $A \subset \mathcal{D}(K)$.*

Theorem 102

Let $\Omega \subset \mathbb{R}^d$ be open. Set

$$\mathcal{U} = \{U \subset \mathcal{D}(\Omega);\ U \text{ absolutely convex}, \\ U \cap \mathcal{D}(K) \in \tau_K(0) \text{ for every compact } K \subset \Omega\}.$$

Then $\mathcal{U}$ is a basis of neighbourhoods of 0 for a Hausdorff locally convex topology τ on $\mathcal{D}(\Omega)$ which has the following properties:

(a) $\tau_\rho \restriction_{\mathcal{D}(\Omega)} \subset \tau$.

(b) *$\mathcal{D}(K)$ is a closed subspace of $(\mathcal{D}(\Omega), \tau)$ and $\tau \restriction_{\mathcal{D}(K)} = \tau_K$ for every compact $K \subset \Omega$.*

(c) *If $A \subset (\mathcal{D}(\Omega), \tau)$ is bounded, then there exists a compact $K \subset \Omega$ such that $A \subset \mathcal{D}(K)$.*

(d) *Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Then $\varphi_n \to \varphi$ in τ if and only if there exists a compact $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for each $n \in \mathbb{N}$ and $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on $\mathbb{R}^d$ for each multi-index α of length d.*

Theorem 102

Let $\Omega \subset \mathbb{R}^d$ be open. Set

$$\mathcal{U} = \{U \subset \mathcal{D}(\Omega);\ U \text{ absolutely convex}, \quad U \cap \mathcal{D}(K) \in \tau_K(0) \text{ for every compact } K \subset \Omega\}.$$

Then $\mathcal{U}$ is a basis of neighbourhoods of 0 for a Hausdorff locally convex topology τ on $\mathcal{D}(\Omega)$ which has the following properties:

(a) $\tau_\rho \restriction_{\mathcal{D}(\Omega)} \subset \tau$.

(b) *$\mathcal{D}(K)$ is a closed subspace of $(\mathcal{D}(\Omega), \tau)$ and $\tau \restriction_{\mathcal{D}(K)} = \tau_K$ for every compact $K \subset \Omega$.*

(c) *If $A \subset (\mathcal{D}(\Omega), \tau)$ is bounded, then there exists a compact $K \subset \Omega$ such that $A \subset \mathcal{D}(K)$.*

(d) *Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Then $\varphi_n \to \varphi$ in τ if and only if there exists a compact $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for each $n \in \mathbb{N}$ and $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on $\mathbb{R}^d$ for each multi-index α of length d.*

(e) *If Ω is non-empty, then $(\mathcal{D}(\Omega), \tau)$ is of the first category in itself.*

T6, T101(a)

Proposition 103

Let $\Omega \subset \mathbb{R}^d$ be open, let Y be a locally convex space, and let $T\colon (\mathcal{D}(\Omega), \tau) \to Y$ be linear. The following statements are equivalent:

(i) *T is continuous.*

(ii) *The set $\{T(\varphi_n);\ n \in \mathbb{N}\}$ is bounded for every sequence $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ converging to 0 in τ.*

(iii) *For every compact $K \subset \Omega$ the restriction $T\restriction_{\mathcal{D}(K)}$ is continuous.*

Definition 104

Let $\Omega \subset \mathbb{R}^d$ be open. Continuous linear functionals on $(\mathcal{D}(\Omega), \tau)$ are called distributions on Ω. The space of all distributions on Ω is therefore the space $\mathcal{D}(\Omega)^* = (\mathcal{D}(\Omega), \tau)^*$.

Theorem 105

Let $\Omega \subset \mathbb{R}^d$ be open and let $\Lambda\colon \mathcal{D}(\Omega) \to \mathbb{K}$ be linear. Then $\Lambda \in \mathcal{D}(\Omega)^*$ if and only if for every compact $K \subset \Omega$ there exist $N \in \mathbb{N}_0$ and $C \geq 0$ such that $|\Lambda(\varphi)| \leq C\|\varphi\|_N$ for every $\varphi \in \mathcal{D}(K)$.

Theorem 105

Let $\Omega \subset \mathbb{R}^d$ be open and let $\Lambda\colon \mathcal{D}(\Omega) \to \mathbb{K}$ be linear. Then $\Lambda \in \mathcal{D}(\Omega)^*$ if and only if for every compact $K \subset \Omega$ there exist $N \in \mathbb{N}_0$ and $C \geq 0$ such that $|\Lambda(\varphi)| \leq C\|\varphi\|_N$ for every $\varphi \in \mathcal{D}(K)$.

Definition 106

Let $\Omega \subset \mathbb{R}^d$ be open and $\Lambda \in \mathcal{D}(\Omega)^*$. If there exists $N \in \mathbb{N}_0$ such that for every compact $K \subset \Omega$ there exists $C \geq 0$ such that $|\Lambda(\varphi)| \leq C\|\varphi\|_N$ for every $\varphi \in \mathcal{D}(K)$, then the smallest such N is called the order of the distribution Λ.

Theorem 105

Let $\Omega \subset \mathbb{R}^d$ be open and let $\Lambda\colon \mathcal{D}(\Omega) \to \mathbb{K}$ be linear. Then $\Lambda \in \mathcal{D}(\Omega)^*$ if and only if for every compact $K \subset \Omega$ there exist $N \in \mathbb{N}_0$ and $C \geq 0$ such that $|\Lambda(\varphi)| \leq C\|\varphi\|_N$ for every $\varphi \in \mathcal{D}(K)$.

Definition 106

Let $\Omega \subset \mathbb{R}^d$ be open and $\Lambda \in \mathcal{D}(\Omega)^*$. If there exists $N \in \mathbb{N}_0$ such that for every compact $K \subset \Omega$ there exists $C \geq 0$ such that $|\Lambda(\varphi)| \leq C\|\varphi\|_N$ for every $\varphi \in \mathcal{D}(K)$, then the smallest such N is called the order of the distribution Λ. If no such N exists, then the order of Λ is defined as infinity.

3. Operations with distributions

Lemma 107

Let $k \in \mathbb{N}$, suppose that $f \in C^k(\mathbb{R}^d)$ has all partial derivatives up to order k bounded and let $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$. Then

$$\int_{\mathbb{R}^d} D^\alpha f \varphi \, \mathrm{d}\lambda = (-1)^{|\alpha|} \int_{\mathbb{R}^d} f D^\alpha \varphi \, \mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Definition 108

Let $\Omega \subset \mathbb{R}^d$ be open and $\Lambda \in \mathcal{D}(\Omega)^*$. For a multi-index α of length d we define the derivative D^α of the distribution Λ as a functional on $\mathcal{D}(\Omega)$ given by the formula

$$(D^\alpha \Lambda)(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi).$$

Definition 108

Let $\Omega \subset \mathbb{R}^d$ be open and $\Lambda \in \mathcal{D}(\Omega)^*$. For a multi-index α of length d we define the derivative D^α of the distribution Λ as a functional on $\mathcal{D}(\Omega)$ given by the formula

$$(D^\alpha \Lambda)(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi).$$

For a function $f \in C^\infty(\Omega)$ we define the product of the function f and the distribution Λ as a functional on $\mathcal{D}(\Omega)$ given by the formula

$$(f\Lambda)(\varphi) = \Lambda(f\varphi).$$

Proposition 109

Let $\Omega \subset \mathbb{R}^d$ be open, $\Lambda \in \mathcal{D}(\Omega)^*$, $\alpha \in \mathbb{N}_0^d$, and $f \in C^\infty(\Omega)$. Then the following hold:

(a) $D^\alpha \Lambda \in \mathcal{D}(\Omega)^*$.

