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Non-integral generalizations of Lukasiewicz logic

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Title: Non-integral generalizations of Lukasiewicz logic

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Abstract: This thesis investigates non-integral generalizations of Łukasiewicz logic. First, we introduce the class of superabelian logics, formally define unbounded Łukasiewicz logic and provide the corresponding completeness theorem. Second, we analyse the subvarieties of pointed Abelian ℓ-groups, which allows us to completely categorize the finitary extensions of unbounded Łukasiewicz logic. Third, we investigate the computational complexity of this logic. Finally, we focus on a modal extension of unbounded Łukasiewicz logic, developing its corresponding real-valued Kripke semantics and establishing an algebraic completeness theorem.

Keywords: Abelian logic, Łukasiewicz logic, Algebraic logic, infinitary extensions, Abelian ℓ-groups, complexity, modal logics, variety, pointed Abelian ℓ-groups

Název práce: Neintegrální zobecnění Lukasiewiczovy logiky

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Abstrakt: Tato disertační práce zkoumá neintegrální zobecnění Łukasiewiczovy logiky. Nejprve zavedeme třídu superabelovských logik, formálně definujeme neomezenou Łukasiewiczovu logiku a dokážeme příslušnou větu o úplnosti. Dále popíšeme podvariety bodových abelovských ℓ-grup, což nám umožní klasifikovat finitární rozšíření neomezené Łukasiewiczovy logiky. Zatřetí zkoumáme výpočetní složitost této logiky. Nakonec se zaměříme na modální verzi neomezené Łukasiewiczovy logiky definované pomocí Kripkeho sémantiky, definujeme korespondující třídu algebraických modelů a dokážeme větu o úplnosti.

Klíčová slova: Abelovská logika, Łukasiewiczova logika, algebraická logika, infinitarní rozšíření, abelovské ℓ-grupy, složitost, modální logiky, variety, bodové abelovské ℓ-grupy

Contents

Introduction

Łukasiewicz logic, in its infinitely-valued version, was introduced by Łukasiewicz and Tarski [71] in 1930 and has since established itself as one of the most prominent non-classical logics. It is a foundational member of the family of many-valued systems designed to model vagueness; indeed, within the framework of continuous t-norms, it stands alongside Gödel and Product logics as one of the three fundamental fuzzy logics [53]. Furthermore, its rich structure has forged deep connections with diverse areas of mathematics, including continuous model theory, error-correcting codes, geometry, ℓ-group theory, and algebraic probability theory [21, 44, 68, 84].

The equivalent algebraic semantics for Łukasiewicz logic is provided by the variety of MV-algebras. Originally axiomatized by Chang [20], these structures have been extensively investigated [21, 53, 79, 82, 26]. The equational theory of this class is fully characterized: the lattice of subvarieties of MV-algebras, corresponding to the axiomatic extensions of Łukasiewicz logic, was completely classified by Komori [65]. In contrast to the subvariety lattice, the quasiequational structure has maximal complexity as Adams and Dziobiak demonstrated that the lattice of subquasivarieties of MV-algebras is Q-universal [72].

One of the defining features of Łukasiewicz logic is known algebraically as integrality (the maximal truth value serves as the identity for the strong conjunction) or proof-theoretically as weakening (one can derive $\varphi \to \psi$ from ψ). There are numerous compelling reasons to omit these conditions. Proof-theoretically, the rejection of weakening is the mechanism used to avoid fallacies of relevance and enforce meaning-containment within implications [5, 35]. Semantically, dropping integrality is increasingly recognized as a requirement for accurately modelling natural language, as it allows formal systems to capture the unbounded nature of comparative adjectives and gradable predicates without encountering the limitations of standard fuzzy or classical bivalent frameworks [24, 86, 77].

Another logic, very important for this work, is Abelian logic. In contrast with Łukasiewicz logic, Abelian logic is a contraclassical, paraconsistent logic. This logic was independently introduced by Casari [18] and Meyer and Slaney [81]. The contraclassicality of Abelian logic arises from the properties of the implication, namely from the relevance axiom

$$((\varphi \to \psi) \to \psi) \to \varphi.$$

The primary motivation for investigating Abelian logic lies in its equivalent algebraic semantics: the variety of Abelian lattice-ordered groups (ℓ-groups). Unlike the algebraic semantics of many non-classical logics, which are often synthesized specifically to model a given calculus, Abelian ℓ-groups possess a remarkably well-behaved and mathematically intuitive structure. Furthermore, they represent a deeply established class within abstract algebra, having been extensively studied independently of their applications in logical systems [10, 6, 32].

While Abelian logic was originally introduced as a playground for relevance logicians, it is closely related to Łukasiewicz logic. Specifically, the negative cone of an Abelian ℓ-group, when truncated by any strictly negative element, yields an MV-algebra. Conversely, given an MV-algebra, one can use the construction of good sequences to reconstruct an Abelian ℓ-group that contains the MV-algebra as this truncated interval. This structural relationship is fully formalized by Mundici's theorem, which establishes a categorical equivalence between the category of Abelian ℓ-groups with a strong unit and the category of MV-algebras [21].

These results strongly support using Abelian logic as a starting point for investigating non-integral generalizations of Łukasiewicz logic, motivating our definition of superabelian logics (see Chapter 1).

However, a critical difference arises in whether the constants for truth ($\mathtt{t}$) and falsity ($\mathtt{f}$) coincide. In Łukasiewicz logic, $\mathtt{t}$ and $\mathtt{f}$ are strictly distinct. By defining negation as:

$$\neg\varphi := \varphi \to \mathtt{f},$$

we ensure that the negation of truth is strictly false ($\neg\mathtt{t} = \mathtt{f}$), because $\mathtt{f}$ is evaluated strictly below $\mathtt{t}$. In contrast, Abelian logic features only a single constant ($\mathtt{t}$). While we can apply the exact same definition for negation, the lack of a distinct second constant forces $\mathtt{f}$ to coincide with $\mathtt{t}$, effectively reducing the definition to:

$$\neg\varphi := \varphi \to \mathtt{t}.$$

Consequently, $\neg\mathtt{t} = \mathtt{t} \to \mathtt{t} = \mathtt{t}$ becomes a valid formula, forcing truth to be its own negation. Thus, Abelian logic cannot express the negation of Łukasiewicz logic, which strictly maps truth to falsity.

To resolve this limitation, we expand the language of Abelian logic by introducing an additional constant. This allows us to define a negation that corresponds exactly to the negation in Łukasiewicz logic, resulting in unbounded Łukasiewicz logic.

The content of this thesis

The first chapter follows the paper "Superabelian Logics", which was recently published in *The Review of Symbolic Logic* [27]. In this work, we define superabelian logics as weakly implicative logics expanding Abelian logic. We develop a general framework for these logics, focusing on the conditions for semilinearity. We then introduce several prominent infinitary extensions of Abelian logic and prove that there are, in total, 2^{2^ω} such infinitary extensions. Subsequently, we provide the definition and completeness theorem for unbounded Łukasiewicz logic, together with its infinitary version. We conclude the chapter by discussing other possible non-integral generalizations of Łukasiewicz logic.

The second chapter is based on the currently submitted preprint "Subvarieties of Pointed Abelian ℓ-groups" [61]. While this chapter is written in the language of universal algebra, the algebraic classes investigated herein correspond to unbounded versions of various extensions of Łukasiewicz logic. We first fully classify the lattice of such subvarieties, achieving a structural parallel to Komori's characterization of MV-algebras, and provide their explicit equational axiomatizations. We conclude the chapter by characterizing the subquasivarieties generated by chains, a result that corresponds logically to categorizing the semilinear finitary extensions of pointed Abelian logic.

The third chapter is based on a conference paper “Satisfiability in Łukasiewicz Logic and Its Unbounded Relative” published in the proceedings of *34th EACSL Annual Conference on Computer Science Logic* [56]. In this paper, we investigate the computational complexity of unbounded Łukasiewicz logic. Specifically, we demonstrate that the existential theory of its standard real-valued semantics, the additive ℓ-group on the reals expanded with a distinguished element -1, is NP-complete. This result is obtained via a polynomial-time reduction to the existential theory of the standard MV-algebra on the reals, establishing a complexity upper bound for the satisfiability problem and the finite consequence relation of the logic.

The fourth chapter is based on a paper “Pointed Modal Abelian Logic, Algebraically” accepted to the proceedings of *Advances in Modal Logic 2026* [62]. In this work, we investigate a modal extension of unbounded Łukasiewicz logic via real-valued Kripke semantics, evaluating formulas over its standard model ($\mathbb{R}_{-1}$). To capture this system algebraically, we introduce the variety of negatively pointed modal Abelian ℓ-groups. We conclude this chapter by establishing an infinitary algebraic completeness theorem that fully characterizes the valid formulas of this modal logic.

1 Superabelian logics

1.1 Introduction

Robert Meyer and John Slaney, at a meeting of the Australasian Association for Logic of 1979,[1] presented a new logic system, *Abelian logic*, as a particular relevant logic in a language consisting of binary connectives $\to$, &, $\vee$, and $\wedge$ and the truth constant $\mathtt{t}$.[2] Special features of Abelian logic are (1) the *axiom of relativity* $((\varphi \to \psi) \to \psi) \to \varphi$, a generalization of the double negation elimination law in which any formula ψ can take the place of falsity (2) its equivalent algebraic semantics, the variety of Abelian lattice-ordered groups (Abelian ℓ-groups for short; see the next section for details).

While one needs to work with all Abelian ℓ-groups to get a sound and complete semantics for the full consequence relation of Abelian logic, if we only want to speak about its tautologies (or even the consequence relation restricted to finite sets of premises), it suffices to restrict to any given non-trivial Abelian ℓ-group, in particular the usual additive group of the real numbers.