Proposition 109

Let $\Omega \subset \mathbb{R}^d$ *be open,* $\Lambda \in \mathcal{D}(\Omega)^*$, $\alpha \in \mathbb{N}_0^d$, *and* $f \in C^\infty(\Omega)$. *Then the following hold:*

(a) $D^\alpha \Lambda \in \mathcal{D}(\Omega)^*$.

(b) $f\Lambda \in \mathcal{D}(\Omega)^*$.

Proposition 109

Let $\Omega \subset \mathbb{R}^d$ *be open,* $\Lambda \in \mathcal{D}(\Omega)^*$, $\alpha \in \mathbb{N}_0^d$, *and* $f \in C^\infty(\Omega)$. *Then the following hold:*

(a) $D^\alpha \Lambda \in \mathcal{D}(\Omega)^*$.

(b) $f\Lambda \in \mathcal{D}(\Omega)^*$.

(c) *If* $g \in L_1^{\mathrm{loc}}(\Omega)$, *then* $f\Lambda_g = \Lambda_{fg}$.

Proposition 109

Let $\Omega \subset \mathbb{R}^d$ be open, $\Lambda \in \mathcal{D}(\Omega)^*$, $\alpha \in \mathbb{N}_0^d$, and $f \in C^\infty(\Omega)$. Then the following hold:

(a) $D^\alpha \Lambda \in \mathcal{D}(\Omega)^*$.

(b) $f\Lambda \in \mathcal{D}(\Omega)^*$.

(c) *If* $g \in L_1^{\mathrm{loc}}(\Omega)$, *then* $f\Lambda_g = \Lambda_{fg}$.

(d) *If* $g \in C^{|\alpha|}(\Omega)$, *then* $D^\alpha \Lambda_g = \Lambda_{D^\alpha g}$.

Proposition 109

Let $\Omega \subset \mathbb{R}^d$ *be open,* $\Lambda \in \mathcal{D}(\Omega)^*$, $\alpha \in \mathbb{N}_0^d$, *and* $f \in C^\infty(\Omega)$. *Then the following hold:*

(a) $D^\alpha \Lambda \in \mathcal{D}(\Omega)^*$.

(b) $f\Lambda \in \mathcal{D}(\Omega)^*$.

(c) *If* $g \in L_1^{\mathrm{loc}}(\Omega)$, *then* $f\Lambda_g = \Lambda_{fg}$.

(d) *If* $g \in C^{|\alpha|}(\Omega)$, *then* $D^\alpha \Lambda_g = \Lambda_{D^\alpha g}$.

(e) *If* $d = 1$, $\Omega = (a, b)$, *and* $g \in L_1^{\mathrm{loc}}((a, b))$, *then*

- $\Lambda_g' = \Lambda_h$, *where* $h \in L_1^{\mathrm{loc}}((a, b))$, *if and only if h is the weak derivative of g;*
- $\Lambda_g' = \Lambda_\mu$, *where* μ *is a Borel complex measure on* (a, b), *if and only if* μ *is the weak derivative of g.*

Fact 110

Let $\alpha \in \mathbb{N}_0^d$*. Then there exist constants* $c_\beta^\alpha \in \mathbb{N}$*,* $\beta \in \mathbb{N}_0^d$*,* $\beta \leq \alpha$ *(the inequality of vectors is understood coordinatewise) such that for every open* $\Omega \subset \mathbb{R}^d$ *and every* $f, g \in C^{|\alpha|}(\Omega)$ *the following holds:*

$$D^\alpha(fg) = \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \leq \alpha}} c_\beta^\alpha D^\beta f D^{\alpha-\beta} g.$$

4. The space of distributions

Proposition 111

Let $\Omega \subset \mathbb{R}^d$ be open.

(a) *Let $\alpha \in \mathbb{N}_0^d$ and $g \in C^\infty(\Omega)$. Then the mappings $\Lambda \mapsto D^\alpha \Lambda$ and $\Lambda \mapsto g\Lambda$ are continuous linear mappings of the space $(\mathcal{D}(\Omega)^*, w^*)$ into itself.*

Proposition 111

Let $\Omega \subset \mathbb{R}^d$ be open.

(a) *Let $\alpha \in \mathbb{N}_0^d$ and $g \in C^\infty(\Omega)$. Then the mappings $\Lambda \mapsto D^\alpha \Lambda$ and $\Lambda \mapsto g\Lambda$ are continuous linear mappings of the space $(\mathcal{D}(\Omega)^*, w^*)$ into itself.*

(b) *If $f_n, f \in L_1^{\mathrm{loc}}(\Omega)$ and if $\int_K |f_n - f| \,\mathrm{d}\lambda \to 0$ for each compact $K \subset \Omega$, then $\Lambda_{f_n} \to \Lambda_f$.*

Proposition 111

Let $\Omega \subset \mathbb{R}^d$ be open.

(a) *Let $\alpha \in \mathbb{N}_0^d$ and $g \in C^\infty(\Omega)$. Then the mappings $\Lambda \mapsto D^\alpha \Lambda$ and $\Lambda \mapsto g\Lambda$ are continuous linear mappings of the space $(\mathcal{D}(\Omega)^*, w^*)$ into itself.*

(b) *If $f_n, f \in L_1^{\mathrm{loc}}(\Omega)$ and if $\int_K |f_n - f| \, \mathrm{d}\lambda \to 0$ for each compact $K \subset \Omega$, then $\Lambda_{f_n} \to \Lambda_f$.*

(c) *The mapping $\varphi \mapsto \Lambda_\varphi$ is a one-to-one continuous linear mapping of $(\mathcal{D}(\Omega), \rho)$ into $(\mathcal{D}(\Omega)^*, w^*)$.*

Proposition 111

Let $\Omega \subset \mathbb{R}^d$ be open.

(a) *Let $\alpha \in \mathbb{N}_0^d$ and $g \in C^\infty(\Omega)$. Then the mappings $\Lambda \mapsto D^\alpha \Lambda$ and $\Lambda \mapsto g\Lambda$ are continuous linear mappings of the space $(\mathcal{D}(\Omega)^*, w^*)$ into itself.*

(b) *If $f_n, f \in L_1^{\text{loc}}(\Omega)$ and if $\int_K |f_n - f| \, \mathrm{d}\lambda \to 0$ for each compact $K \subset \Omega$, then $\Lambda_{f_n} \to \Lambda_f$.*

(c) *The mapping $\varphi \mapsto \Lambda_\varphi$ is a one-to-one continuous linear mapping of $(\mathcal{D}(\Omega), \rho)$ into $(\mathcal{D}(\Omega)^*, w^*)$.*

(d) *If $1 \leq p \leq \infty$ and $f_n \to f$ in $L_p(\Omega)$, then $\Lambda_{f_n} \to \Lambda_f$.*

Theorem 112

Let $\Omega \subset \mathbb{R}^d$ *be open and let* $\{\Lambda_n\}$ *be a sequence in* $\mathcal{D}(\Omega)^*$ *such that* $\Lambda(\varphi) = \lim_{n\to\infty} \Lambda_n(\varphi)$ *exists for every* $\varphi \in \mathcal{D}(\Omega)$*. Then* $\Lambda \in \mathcal{D}(\Omega)^*$*.*

5. The support of a distribution

5. The support of a distribution

Definition 113

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. We say that an open set $G \subset \Omega$ is null for Λ if $\Lambda(\varphi) = 0$ for every $\varphi \in \mathcal{D}(G)$.

5. The support of a distribution

Definition 113

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. We say that an open set $G \subset \Omega$ is null for Λ if $\Lambda(\varphi) = 0$ for every $\varphi \in \mathcal{D}(G)$.