The tradition of endowing logical systems with an algebraic semantics based on algebras over real numbers is much older. Indeed, Łukasiewicz logic (which can be defined in the language of Abelian logic expanded by a truth constant $\mathtt{f}$ for falsum) in its infinitely-valued version was introduced in 1930 by Łukasiewicz and Tarski [71] and since then it has proved to be one of the most prominent members of the family of many-valued logics often used to model some aspects of vagueness. Also, it has deep connections with other areas of mathematics such as continuous model theory, error-correcting codes, geometry, algebraic probability theory, etc. [21, 44, 68, 84].

Let us now introduce the real-valued semantics of these two logics side-by-side, to see their close relationship.

	Abelian logic	Łukasiewicz logic
truth-values	$\mathbb{R}$	$[0,1]$
truth-definition	$e(\varphi) \geq 0$	$e(\varphi) = 1$
$e(\varphi \to \psi)$	$e(\psi) - e(\varphi)$	$\min\{1, 1 + e(\psi) - e(\varphi)\}$
$e(\varphi \,\&\, \psi)$	$e(\varphi) + e(\psi)$	$\max\{0, e(\varphi) + e(\psi) - 1\}$
$e(\varphi \wedge \psi)$	$\min\{e(\varphi), e(\psi)\}$	$\min\{e(\varphi), e(\psi)\}$
$e(\varphi \vee \psi)$	$\max\{e(\varphi), e(\psi)\}$	$\max\{e(\varphi), e(\psi)\}$
$e(\mathtt{t})$	0	1
$e(\mathtt{f})$	-	0

We say that a formula φ is a tautology of any of these two logics, if the truth definition holds for all evaluations (we get to consequence relations later). The tautologies of Łukasiewicz logic form a proper subset of the classical tautologies (for example, the evaluation $e(p) = 0.5$ shows that the classical tautology $p \vee (p \to \mathtt{f})$ is

[1]Their work was only published ten years later in [81], the same year in which Ettore Casari also published an independent discovery of the same logic motivated by the study of comparative adjectives in natural language [18].

[2]Note that the original paper uses two truth constants $\mathtt{t}$ and $\mathtt{f}$ which, however, were assumed to coincide. Later we give convincing reasons for choosing otherwise here.

not a tautology of Łukasiewicz logic). In contrast, Abelian logic is contraclassical because, for instance, the formula $((p \to q) \to q) \to p$ is a tautology (while in classical logic we can take the evaluation e such that $e(p) = 0$ and $e(q) = 1$, which makes it false). Observe also that Abelian logic is consistent: for instance, the evaluation $e(p) = e(q) = 1$ shows that $q \to (p \to q)$ is not a tautology.

Recently, a variation of Łukasiewicz logic, that we will call here *Łukasiewicz unbound logic*, has been introduced in [24], with philosophical and linguistic motivations related to a finer analysis of reasoning with vagueness and graded predicates. This logic shares the language with Łukasiewicz logic and so we can again present their semantics side-by-side for comparison (note that the semantics of the unbound logic uses all real numbers as truth degrees and hence it allows us to avoid "truncations" in the definition of evaluations):

	Łukasiewicz unbound logic	Łukasiewicz logic
truth-values	$\mathbb{R}$	$[0, 1]$
truth-definition	$e(\varphi) \geq 1$	$e(\varphi) = 1$
$e(\varphi \to \psi)$	$1 + e(\psi) - e(\varphi)$	$\min\{1, 1 + e(\psi) - e(\varphi)\}$
$e(\varphi \mathbin{\&} \psi)$	$e(\varphi) + e(\psi) - 1$	$\max\{0, e(\varphi) + e(\psi) - 1\}$
$e(\varphi \wedge \psi)$	$\min\{e(\varphi), e(\psi)\}$	$\min\{e(\varphi), e(\psi)\}$
$e(\varphi \vee \psi)$	$\max\{e(\varphi), e(\psi)\}$	$\max\{e(\varphi), e(\psi)\}$
$e(\mathtt{t})$	1	1
$e(\mathtt{f})$	0	0

The first starting point of this paper is the observation that Łukasiewicz unbound logic can be seen as an expansion of Abelian logic (in the language expanded with $\mathtt{f}$): indeed we only need to use the isomorphism $f(x) = x - 1$ to obtain a "shifted" version of the semantics of Łukasiewicz unbound logic:

	Abelian logic	Łukasiewicz unbound logic, shifted
truth-values	$\mathbb{R}$	$\mathbb{R}$
truth-definition	$e(\varphi) \geq 0$	$e(\varphi) \geq 0$
$e(\varphi \to \psi)$	$e(\psi) - e(\varphi)$	$e(\psi) - e(\varphi)$
$e(\varphi \mathbin{\&} \psi)$	$e(\varphi) + e(\psi)$	$e(\varphi) + e(\psi)$
$e(\varphi \wedge \psi)$	$\min\{e(\varphi), e(\psi)\}$	$\min\{e(\varphi), e(\psi)\}$
$e(\varphi \vee \psi)$	$\max\{e(\varphi), e(\psi)\}$	$\max\{e(\varphi), e(\psi)\}$
$e(\mathtt{t})$	0	0
$e(\mathtt{f})$	-	-1

It is trivial to see that tautologies given by the original and the "shifted" semantics coincide and thus any tautology of Abelian logic is also a tautology of Łukasiewicz unbound logic (its relation to Łukasiewicz logic is more complex and is explored in Theorem 1.44).[3] This observation induces a rich family of logics *expanding* Abelian logic in the language with just one additional truth constant.