Theorem 114

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. The set $G = \bigcup\{H \subset \Omega;\ H \text{ is null for } \Lambda\}$ is null for Λ and it is the largest null set for Λ, i.e. if $H \subset \Omega$ is null for Λ, then $H \subset G$.

5. The support of a distribution

Definition 113

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. We say that an open set $G \subset \Omega$ is null for Λ if $\Lambda(\varphi) = 0$ for every $\varphi \in \mathcal{D}(G)$.

Theorem 114

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. The set $G = \bigcup\{H \subset \Omega;\ H \text{ is null for } \Lambda\}$ is null for Λ and it is the largest null set for Λ, i.e. if $H \subset \Omega$ is null for Λ, then $H \subset G$.

Definition 115

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. The support of the distribution Λ is defined as
$\operatorname{supp} \Lambda = \Omega \setminus G$, where G is the largest null set for Λ.

Theorem 116

Let $\Omega \subset \mathbb{R}^d$ *be open and let* Λ *be a distribution on* Ω*.*

(a) *If* $f \in C(\Omega)$*, then* $\operatorname{supp} \Lambda_f = \operatorname{supp} f$*.*

Theorem 116

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω.

(a) *If $f \in C(\Omega)$, then* $\operatorname{supp} \Lambda_f = \operatorname{supp} f$.

(b) *If μ is a Borel complex measure on Ω, then* $\operatorname{supp} \Lambda_\mu = \operatorname{supp} \mu$.

Theorem 116

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω.

(a) *If $f \in C(\Omega)$, then* $\operatorname{supp} \Lambda_f = \operatorname{supp} f$.

(b) *If μ is a Borel complex measure on Ω, then* $\operatorname{supp} \Lambda_\mu = \operatorname{supp} \mu$.

(c) *If* $\operatorname{supp} \Lambda$ *is compact, then there exist $N \in \mathbb{N}_0$ and $C \geq 0$ such that $|\Lambda(\varphi)| \leq C\|\varphi\|_N$ for every $\varphi \in \mathcal{D}(\Omega)$. In particular, Λ is of a finite order.*

Theorem 116

Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω.

(a) *If $f \in C(\Omega)$, then* $\operatorname{supp} \Lambda_f = \operatorname{supp} f$.

(b) *If μ is a Borel complex measure on Ω, then* $\operatorname{supp} \Lambda_\mu = \operatorname{supp} \mu$.

(c) *If* $\operatorname{supp} \Lambda$ *is compact, then there exist $N \in \mathbb{N}_0$ and $C \geq 0$ such that $|\Lambda(\varphi)| \leq C\|\varphi\|_N$ for every $\varphi \in \mathcal{D}(\Omega)$. In particular, Λ is of a finite order.*

(d) $\operatorname{supp} \Lambda = \{z\}$ *for $z \in \Omega$ if and only if $\Lambda = \sum_{|\alpha| \leq N} c_\alpha D^\alpha \Lambda_{\delta_z}$ for some $N \in \mathbb{N}_0$ and constants c_α, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq N$ not all zero.*

6. Schwartz space

Lemma 117

For $N \in \mathbb{N}$ the function $x \mapsto (1 + \|x\|^2)^N$ is a polynomial on $\mathbb{R}^d$. For every polynomial P on $\mathbb{R}^d$ there exist $N \in \mathbb{N}$ and $C > 0$ such that $|P(x)| \leq C(1 + \|x\|^2)^N$ for each $x \in \mathbb{R}^d$.

Definition 118

The Schwartz space on $\mathbb{R}^d$ is defined as follows:

$$\mathcal{S}_d = \{f \in C^\infty(\mathbb{R}^d, \mathbb{C});\ PD^\alpha f \text{ is bounded for each } \alpha \in \mathbb{N}_0^d \text{ and each polynomial } P \text{ on } \mathbb{R}^d\}.$$

Lemma 119

Let $d \in \mathbb{N}$*,* $1 \leq p < \infty$*,* $N > \frac{d}{2p}$*, and* $h(x) = \frac{1}{(1+\|x\|^2)^N}$ *for* $x \in \mathbb{R}^d$*. Then* $h \in L_p(\mathbb{R}^d)$*.*

Proposition 120

The Schwartz space has the following properties:

(a) $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}_d \subset C_0(\mathbb{R}^d) \cap \bigcap_{1 \le p < \infty} L_p(\mathbb{R}^d)$.

Proposition 120

The Schwartz space has the following properties:

(a) $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}_d \subset C_0(\mathbb{R}^d) \cap \bigcap_{1 \le p < \infty} L_p(\mathbb{R}^d)$.

(b) *If* $f \in \mathcal{S}_d$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^d$, *and* $h(x) = f(ax + b)$, *then* $h \in \mathcal{S}_d$.

Proposition 120

The Schwartz space has the following properties:

(a) $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}_d \subset C_0(\mathbb{R}^d) \cap \bigcap_{1 \le p < \infty} L_p(\mathbb{R}^d)$.

(b) *If* $f \in \mathcal{S}_d$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^d$, *and* $h(x) = f(ax + b)$, *then* $h \in \mathcal{S}_d$.

(c) *If* $f \in \mathcal{S}_d$ *and* α *is a multi-index of length* d, *then* $D^\alpha f \in \mathcal{S}_d$.

Proposition 120

The Schwartz space has the following properties:

(a) $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}_d \subset C_0(\mathbb{R}^d) \cap \bigcap_{1 \le p < \infty} L_p(\mathbb{R}^d)$.

(b) *If* $f \in \mathcal{S}_d$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^d$, *and* $h(x) = f(ax + b)$, *then* $h \in \mathcal{S}_d$.

(c) *If* $f \in \mathcal{S}_d$ *and* α *is a multi-index of length* d, *then* $D^\alpha f \in \mathcal{S}_d$.

(d) *If* $f \in \mathcal{S}_d$ *and if* $g \in C^\infty(\mathbb{R}^d)$ *is bounded and has all partial derivatives of all orders bounded (in particular, if* $g \in \mathcal{S}_d$*), then* $fg \in \mathcal{S}_d$.

Proposition 120

The Schwartz space has the following properties:

(a) $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}_d \subset C_0(\mathbb{R}^d) \cap \bigcap_{1 \le p < \infty} L_p(\mathbb{R}^d)$.

(b) *If* $f \in \mathcal{S}_d$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^d$, *and* $h(x) = f(ax + b)$, *then* $h \in \mathcal{S}_d$.

(c) *If* $f \in \mathcal{S}_d$ *and* α *is a multi-index of length* d, *then* $D^\alpha f \in \mathcal{S}_d$.

(d) *If* $f \in \mathcal{S}_d$ *and if* $g \in C^\infty(\mathbb{R}^d)$ *is bounded and has all partial derivatives of all orders bounded (in particular, if* $g \in \mathcal{S}_d$*), then* $fg \in \mathcal{S}_d$.

(e) *If* $f \in \mathcal{S}_d$ *and* $P \colon \mathbb{R}^d \to \mathbb{C}$ *is a polynomial, then* $Pf \in \mathcal{S}_d$.

For $N \in \mathbb{N}_0$ and $f \in \mathcal{S}_d$ put

$$\nu_N(f) = \max_{|\alpha| \leq N} \left\| x \mapsto (1 + \|x\|^2)^N D^\alpha f(x) \right\|_\infty.$$

For $N \in \mathbb{N}_0$ and $f \in \mathcal{S}_d$ put

$$\nu_N(f) = \max_{|\alpha| \le N} \left\| x \mapsto (1 + \|x\|^2)^N D^\alpha f(x) \right\|_\infty .$$

Denote by σ the Hausdorff locally convex topology on $\mathcal{S}_d$ generated by the system $\{\nu_N\}_{N=0}^{\infty}$. This topology is metrisable by the metric from Lemma 50.

Theorem 121

The metric from Lemma 50 corresponding to the system $\{\nu_N\}_{N=0}^{\infty}$ *is complete. The space* $(\mathcal{S}_d, \sigma)$ *is then a Fréchet space.*

Theorem 121

The metric from Lemma 50 corresponding to the system $\{\nu_N\}_{N=0}^{\infty}$ is complete. The space $(\mathcal{S}_d, \sigma)$ is then a Fréchet space. The topology σ has the following properties:

(a) *Let $\{f_n\}$ be a sequence in $\mathcal{S}_d$ and $f \in \mathcal{S}_d$. The following statements are equivalent:*

  (i) $f_n \to f$ *in the topology* σ.

  (ii) $(1 + \|x\|^2)^N D^\alpha f_n \to (1 + \|x\|^2)^N D^\alpha f$ *uniformly on* $\mathbb{R}^d$ *for every* $N \in \mathbb{N}_0$ *and every multi-index* α *of length* d.

  (iii) $PD^\alpha f_n \to PD^\alpha f$ *uniformly on* $\mathbb{R}^d$ *for every polynomial* P *and every multi-index* α *of length* d.

Theorem 121

The metric from Lemma 50 corresponding to the system $\{\nu_N\}_{N=0}^{\infty}$ is complete. The space $(\mathcal{S}_d, \sigma)$ is then a Fréchet space. The topology σ has the following properties:

(a) *Let $\{f_n\}$ be a sequence in $\mathcal{S}_d$ and $f \in \mathcal{S}_d$. The following statements are equivalent:*

  (i) $f_n \to f$ *in the topology* σ.

  (ii) $(1 + \|x\|^2)^N D^\alpha f_n \to (1 + \|x\|^2)^N D^\alpha f$ *uniformly on* $\mathbb{R}^d$ *for every* $N \in \mathbb{N}_0$ *and every multi-index* α *of length* d.

  (iii) $PD^\alpha f_n \to PD^\alpha f$ *uniformly on* $\mathbb{R}^d$ *for every polynomial* P *and every multi-index* α *of length* d.

(b) *If* $f_n \to f$ *in the space* $(\mathcal{S}_d, \sigma)$, *then* $f_n \to f$ *in* $L_p(\mathbb{R}^d)$ *for each* $1 \le p < \infty$.

Theorem 121

The metric from Lemma 50 corresponding to the system $\{\nu_N\}_{N=0}^{\infty}$ is complete. The space $(\mathcal{S}_d, \sigma)$ is then a Fréchet space. The topology σ has the following properties:

(a) *Let $\{f_n\}$ be a sequence in $\mathcal{S}_d$ and $f \in \mathcal{S}_d$. The following statements are equivalent:*

  (i) $f_n \to f$ *in the topology* σ.

  (ii) $(1 + \|x\|^2)^N D^\alpha f_n \to (1 + \|x\|^2)^N D^\alpha f$ *uniformly on* $\mathbb{R}^d$ *for every* $N \in \mathbb{N}_0$ *and every multi-index* α *of length* d.

  (iii) $PD^\alpha f_n \to PD^\alpha f$ *uniformly on* $\mathbb{R}^d$ *for every polynomial* P *and every multi-index* α *of length* d.

(b) *If* $f_n \to f$ *in the space* $(\mathcal{S}_d, \sigma)$, *then* $f_n \to f$ *in* $L_p(\mathbb{R}^d)$ *for each* $1 \le p < \infty$.

(c) *If* α *is a multi-index of length* d, P *is a polynomial on* $\mathbb{R}^d$, *and* $g \in \mathcal{S}_d$, *then the mappings* $f \mapsto D^\alpha f$, $f \mapsto Pf$, *and* $f \mapsto gf$ *are continuous linear mappings from* $(\mathcal{S}_d, \sigma)$ *to* $(\mathcal{S}_d, \sigma)$.

T48(c), L117, L50; L119; T25, F110

Proposition 122

Let $f \in \mathcal{S}_d$ *and* $\alpha \in \mathbb{N}_0^d$.

(a) $\widehat{D^\alpha f}(t) = (it)^\alpha \widehat{f}(t)$ *for every* $t \in \mathbb{R}^d$.

Proposition 122

Let $f \in \mathcal{S}_d$ *and* $\alpha \in \mathbb{N}_0^d$.

(a) $\widehat{D^\alpha f}(t) = (it)^\alpha \widehat{f}(t)$ *for every* $t \in \mathbb{R}^d$.

(b) $D^\alpha \widehat{f} = \widehat{m_\alpha f}$, *where* $m_\alpha(x) = (-ix)^\alpha$.

Theorem 123

The Fourier transform is an isomorphism of the space $(\mathcal{S}_d, \sigma)$ *onto itself. Moreover, if* $f \in \mathcal{S}_d$, *then*

$$\widehat{\widehat{f}}(x) = f(-x) \text{ for every } x \in \mathbb{R}^d \quad \text{and} \quad \widehat{\widehat{\widehat{\widehat{f}}}} = f.$$

7. Tempered distributions

Lemma 124

Let $K \subset \mathbb{R}^d$ be compact. Then $\sigma \restriction_{\mathcal{D}(K)} = \tau_K$.

Lemma 124

Let $K \subset \mathbb{R}^d$ be compact. Then $\sigma \restriction_{\mathcal{D}(K)} = \tau_K$.

Proposition 125

The subspace $\mathcal{D}(\mathbb{R}^d)$ is dense in $(\mathcal{S}_d, \sigma)$ and $\sigma \restriction_{\mathcal{D}(\mathbb{R}^d)} \subset \tau$. In other words, the embedding Id: $(\mathcal{D}(\mathbb{R}^d), \tau) \to (\mathcal{S}_d, \sigma)$ is continuous and onto a dense subset.

Definition 126

Distributions on $\mathbb{R}^d$ which are restrictions of functionals from $(\mathcal{S}_d, \sigma)^*$ are called tempered distributions.

Definition 126

Distributions on $\mathbb{R}^d$ which are restrictions of functionals from $(\mathcal{S}_d, \sigma)^*$ are called tempered distributions.

Theorem 127

Let $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$. The following statements are equivalent:

(i) *Λ is tempered.*

(ii) *Λ is continuous also in the (weaker) topology σ.*

(iii) *There exist $N \in \mathbb{N}_0$ and $C \geq 0$ such that $|\Lambda(\varphi)| \leq C\nu_N(\varphi)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$.*

Proposition 128

Let Λ be a tempered distribution on $\mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $g \in \mathcal{S}_d$, and let P be a polynomial on $\mathbb{R}^d$. Then $D^\alpha \Lambda$, $g\Lambda$, and $P\Lambda$ are also tempered distributions and the formulas

- $D^\alpha \Lambda(f) = (-1)^{|\alpha|} \Lambda(D^\alpha f)$,
- $(g\Lambda)(f) = \Lambda(gf)$, *and*
- $(P\Lambda)(f) = \Lambda(Pf)$

hold for every $f \in \mathcal{S}_d$.

Proposition 128

Let Λ be a tempered distribution on $\mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $g \in \mathcal{S}_d$, and let P be a polynomial on $\mathbb{R}^d$. Then $D^\alpha \Lambda$, $g\Lambda$, and $P\Lambda$ are also tempered distributions and the formulas

- $D^\alpha \Lambda(f) = (-1)^{|\alpha|} \Lambda(D^\alpha f)$,
- $(g\Lambda)(f) = \Lambda(gf)$, *and*
- $(P\Lambda)(f) = \Lambda(Pf)$

hold for every $f \in \mathcal{S}_d$. Further, the mappings $\Lambda \mapsto D^\alpha \Lambda$, $\Lambda \mapsto g\Lambda$, and $\Lambda \mapsto P\Lambda$ are continuous linear mappings from the space $(\mathcal{S}_d^*, w^*)$ into itself.