The second starting point of the paper is the observation that, despite the fact that Abelian logic has no *finitary* consistent extensions in the original language, it has a vast family of *infinitary* extensions.

[3]Note that this is the reason why we do not allow $\mathtt{f}$ to be present in the language of Abelian logic and coincide with $\mathtt{t}$ as this would make $\mathtt{t} \to \mathtt{f}$ one of its tautologies which is clearly not a tautology of Łukasiewicz unbound logic.

Thus, the main goal of this article is twofold: (1) clarify the exact relations between Abelian, Łukasiewicz and Łukasiewicz unbound logics and (2) study two prominent families of expansions of Abelian logic: infinitary extensions in the original language and expansions in the language with `f`.

The paper is organized as follows. After this introduction, in Section 1.2 we formally introduce Abelian logic Ab and the theoretical framework to study its expansions. We choose the framework of (abstract) algebraic logic (see e.g. [42, 41]) to develop our study. More precisely, since Ab is a weakly implicative logic in the sense of [23], we capitalize mostly on the kind of abstract algebraic logic presented in the monograph [28], narrowing down its results and reformulating them in a convenient way for our purposes. We also need to employ some tools of universal algebra which can be found e.g. in the monographs [10, 15, 45, 49, 74]. In particular, in this approach, we have that axiomatic extensions correspond to subvarieties of algebras, finitary extensions correspond to subquasivarieties, and infinitary extensions to generalized subquasivarieties. Since, as mentioned above, every non-trivial Abelian ℓ-group generates the whole class as a quasivariety, it turns out that there are no non-trivial subquasivarieties, and hence the only finitary extension of Ab is the inconsistent logic (i.e. the logic in which all formulas are tautologies). While this section is thus mostly preliminary, it also presents a new, purely syntactic proof of the semilinearity of Abelian logic in Theorem 1.12, which, to the best of our knowledge, has not appeared in this form before.

In contrast, the study of its infinitary extensions is much richer, for which in Section 1.3 we just scratch the surface. In particular, we give an infinitary rule that axiomatizes the extension of Ab corresponding to the generalized quasivariety generated by $\boldsymbol{R}$. Moreover, we provide 2^{ω} semilinear infinitary extensions of Ab, with corresponding completeness theorems, showing $\models_{\boldsymbol{R}} \subsetneq \models_{\boldsymbol{Q}}$ and that there are $2^{2^{\omega}}$ infinitary extensions of Ab. At the end of the section, we discuss a possible axiomatization of the logic generated by $\boldsymbol{Z}$.

After that, in Section 1.4, we introduce the pointed Abelian logic pAb, which gives rise to a dramatically different landscape, since here sub(quasi)varieties of algebras are plentiful. In particular, we prove that pAb is finitely strongly complete with respect to $\{\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1\}$.

In Section 1.5 we study the Łukasiewicz unbound logics Lu and Lu_∞, we axiomatize them as extensions of pAb, and we show there is a translation of Łukasiewicz logic into these logics. Following this, we generalize the methods to axiomatize and provide completeness results for a family of other prominent extensions of pAb, which are similar to Lu. We conclude the paper with Section 1.6 discussing possible avenues of further research.

1.2 Abelian logic and its expansions

Let $Fm_{\mathcal{L}}$ be the set of all formulas built from a propositional language $\mathcal{L}$ and a countable set of variables, and let $\boldsymbol{Fm}_{\mathcal{L}}$ be the absolutely free algebra of type $\mathcal{L}$ defined on $Fm_{\mathcal{L}}$.

Definition 1.1. *We say that a relation* L *between sets of formulas and formulas in a language $\mathcal{L}$ is a* logic *in $\mathcal{L}$ when it satisfies the following conditions for each $\Gamma \cup \Delta \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$:*

- $\{\varphi\} \vdash_{\mathrm{L}} \varphi$. *(Reflexivity)*
- *If* $\Gamma \vdash_{\mathrm{L}} \varphi$ *and* $\Gamma \subseteq \Delta$, *then* $\Delta \vdash_{\mathrm{L}} \varphi$. *(Monotonicity)*
- *If* $\Delta \vdash_{\mathrm{L}} \psi$ *for each* $\psi \in \Gamma$ *and* $\Gamma \vdash_{\mathrm{L}} \varphi$, *then* $\Delta \vdash_{\mathrm{L}} \varphi$. *(Cut)*
- *If* $\Gamma \vdash_{\mathrm{L}} \varphi$, *then* $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ *for each substitution* σ, *i.e. endomorphisms on* $\boldsymbol{Fm}_{\mathcal{L}}$. *(Structurality)*