Definition 129

The Fourier transform of a tempered distribution Λ on $\mathbb{R}^d$ is defined by the formula $\widehat{\Lambda}(f) = \Lambda(\widehat{f})$ for $f \in \mathcal{S}_d$.

Definition 129

The Fourier transform of a tempered distribution Λ on $\mathbb{R}^d$ is defined by the formula $\widehat{\Lambda}(f) = \Lambda(\widehat{f})$ for $f \in \mathcal{S}_d$.

Theorem 130

(a) *If $g \in L_1(\mathbb{R}^d)$, then $\Lambda_{\widehat{g}}$ is a tempered distribution and $\widehat{\Lambda_g} = \Lambda_{\widehat{g}}$.*

T123, P120(a), P128, P122

Definition 129

The Fourier transform of a tempered distribution Λ on $\mathbb{R}^d$ is defined by the formula $\widehat{\Lambda}(f) = \Lambda(\widehat{f})$ for $f \in \mathcal{S}_d$.

Theorem 130

(a) *If $g \in L_1(\mathbb{R}^d)$, then $\Lambda_{\widehat{g}}$ is a tempered distribution and $\widehat{\Lambda_g} = \Lambda_{\widehat{g}}$. If $g \in L_2(\mathbb{R}^d)$, then $\widehat{\Lambda_g} = \Lambda_{F(g)}$, where F is the extension of the Fourier transform from Plancherel's theorem.*

Definition 129

The Fourier transform of a tempered distribution Λ on $\mathbb{R}^d$ is defined by the formula $\widehat{\Lambda}(f) = \Lambda(\widehat{f})$ for $f \in \mathcal{S}_d$.

Theorem 130

(a) *If $g \in L_1(\mathbb{R}^d)$, then $\Lambda_{\widehat{g}}$ is a tempered distribution and $\widehat{\Lambda_g} = \Lambda_{\widehat{g}}$. If $g \in L_2(\mathbb{R}^d)$, then $\widehat{\Lambda_g} = \Lambda_{F(g)}$, where F is the extension of the Fourier transform from Plancherel's theorem.*

(b) *If Λ is a tempered distribution on $\mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$, then*

- $\widehat{D^\alpha \Lambda} = s_\alpha \widehat{\Lambda}$, *where* $s_\alpha(x) = (ix)^\alpha$, *and*
- $D^\alpha \widehat{\Lambda} = \widehat{m_\alpha \Lambda}$, *where* $m_\alpha(x) = (-ix)^\alpha$.

Definition 129

The Fourier transform of a tempered distribution Λ on $\mathbb{R}^d$ is defined by the formula $\widehat{\Lambda}(f) = \Lambda(\widehat{f})$ for $f \in \mathcal{S}_d$.

Theorem 130

(a) *If $g \in L_1(\mathbb{R}^d)$, then $\Lambda_{\widehat{g}}$ is a tempered distribution and $\widehat{\Lambda_g} = \Lambda_{\widehat{g}}$. If $g \in L_2(\mathbb{R}^d)$, then $\widehat{\Lambda_g} = \Lambda_{F(g)}$, where F is the extension of the Fourier transform from Plancherel's theorem.*

(b) *If Λ is a tempered distribution on $\mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$, then*
- $\widehat{D^\alpha \Lambda} = s_\alpha \widehat{\Lambda}$, *where* $s_\alpha(x) = (ix)^\alpha$, *and*
- $D^\alpha \widehat{\Lambda} = \widehat{m_\alpha \Lambda}$, *where* $m_\alpha(x) = (-ix)^\alpha$.

(c) *The Fourier transform $\mathcal{F}$ of tempered distributions is an isomorphism of the space $(\mathcal{S}_d^*, w^*)$ onto itself. The following holds: $\mathcal{F}^4 = Id$.*

III. The Bochner integral

1. Measurable mappings

Proposition 131

Let Ω be a measurable space and X a metric space. Then the pointwise limit of a sequence of measurable mappings from Ω to X is a measurable mapping.

Definition 132

Let Ω and X be sets. A mapping $f\colon \Omega \to X$ is called simple if $f(\Omega)$ is a finite set.

Definition 132

Let Ω and X be sets. A mapping $f\colon \Omega \to X$ is called simple if $f(\Omega)$ is a finite set.

Theorem 133

Let Ω be a measurable space and X a separable metric space. Then $f\colon \Omega \to X$ is measurable if and only if it is a pointwise limit of a sequence of simple measurable mappings from Ω to X.

If (Ω, μ) is a measure space and M a set, then the symbol $Æ(\Omega, M)$ denotes the set of all mappings defined μ-a. e. on Ω with values in M.

If (Ω, μ) is a measure space and *M* a set, then the symbol Æ(Ω, M) denotes the set of all mappings defined μ-a. e. on Ω with values in *M*.

Definition 134 (Salomon Bochner (1933))

Let (Ω, μ) be a measure space and *X* a metric space. A mapping from Æ(Ω, X) is called strongly measurable (or Bochner measurable) with respect to μ if it is a μ-a. e. pointwise limit of a sequence of simple measurable mappings from Ω to *X*.

If (Ω, μ) is a measure space and M a set, then the symbol $Æ(\Omega, M)$ denotes the set of all mappings defined μ-a. e. on Ω with values in M.

Definition 134 (Salomon Bochner (1933))

Let (Ω, μ) be a measure space and X a metric space. A mapping from $Æ(\Omega, X)$ is called strongly measurable (or Bochner measurable) with respect to μ if it is a μ-a. e. pointwise limit of a sequence of simple measurable mappings from Ω to X.

Lemma 135

Let (Ω, μ) be a space with a complete measure, X a metric space, and $f \in Æ(\Omega, X)$. Then f is strongly measurable if and only if it is measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.

Corollary 136

Let (Ω, μ) be a space with a complete measure, X a metric space, and $\{f_n\} \subset Æ(\Omega, X)$ a sequence of strongly measurable mappings that converges pointwise a. e. to $f \in Æ(\Omega, X)$. Then f is strongly measurable.

Lemma 137

Let Ω and X be measurable spaces. Assume that X is also a vector space over $\mathbb{K}$, $f, g \colon \Omega \to X$ are simple measurable mappings, and $\alpha \in \mathbb{K}$. Then $f + g$ and αf are simple measurable mappings.

Lemma 137

Let Ω and X be measurable spaces. Assume that X is also a vector space over $\mathbb{K}$, $f, g\colon \Omega \to X$ are simple measurable mappings, and $\alpha \in \mathbb{K}$. Then $f + g$ and αf are simple measurable mappings.

Corollary 138

Let (Ω, μ) be a measure space, X a normed linear space over $\mathbb{K}$, $f, g \in$ Æ(Ω, X) strongly measurable, and $\alpha \in \mathbb{K}$. Then $f + g$ and αf are strongly measurable mappings.

Definition 139 (Izrail Moisejevič Gelfand (1938), Billy James Pettis (1938))

Let (Ω, μ) be a measure space and X a normed linear space. A mapping $f \in$ Æ(Ω, X) s called weakly measurable if $\phi \circ f$ is a measurable function for every $\phi \in X^*$.

Definition 140

Let X be a normed linear space. We say that $A \subset B_{X^*}$ is 1-norming if $\|x\| = \sup_{f \in A} |f(x)|$ for every $x \in X$.

Definition 140

Let X be a normed linear space. We say that $A \subset B_{X^*}$ is 1-norming if $\|x\| = \sup_{f \in A} |f(x)|$ for every $x \in X$.

Lemma 141

Let X be a normed linear space. If $A \subset B_{X^*}$ is w^*-dense in B_{X^*}, then A is 1-norming.

Lemma 142

Let X be a normed linear space and let $A \subset B_{X^*}$ be 1-norming. Then $A_\circ = B_X$.

Lemma 142

Let X be a normed linear space and let $A \subset B_{X^*}$ be 1-norming. Then $A_\circ = B_X$. More generally, $B(x,r) = \bigcap_{f \in A}\{y \in X;\ |f(y) - f(x)| \leq r\}$ for every $x \in X$ and $r > 0$.

Lemma 142

Let X be a normed linear space and let $A \subset B_{X^*}$ be 1-norming. Then $A_\circ = B_X$. More generally, $B(x, r) = \bigcap_{f \in A} \{y \in X;\ |f(y) - f(x)| \leq r\}$ for every $x \in X$ and $r > 0$.

Lemma 143

Let X be a normed linear space and $A \subset X$. Then $\operatorname{span}_{\mathbb{Q}} A$ is dense in $\operatorname{span} A$ and $B_X \cap \operatorname{span}_{\mathbb{Q}} A$ is dense in $B_X \cap \operatorname{span} A$.

Lemma 142

Let X be a normed linear space and let $A \subset B_{X^*}$ be 1-norming. Then $A_\circ = B_X$. More generally, $B(x,r) = \bigcap_{f \in A} \{y \in X;\ |f(y) - f(x)| \leq r\}$ for every $x \in X$ and $r > 0$.

Lemma 143

Let X be a normed linear space and $A \subset X$. Then $\operatorname{span}_{\mathbb{Q}} A$ is dense in $\operatorname{span} A$ and $B_X \cap \operatorname{span}_{\mathbb{Q}} A$ is dense in $B_X \cap \operatorname{span} A$. It follows that if $M \subset X$ is separable, then $\overline{\operatorname{span}}\, M$ is also separable.

Theorem 144

Let (Ω, μ) be a space with a complete measure, X a normed linear space, and $f \in$ Æ(Ω, X). The following statements are equivalent:

(i) *f is strongly measurable.*

Theorem 144

Let (Ω, μ) be a space with a complete measure, X a normed linear space, and $f \in Æ(\Omega, X)$. The following statements are equivalent:

(i) *f is strongly measurable.*

(ii) *f is measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.*

Theorem 144

Let (Ω, μ) be a space with a complete measure, X a normed linear space, and $f \in Æ(\Omega, X)$. The following statements are equivalent:

(i) *f is strongly measurable.*

(ii) *f is measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.*

(iii) *f is weakly measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.*

Theorem 144

Let (Ω, μ) be a space with a complete measure, X a normed linear space, and $f \in Æ(\Omega, X)$. The following statements are equivalent:

(i) *f is strongly measurable.*

(ii) *f is measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.*

(iii) *f is weakly measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.*

(iv) *There exist $E \subset \Omega$, $Y \subset X$ a separable subspace, and $A \subset B_{Y^*}$ countable such that $\mu(E) = 0$, $f(\Omega \setminus E) \subset Y$, $B_{Y^*} \cap \operatorname{span} A$ is w^*-dense in B_{Y^*}, and $\phi \circ f$ is measurable for each $\phi \in A$.*

Proposition 145

Let (Ω, μ) be a space with a complete measure, X a normed linear space, and let $f \in$ Æ(Ω, X) be strongly measurable. If $\phi \circ f = 0$ a. e. for every $\phi \in X^*$, then $f = 0$ a. e.

L135, L143, C85, P88, L141

2. The Bochner integral

Definition 146

Let (Ω, μ) be a measure space and X a normed linear space. A mapping $f\colon \Omega \to X$ is called a step mapping if it is simple, measurable, and $\mu(f^{-1}(x)) < +\infty$ for each $x \in f(\Omega) \setminus \{0\}$.

Definition 146

Let (Ω, μ) be a measure space and X a normed linear space. A mapping $f\colon \Omega \to X$ is called a step mapping if it is simple, measurable, and $\mu(f^{-1}(x)) < +\infty$ for each $x \in f(\Omega) \setminus \{0\}$.

Definition 147

Let (Ω, μ) be a measure space, X a normed linear space, and $f\colon \Omega \to X$ a step mapping. Then for each measurable $E \subset \Omega$ we define the Bochner integral of f over E as

$$\int_E f \,\mathrm{d}\mu = \sum_{x \in f(\Omega)\setminus\{0\}} \mu(f^{-1}(x) \cap E)x.$$

Lemma 148

Let (Ω, μ) be a measure space, X a normed linear space, and $f\colon \Omega \to X$ a step mapping. If $A, B \subset \Omega$ are disjoint measurable sets, then $\int_{A\cup B} f \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu + \int_B f \,\mathrm{d}\mu$.

Lemma 148

Let (Ω, μ) be a measure space, X a normed linear space, and $f \colon \Omega \to X$ a step mapping. If $A, B \subset \Omega$ are disjoint measurable sets, then $\int_{A \cup B} f \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu + \int_B f \,\mathrm{d}\mu$.

Theorem 149

Let (Ω, μ) be a measure space, X a normed linear space over $\mathbb{K}$, $f, g \colon \Omega \to X$ step mappings, and $\alpha \in \mathbb{K}$. Then $f + g$ and αf are step mappings and $\int_E (f + g) \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu + \int_E g \,\mathrm{d}\mu$ and $\int_E \alpha f \,\mathrm{d}\mu = \alpha \int_E f \,\mathrm{d}\mu$ for every meaurable $E \subset \Omega$.

If (Ω, μ) is a measure space, X a normed linear space, and $f \in$ Æ(Ω, X) a measurable mapping, then the function $t \mapsto \|f(t)\|$ is measurable on Ω, since it is a composition of a continuous function $\|\cdot\|$ with a measurable mapping f. This function will be denoted by $\|f\|$.

If (Ω, μ) is a measure space, X a normed linear space, and $f \in Æ(\Omega, X)$ a measurable mapping, then the function $t \mapsto \|f(t)\|$ is measurable on Ω, since it is a composition of a continuous function $\|\cdot\|$ with a measurable mapping f. This function will be denoted by $\|f\|$.

Lemma 150

Let (Ω, μ) be a measure space, X a normed linear space, and $f \colon \Omega \to X$ a simple measurable mapping. Then f is a step mapping if and only if $\int_\Omega \|f\| \,\mathrm{d}\mu < +\infty$. In this case $\left\| \int_E f \,\mathrm{d}\mu \right\| \le \int_E \|f\| \,\mathrm{d}\mu$ for every measurable $E \subset \Omega$.