A logic L *is said to be* finitary *if furthermore*

- *If* $\Gamma \vdash_{\mathrm{L}} \varphi$, *then there is a* finite $\Gamma' \subseteq \Gamma$ *such that* $\Gamma' \vdash_{\mathrm{L}} \varphi$. *(Finitarity)*

As a matter of convention we say that L *is* infinitary *if it is not finitary.*

A *consecution* in a language $\mathcal{L}$ is a tuple written as $\Gamma \blacktriangleright \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$; it is said to be *finitary* if Γ is finite and *infinitary* otherwise, we also identify consecutions $\emptyset \blacktriangleright \varphi$ with just the formula φ. Note that any logic can be seen as a set of consecutions and thus whenever $\Gamma \vdash_{\mathrm{L}} \varphi$, we say that the consecution $\Gamma \blacktriangleright \varphi$ is *valid* in L or that L *satisfies* $\Gamma \blacktriangleright \varphi$.

A set of consecutions closed under arbitrary substitutions (i.e. endomorphisms on $\boldsymbol{Fm}_{\mathcal{L}}$, which are extended to consecutions in an obvious way) is called an *axiomatic system*. An element $\Gamma \blacktriangleright \varphi$ of the set is called an *axiom*, a *finitary rule* or an *infinitary rule* depending on the cardinality of Γ. An axiomatic system is called *infinitary* if it contains at least one infinitary rule, otherwise it is called *finitary*.

Given an axiomatic system $\mathcal{AS}$ we write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in $\mathcal{AS}$. Note that, as we allow infinitary rules, our notion of proof cannot be the usual finite sequence of formulas, but it has to be a tree with no infinite branches (for details see [28]). We say that $\mathcal{AS}$ is a presentation (an axiomatic system) of logic L if $\mathrm{L} = \vdash_{\mathcal{AS}}$. Consequently, we can trivially observe that every (finitary) logic possesses a (finitary) presentation.

In this paper we deal with several logics in different languages with rather intricate interrelations. Therefore, we need to introduce the formal notions of extension and expansion.

Definition 1.2. *Let* $\mathcal{L}_1 \subseteq \mathcal{L}_2$ *be two languages and* L_i *a logic in* $\mathcal{L}_i$ *for* $i \in \{1, 2\}$. *We say that* L_2 *is*

- *the* expansion of L_1 by a set of consecutions $\mathcal{S}$ *in* $\mathcal{L}_2$ *if it is the least logic in* $\mathcal{L}_2$ *containing* L_1 *and* $\mathcal{S}$ *(i.e.* L_2 *is axiomatized by all substitutional instances in* $\mathcal{L}_2$ *of consecutions from* $\mathcal{S} \cup \mathcal{AS}$, *for any presentation* $\mathcal{AS}$ *of* L_1*). We denote* L_2 *by* $\mathrm{L}_1 + \mathcal{S}$.

- *an (axiomatic)* expansion of L_1 *if it is the expansion of* L_1 *by some set of* consecutions *(resp. axioms).*

- *an* extension of L_1 *if* L_2 *is an expansion of* L_1 *and* $\mathcal{L}_1 = \mathcal{L}_2$.

(sf)	$(\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi))$	suffixing
(e)	$(\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$	exchange
(id)	$\varphi \to \varphi$	identity
(rel)	$((\varphi \to \psi) \to \psi) \to \varphi$	relativity axiom
(ub_1)	$\varphi \to \varphi \vee \psi$	upper bound
(ub_2)	$\psi \to \varphi \vee \psi$	upper bound
(lb_1)	$\varphi \wedge \psi \to \varphi$	lower bound
(lb_2)	$\varphi \wedge \psi \to \psi$	lower bound
(sup)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$	supremality
(inf)	$(\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$	infimality
(push)	$\varphi \to (\mathtt{t} \to \varphi)$	push
(pop)	$(\mathtt{t} \to \varphi) \to \varphi$	pop
(res_1)	$(\varphi \to (\psi \to \xi)) \to (\varphi \,\&\, \psi \to \xi)$	residuation
(res_2)	$(\varphi \,\&\, \psi \to \xi) \to (\varphi \to (\psi \to \xi))$	residuation
(MP)	$\varphi, \varphi \to \psi \blacktriangleright \psi$	*modus ponens*
(Adj)	$\varphi, \psi \blacktriangleright \varphi \wedge \psi$	adjunction

Table 1.1 The axiomatic system of Abelian logic

Now we can formally define the *Abelian logic* Ab as the logic in the language[4] $\mathcal{L}_{\mathrm{Ab}} = \{\to, \&, \vee, \wedge, \mathtt{t}\}$ given by the axiomatic system presented in Table 1.1.

The following consecutions are known to be valid in Ab and thus they are valid in all its expansions:

($\mathtt{t}$)	$\mathtt{t}$	
(T)	$\varphi \to \psi, \psi \to \xi \blacktriangleright \varphi \to \xi$	transitivity
($\mathrm{C}_\vee$)	$\varphi \vee \psi \blacktriangleright \psi \vee \varphi$	$\vee$-commutativity
($\mathrm{A}_\vee$)	$(\varphi \vee \psi) \vee \xi \blacktriangleright \varphi \vee (\psi \vee \xi)$	associativity
($\mathrm{I}_\vee$)	$\varphi \vee \varphi \blacktriangleright \varphi$	$\vee$-idempotency
($\mathrm{p}_\vee$)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
($\mathrm{MP}_\vee$)	$\varphi \vee \psi, \varphi \to \psi \blacktriangleright \psi$	

Moreover, one can easily see that Ab is a weakly implicative logic in the sense of [28], i.e., it satisfies (id), (MP), (T) and, for each n-ary c in $\mathcal{L}_{\mathrm{Ab}}$, also

$$(\mathrm{sCNG}_c) \quad \varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{Ab}} c(\xi_1, \dots, \xi_i, \varphi, \dots, \xi_n) \to c(\xi_1, \dots, \xi_i, \psi, \dots, \xi_n) \quad \text{for each } 0 \leq i < n.$$

[4]The literature has equivalent presentations of the logic in other languages, with interdefinable sets of connectives. For example, one can have $\to$ as a primitive connective and use it to define: the constant $\mathtt{t}$ (as $\mathtt{t} := p \to p$), a negation $\neg$ (as $\neg\varphi := \varphi \to \mathtt{t}$), and the conjunction & (as $\varphi \,\&\, \psi := \neg(\varphi \to \neg\psi)$); conversely, from $\neg$ and &, one can retrieve implication as $\varphi \to \psi := \neg(\varphi \,\&\, \neg\psi)$.

Expansions of Ab need not in general satisfy (sCNG$_c$) for connectives added to the language $\mathcal{L}_{\mathrm{Ab}}$. Since we want to focus on weakly implicative logics, we provide the following definition.

Definition 1.3. *A logic* L *in a language* $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{Ab}}$ *is* superabelian *if* L *is an expansion of* Ab *satisfying* (sCNG$_c$) *for each* $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{Ab}}$.

Notice the difference between the notion of superabelian logic and the well known terminology of *superintuitionistic* logics, which are *axiomatic extensions* (instead of *expansions*) of intuitionistic logic. Actually, as we shall soon see, Ab has no non-trivial axiomatic (or even finitary) extension.

There are many natural examples of superabelian logics. In Section 1.3 we will focus on infinitary extensions of Ab and in Sections 1.4 and 1.5 we focus on some extensions of pointed Abelian logic (the least expansion of Abelian logic in the language with the additional constant $\mathtt{f}$). Other natural examples of superabelian logics are the expansions of Ab with connectives corresponding to additional arithmetical operations on real numbers, such as multiplication or division. Other examples would be various types of congruential modal Abelian logics, i.e., logics where:

$$\varphi \to \psi, \psi \to \varphi \blacktriangleright \Box\varphi \to \Box\psi$$

We introduce now the algebraic semantics of Abelian logic and all superabelian logics.

Definition 1.4. *An* Abelian lattice-ordered group *(*Abelian ℓ-group*, see e.g. [43]) is a structure of the form* $\langle A, +, -, \vee, \wedge, 0\rangle$ *such that:*

- $\langle A, +, -, 0\rangle$ *is an Abelian group, i.e. + is a commutative and associative binary operation with unit* 0 *and* $-$ *is the inverse operation (that is, for each* $a \in A$*, we have* $a + (-a) = 0$*),*
- $\langle A, \vee, \wedge\rangle$ *is a lattice, i.e. the binary relation* $\leq$ *defined as* $a \leq b$ *iff* $a \vee b = b$ *iff* $a \wedge b = a$ *turns out to be an order in which, for each* $x, y \in A$*,* $x \vee y$ *and* $x \wedge y$ *are respectively the supremum and the infimum of* $\{x, y\}$*,*
- *it satisfies the monotonicity condition, that is, for each* $a, b, c \in A$*:* $a \leq b$ *implies* $a + c \leq b + c$*.*

It is well known that the defining conditions of Abelian ℓ-groups can be expressed by means of equations, so they form a variety which we denote by $\mathbb{AL}$. Moreover, the lattice reduct of any Abelian ℓ-group is distributive.

In order to match the language that we have chosen to introduce the logic Ab, Abelian ℓ-groups can be equivalently seen as structures $\langle \boldsymbol{A}, +, \to, \vee, \wedge, 0\rangle$ based on the interdefinability: $-a := a \to 0$ and $a \to b := b - a$. To avoid excessive parentheses, we stipulate that the unary operation $-$ has precedence over all binary operations. Natural examples of (linearly ordered) Abelian ℓ-groups are those of the integers $\boldsymbol{Z}$, the rational numbers $\boldsymbol{Q}$, and the real numbers $\boldsymbol{R}$. Note that $a \to b \geq 0$ iff $a \leq b$.