Lemma 151

Let (Ω, μ) be a measure space, X a Banach space, $f \in Æ(\Omega, X)$, and let $f_n \colon \Omega \to X$, $n \in \mathbb{N}$ be a sequence of step mappings such that $\lim_{n\to\infty} \int_\Omega \|f_n(t) - f(t)\| \, \mathrm{d}\mu(t) = 0$*. Then* $\lim_{n\to\infty} \int_E f_n \, \mathrm{d}\mu$ *exists for every measurable $E \subset \Omega$.*

Lemma 151

Let (Ω, μ) *be a measure space,* X *a Banach space,* $f \in Æ(\Omega, X)$*, and let* $f_n \colon \Omega \to X$*,* $n \in \mathbb{N}$ *be a sequence of step mappings such that* $\lim_{n\to\infty} \int_\Omega \|f_n(t) - f(t)\| \,\mathrm{d}\mu(t) = 0$*. Then* $\lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu$ *exists for every measurable* $E \subset \Omega$*. Moreover, if* $g_n \colon \Omega \to X$*,* $n \in \mathbb{N}$ *is a sequence of step mappings with the same property as* $\{f_n\}$*, then* $\lim_{n\to\infty} \int_E g_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu$*.*

Definition 152

Let (Ω, μ) be a measure space, X a Banach space, and $f \in Æ(\Omega, X)$. We say that f is Bochner integrable if there exists a sequence $f_n\colon \Omega \to X$, $n \in \mathbb{N}$ of step mappings such that $\lim_{n\to\infty} \int_\Omega \|f_n - f\| \,\mathrm{d}\mu = 0$. For every measurable $E \subset \Omega$ we then define the Bochner integral of f over E as

$$\int_E f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu.$$

Theorem 153

Let (Ω, μ) be a space with a complete measure, X a Banach space, and $f \in Æ(\Omega, X)$. Then f is Bochner integrable if and only if it is strongly measurable and $\|f\|$ is Lebesgue integrable. In this case $\left\|\int_E f \, \mathrm{d}\mu\right\| \leq \int_E \|f\| \, \mathrm{d}\mu$ for every measurable $E \subset \Omega$.

Theorem 154

Let (Ω, μ) *be a space with a complete measure,* X *a Banach space over* $\mathbb{K}$*,* $f, g \in$ Æ(Ω, X) *Bochner integrable and* $\alpha \in \mathbb{K}$*. Then the mappings* $f + g$ *and* αf *are Bochner integrable and* $\int_E (f + g)\,\mathrm{d}\mu = \int_E f\,\mathrm{d}\mu + \int_E g\,\mathrm{d}\mu$ *and* $\int_E \alpha f\,\mathrm{d}\mu = \alpha \int_E f\,\mathrm{d}\mu$ *for every measurable* $E \subset \Omega$*.*

T149, T153, C138, L135

Theorem 155 (dominated convergence)

Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $\{f_n\} \subset Æ(\Omega, X)$ be a sequence of strongly measurable mappings. Let $f \in Æ(\Omega, X)$ be such that $f_n \to f$ pointwise a. e. and let $g \in L_1(\mu)$ be such that for each $n \in \mathbb{N}$ we have $\|f_n(t)\| \leq g(t)$ for a. a. $t \in \Omega$. Then f_n and f are Bochner integrable and $\lim_{n\to\infty} \int_\Omega \|f_n - f\| \, \mathrm{d}\mu = 0$. In particular, $\int_\Omega f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu$.

Theorem 155 (dominated convergence)

Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $\{f_n\} \subset Æ(\Omega, X)$ be a sequence of strongly measurable mappings. Let $f \in Æ(\Omega, X)$ be such that $f_n \to f$ pointwise a. e. and let $g \in L_1(\mu)$ be such that for each $n \in \mathbb{N}$ we have $\|f_n(t)\| \leq g(t)$ for a. a. $t \in \Omega$. Then f_n and f are Bochner integrable and $\lim\limits_{n\to\infty} \int_\Omega \|f_n - f\| \, \mathrm{d}\mu = 0$. In particular, $\int_\Omega f \, \mathrm{d}\mu = \lim\limits_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu$.

Theorem 156

Let (Ω, μ) be a space with a finite complete measure, X a Banach space, and let $\{f_n\} \subset Æ(\Omega, X)$ be a sequence of strongly measurable mappings. Let $f \in Æ(\Omega, X)$ be Bochner integrable such that $f_n \to f$ uniformly a. e. on Ω. Then $\int_\Omega f \, \mathrm{d}\mu = \lim\limits_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu$.

C136, T153, T154

Theorem 157 (absolute continuity of the Bochner integral)

Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $f \in$ Æ(Ω, X) be Bochner integrable. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\left\| \int_E f \, \mathrm{d}\mu \right\| < \varepsilon$ whenever $E \subset \Omega$ is such that $\mu(E) < \delta$.

Theorem 158

Let (Ω, μ) be a space with a complete measure, X and Y Banach spaces, and let $f \in Æ(\Omega, X)$ be Bochner integrable and $T \in \mathcal{L}(X, Y)$. Then $T \circ f$ is Bochner integrable and

$$\int_E T \circ f \, \mathrm{d}\mu = T\left(\int_E f \, \mathrm{d}\mu\right)$$

for every measurable $E \subset \Omega$.

Fact 159

Let (Ω, μ) *be a space with a complete measure,* X *a Banach space,* $x \in X$*, and* $f \in L_1(\mu)$*. Then* $\int_E f(t)x \, \mathrm{d}\mu(t) = \left(\int_E f \, \mathrm{d}\mu\right)x$ *for every measurable* $E \subset \Omega$*.*

Theorem 160

Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $f \in Æ(\Omega, X)$ be Bochner integrable. If $\int_E f \, \mathrm{d}\mu = 0$ for each measurable $E \subset \Omega$, then $f = 0$ a. e.

Theorem 161

Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $f \in Æ(\Omega, X)$ be Bochner integrable. Then

$$\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu \in \overline{\mathrm{conv}}\, f(E).$$

for each $E \subset \Omega$ of positive measure.

Theorem 162 (Fubini's theorem for the Bochner integral)

Let (Ω_1, μ_1) and (Ω_2, μ_2) be spaces with σ-finite complete measures and let ν be a completion of the product measure $\mu_1 \times \mu_2$. Let X be a Banach space and let $f \in Æ(\Omega_1 \times \Omega_2, X)$ be Bochner integrable with respect to ν. Then for μ_1-a. a. $s \in \Omega_1$ the mapping $t \mapsto f(s, t)$ is Bochner integrable on Ω_2, for μ_2-a. a. $t \in \Omega_2$ the mapping $s \mapsto f(s, t)$ is Bochner integrable on Ω_1;

Theorem 162 (Fubini's theorem for the Bochner integral)

Let (Ω_1, μ_1) and (Ω_2, μ_2) be spaces with σ-finite complete measures and let ν be a completion of the product measure $\mu_1 \times \mu_2$. Let X be a Banach space and let $f \in Æ(\Omega_1 \times \Omega_2, X)$ be Bochner integrable with respect to ν. Then for μ_1-a. a. $s \in \Omega_1$ the mapping $t \mapsto f(s,t)$ is Bochner integrable on Ω_2, for μ_2-a. a. $t \in \Omega_2$ the mapping $s \mapsto f(s,t)$ is Bochner integrable on Ω_1; the mappings $\psi_1(s) = \int_{\Omega_2} f(s,t)\,\mathrm{d}\mu_2(t)$ and $\psi_2(t) = \int_{\Omega_1} f(s,t)\,\mathrm{d}\mu_1(s)$ defined a. e. on Ω_1, resp. Ω_2 are Bochner integrable and

$$\int_{\Omega_1} \psi_1\,\mathrm{d}\mu_1 = \int_{\Omega_1 \times \Omega_2} f\,\mathrm{d}\nu = \int_{\Omega_2} \psi_2\,\mathrm{d}\mu_2.$$

C85, T88, L135, T144, T153, T158

3. The Lebesgue-Bochner spaces

Definition 163

Let (Ω, μ) be a space with a complete measure, X a Banach space, and $1 \leq p \leq \infty$. The symbol $L_p(\mu, X)$ denotes the set of all strongly measurable mappings from Æ(Ω, X) such that $\|f\| \in L_p(\mu)$, factorised by the equality μ-a. e.

3. The Lebesgue-Bochner spaces

Definition 163

Let (Ω, μ) be a space with a complete measure, X a Banach space, and $1 \leq p \leq \infty$. The symbol $L_p(\mu, X)$ denotes the set of all strongly measurable mappings from Æ(Ω, X) such that $\|f\| \in L_p(\mu)$, factorised by the equality μ-a. e.

Further, for $f \in L_p(\mu, X)$ we define $\|f\|_{L_p(\mu,X)} = \left\| t \mapsto \|f(t)\| \right\|_{L_p(\mu)}$.

Theorem 164

Let (Ω, μ) *be a space with a complete measure,* X *a Banach space, and* $1 \le p \le \infty$*.*

(a) $L_p(\mu, X)$ *is a Banach space with the norm* $\|f\|_{L_p(\mu,X)}$*.*

Theorem 164

Let (Ω, μ) be a space with a complete measure, X a Banach space, and $1 \le p \le \infty$.

(a) *$L_p(\mu, X)$ is a Banach space with the norm $\|f\|_{L_p(\mu,X)}$.*

(b) *If X is a Hilbert space, then $L_2(\mu, X)$ is a Hilbert space with the scalar product*

$$\langle f, g\rangle_{L_2(\mu,X)} = \int_\Omega \langle f(t), g(t)\rangle \, \mathrm{d}\mu.$$

Theorem 165

Let (Ω, μ) *be a space with a complete measure,* X *a Banach space, and* $1 \leq p < \infty$*.*

(a) *The set of all step mappings from* Ω *to* X *is dense in* $L_p(\mu, X)$*.*

Theorem 165

Let (Ω, μ) be a space with a complete measure, X a Banach space, and $1 \le p < \infty$.

(a) *The set of all step mappings from Ω to X is dense in $L_p(\mu, X)$.*

(b) *If X and $L_p(\mu)$ are separable, then $L_p(\mu, X)$ is also separable.*

IV. Compact convex sets

Definition 166

Let C be a convex subset of a vector space. We say that $x \in C$ is an extreme point of the set C if x is not an inner point of any segment lying in C, i.e. if $u, v \in C$ and $x = \lambda u + (1 - \lambda)v$ for some $\lambda \in (0, 1)$, then $u = v$.

Definition 166

Let C be a convex subset of a vector space. We say that $x \in C$ is an extreme point of the set C if x is not an inner point of any segment lying in C, i.e. if $u, v \in C$ and $x = \lambda u + (1 - \lambda)v$ for some $\lambda \in (0, 1)$, then $u = v$. The set of all extreme points of C is denoted by $\operatorname{ext} C$.

Definition 167

Let C be a convex subset of a real vector space X. An affine hyperplane $W \subset X$ is called a supporting hyperplane of the set C (at a point $x \in C$), if $W \cap C \neq \emptyset$ (resp. $x \in W \cap C$) and C lies completely in one of the half-spaces determined by W (i.e. there exist a non-zero linear form f on X and $\alpha \in \mathbb{R}$ such that $W = f^{-1}(\alpha)$ and $\sup_C f \leq \alpha$).

Fact 168

Let C be a convex set in a vector space X.

(a) *If $B \subset C$ is convex, then $B \cap \operatorname{ext} C \subset \operatorname{ext} B$.*

Fact 168

Let C be a convex set in a vector space X.

(a) *If $B \subset C$ is convex, then $B \cap \operatorname{ext} C \subset \operatorname{ext} B$.*

(b) *If Y is a vector space and $T\colon X \to Y$ is an affine mapping, then T preserves convex combinations. If T is one-to-one, then $\operatorname{ext} T(C) = T(\operatorname{ext} C)$.*

Fact 168

Let C be a convex set in a vector space X.

(a) *If $B \subset C$ is convex, then $B \cap \operatorname{ext} C \subset \operatorname{ext} B$.*

(b) *If Y is a vector space and $T\colon X \to Y$ is an affine mapping, then T preserves convex combinations. If T is one-to-one, then $\operatorname{ext} T(C) = T(\operatorname{ext} C)$.*

(c) *If X is real and $W \subset X$ is a supporting hyperplane of the set C, then $\operatorname{ext}(C \cap W) = W \cap \operatorname{ext} C$.*

Theorem 169

If C is a compact convex subset of $\mathbb{R}^n$, then each point of the set C is a convex combination of at most $n + 1$ extreme points of the set C. Therefore $C = \operatorname{conv} \operatorname{ext} C$.

Theorem 169

If C is a compact convex subset of $\mathbb{R}^n$, then each point of the set C is a convex combination of at most $n+1$ extreme points of the set C. Therefore $C = \operatorname{conv}\operatorname{ext} C$.

Lemma 170

Let $C \subset \mathbb{R}^n$ be convex. The either $\operatorname{Int} C \neq \emptyset$, or C lies in some affine hyperplane in $\mathbb{R}^n$.

Theorem 169

If C is a compact convex subset of $\mathbb{R}^n$, then each point of the set C is a convex combination of at most $n+1$ extreme points of the set C. Therefore $C = \operatorname{conv} \operatorname{ext} C$.

Lemma 170

Let $C \subset \mathbb{R}^n$ be convex. The either $\operatorname{Int} C \neq \emptyset$, or C lies in some affine hyperplane in $\mathbb{R}^n$.

Corollary 171

If $A \subset \mathbb{R}^n$ and $x \in \operatorname{conv} A$, then there exists at most $(n+1)$-element subset $B \subset A$ such that $x \in \operatorname{conv} B$.

Corollary 172

Let $K \subset \mathbb{R}^n$ be compact. Then $\operatorname{conv} K$ *is also compact.*

Corollary 172

Let $K \subset \mathbb{R}^n$ be compact. Then $\operatorname{conv} K$ is also compact.

Proposition 173

Let X be a Fréchet space and let $K \subset X$ be compact. Then $\overline{\operatorname{conv}} K$ and $\overline{\operatorname{aconv}} K$ are compact.

Definition 174

Let C be a convex subset of a vector space. We say that a non-empty $E \subset C$ is an extreme subset of C if no point of E is a non-trivial convex combination of points from C some of which lie outside of E, i.e. if $\lambda x + (1 - \lambda)y \in E$ for some $x, y \in C$ and $\lambda \in (0, 1)$, then $x, y \in E$.

Definition 175

Let C be a convex subset of a vector space. We say that a function $f\colon C \to \mathbb{R}$ is convex if $f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$ for every $x, y \in C$ and $\lambda \in [0, 1]$.

Lemma 176

Let C be a convex subset of a vector space, let $f : C \to \mathbb{R}$ be convex, and let E be an extreme subset of C. Then the set of points at which f attains its maximum over E is either empty, or an extreme subset of C.

Lemma 176

Let C be a convex subset of a vector space, let $f\colon C \to \mathbb{R}$ be convex, and let E be an extreme subset of C. Then the set of points at which f attains its maximum over E is either empty, or an extreme subset of C.

Lemma 177

Let X be a topological vector space such that X^* separates the points of X (e.g. a Hausdorff locally convex space) and let $C \subset X$ be convex. Then every compact extreme subset of C contains an extreme point of C.

Recall that a real function f on a topological space X is called upper semi-continuous if the set $\{x \in X;\ f(x) \geq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.

Recall that a real function f on a topological space X is called upper semi-continuous if the set $\{x \in X;\ f(x) \geq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.

Theorem 178 (Bauer's maximum principle)

Let X be a topological vector space such that X^* separates the points of X (e.g. a Hausdorff locally convex space), let $K \subset X$ be a non-empty compact convex set and $f\colon K \to \mathbb{R}$ an upper semi-continuous convex function. Then f attains its maximum over K in an extreme point of K.

Theorem 179 (Kreĭn-Milman)

Let X be a topological vector space such that X^* separates the points of X (e.g. a Hausdorff locally convex space) and let $K \subset X$ be compact and convex. Then $K = \overline{\text{conv}}\, \text{ext}\, K$.