Let $\mathcal{L}$ be any propositional language containing $\mathcal{L}_{\mathrm{Ab}}$ and let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra with an Abelian ℓ-group reduct. An $\boldsymbol{A}$-*evaluation* is a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$. A consecution $\Gamma \blacktriangleright \varphi$ is valid in $\boldsymbol{A}$, $\Gamma \models_{\boldsymbol{A}} \varphi$ in symbols, if for every $\boldsymbol{A}$-evaluation e it holds that, whenever $e(\psi) \geq 0$ for every $\psi \in \Gamma$, then also $e(\varphi) \geq 0$.

Given a class $\mathbb{K}$ of $\mathcal{L}$-algebras with an Abelian ℓ-group reduct, we write $\Gamma \models_{\mathbb{K}} \varphi$ if $\Gamma \models_{\boldsymbol{B}} \varphi$ for each $\boldsymbol{B} \in \mathbb{K}$. It is well known that $\models_{\mathbb{K}}$ is a superabelian logic.

Let L be a superabelian logic in the language $\mathcal{L}$. By $\mathbf{Alg}^*(\mathrm{L})$ we denote the class of L-*algebras,* i.e., $\mathcal{L}$-algebras $\boldsymbol{A}$ which have an Abelian ℓ-group reduct and $\mathrm{L} \subseteq \models_{\boldsymbol{A}}$. For the purposes of this paper, we sometimes write that $\boldsymbol{A}$ is a *model* of L instead of saying that $\boldsymbol{A}$ is an L-algebra.

The usual techniques of algebraic logic allow us to prove that $\mathbf{Alg}^*(\mathrm{Ab}) = \mathbb{AL}$ and, for any superabelian logic L, we have the following completeness theorem: for each $\Gamma \cup \{\varphi\}$, we have

$$\Gamma \vdash_{\mathrm{L}} \varphi \qquad \text{iff} \qquad \Gamma \models_{\mathbf{Alg}^*(\mathrm{L})} \varphi.$$

Furthermore, one can easily check that the following consecutions are valid in Ab:

$$\varphi \blacktriangleright \mathtt{t} \wedge \varphi \to \mathtt{t}$$

$$\varphi \blacktriangleright \mathtt{t} \to \mathtt{t} \wedge \varphi$$

$$\{\mathtt{t} \wedge \varphi \to \mathtt{t}, \mathtt{t} \to \mathtt{t} \wedge \varphi\} \blacktriangleright \varphi$$

Taken together, these consecutions give the condition (Alg) from [28, Theorem 2.9.5], which guarantees that Ab (resp. any superabelian logic L) is algebraically implicative (that is, weakly implicative and algebraizable in the sense of Blok and Pigozzi [13]) and the class $\mathbb{AL}$ (resp. $\mathbf{Alg}^*(\mathrm{L})$) is its equivalent algebraic semantics. Therefore, there is a dual order isomorphism between the lattices of finitary extensions of Ab and subquasivarieties of $\mathbb{AL}$.

We are interested, for a given superabelian logic L, in completeness theorems with respect to various (classes of) L-algebras, instead of the whole equivalent algebraic semantics. For reasons apparent later, we distinguish two kinds of completeness theorems.

Definition 1.5. *A superabelian logic* L *enjoys the* (Finite) Strong $\mathbb{K}$-Completeness *with respect to a class* $\mathbb{K}$ *of* L-*algebras,* (F)S$\mathbb{K}$C *for short, if for every (finite) set* $\Gamma \cup \{\varphi\}$ *of formulas,*

$$\Gamma \vdash_{\mathrm{L}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbb{K}} \varphi.$$

A particularly interesting class of L-algebras is that of those whose underlying lattice is linearly ordered; following the common notation in the literature (see [28]), we denote this class as $\mathbf{Alg}^{\ell}(\mathrm{L})$ and introduce the following central notion.

Definition 1.6. *A superabelian logic* L *is* semilinear *if it has the Strong* $\mathbf{Alg}^{\ell}(\mathrm{L})$-*Completeness.*

As we will see later in Example 1.5, not all superabelian logics have to be semilinear. There is a very useful characterization of semilinearity among superabelian logics with a countable presentation (which is, of course, the case for Ab, all finitary superabelian logics and, as we will see later, even some prominent infinitary ones) using the notion of a strong disjunction (see [28, Definition 5.1.2] and note that the definition also requires the validity of $\varphi \blacktriangleright \varphi \vee \psi$ and $\psi \blacktriangleright \varphi \vee \psi$, which is clearly the case in any superabelian logic).

Definition 1.7. *The lattice connective* $\vee$ *is a* strong disjunction *in a superabelian logic* L *if for each set* $\Gamma \cup \Phi \cup \Psi \cup \{\xi\}$ *of formulas,*

$$\frac{\Gamma, \Phi \vdash_{\mathrm{L}} \xi \qquad \Gamma, \Psi \vdash_{\mathrm{L}} \xi}{\Gamma, \Phi \vee \Psi \vdash_{\mathrm{L}} \xi},$$

where $\Phi \vee \Psi$ *is short for* $\{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\}$.

The next theorem follows directly from [28, Theorems 6.2.2 and 5.5.14].

Theorem 1.8. *Let* L *be a superabelian logic with a countable presentation. Then,* L *is semilinear iff* $\vee$ *is a strong disjunction.*

The power of the theorem above is that there is an easy syntactical way (see Corollary 1.13) to check whether $\vee$ is a strong disjunction using a simple syntactical construction.

Definition 1.9. *Let* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$. *The* $\vee$-form *of a consecution* $\Gamma \blacktriangleright \varphi$ *is the set of the consecutions*

$$(\Gamma \blacktriangleright \varphi)^{\vee} = \{\Gamma \vee \psi \blacktriangleright \varphi \vee \psi \mid \psi \in Fm_{\mathcal{L}}\}.$$

Lemma 1.10. *Let* L *be a superabelian logic,* $\boldsymbol{A}$ *an* L*-algebra, and* $\Gamma \cup \{\varphi\}$ *a set of formulas. If we have* $\Gamma \vee \psi \vDash_{\boldsymbol{A}} \varphi \vee \psi$ *for each* ψ*, then we also have* $\Gamma \vDash_{\boldsymbol{A}} \varphi$*. Moreover, if* $\boldsymbol{A} \in \mathbf{Alg}^{\ell}(\mathrm{L})$*, the converse implication also holds.*

Proof. Take an evaluation e such that, for each $\xi \in \Gamma$, we have $e(\xi) \geq 0$. Then clearly $e(\xi \vee \varphi) \geq 0$ ($\boldsymbol{A}$ has a lattice reduct). Thus by our assumption for $\psi = \varphi$, we know that $e(\varphi) = e(\varphi \vee \varphi) \geq 0$.

For the converse implication, take an evaluation e such that $e(\xi \vee \psi) \geq 0$ for each $\xi \in \Gamma$. Since we assume that $\boldsymbol{A}$ is linearly ordered, we can derive that either $e(\psi) \geq 0$ or, for all $\xi \in \Gamma$, we have $e(\xi) \geq 0$. In the latter case, the assumption $\Gamma \vDash_{\boldsymbol{A}} \varphi$ gives us $e(\varphi) \geq 0$ and so in both cases we have $e(\varphi \vee \psi) \geq 0$. $\square$

The following theorem is a particular version of [28, Theorem 5.2.6] (clearly, the consecutions ($\mathrm{I}_{\vee}$) and ($\mathrm{C}_{\vee}$) mentioned in that theorem are valid in each superabelian logic).

Theorem 1.11. *Let* L *be a superabelian logic with a presentation* $\mathcal{AS}$*. Then, the following are equivalent:*

1. $\vee$ *is a strong disjunction.*
2. $R^{\vee} \subseteq \mathrm{L}$ *for each consecution* $R \in \mathcal{AS}$.
3. $R^{\vee} \subseteq \mathrm{L}$ *for each consecution* $R \in \mathrm{L}$.

Now we are ready to give the promised direct alternative proof of semilinearity of Ab, which originally was established using a completeness theorem (see e.g. [28, Example 6.2.3]).

Theorem 1.12. Ab *is a semilinear logic with strong disjunction* $\vee$.

Proof. As Ab has a countable presentation, it suffices to prove that $\vee$ is a strong disjunction which, thanks to Theorem 1.11, follows if we prove that $(\mathrm{Adj})^{\vee}$ and $(\mathrm{MP})^{\vee}$ are valid in Ab.

For the first one, let $\boldsymbol{A}$ be an Abelian ℓ-group and e an $\boldsymbol{A}$-evaluation such that $e(\varphi \vee \xi) \geq 0$ and $e(\psi \vee \xi) \geq 0$. Then, by the assumptions and distributivity of the lattice operations, we have:

$$e((\varphi \wedge \psi) \vee \xi) = e((\varphi \vee \xi) \wedge (\psi \vee \xi)) = e(\varphi \vee \xi) \wedge e(\psi \vee \xi) \geq 0.$$

To prove the validity of $(\mathrm{MP})^{\vee}$, consider an evaluation e such that $0 \leq e(\varphi \vee \xi)$ and $0 \leq e((\varphi \to \psi) \vee \xi)$. This gives us $0 = e(\varphi \vee \xi) \wedge 0 = e((\varphi \to \psi) \vee \xi) \wedge 0$. Using the distributivity of the lattice operations, we obtain

$$0 = e((\varphi \wedge \mathtt{t}) \vee (\xi \wedge \mathtt{t})) = e(((\varphi \to \psi) \wedge \mathtt{t}) \vee (\xi \wedge \mathtt{t})).$$

For simplicity, let us set $e(\varphi \wedge \mathtt{t}) = a$, $e(\psi) = b$, $e(\xi \wedge \mathtt{t}) = c$ and $e((\varphi \to \psi) \wedge \mathtt{t}) = d$. As in ℓ-groups it holds $(a \vee b) + c = (a + c) \vee (b + c)$, we obtain

$$0 = (a \vee c) + (d \vee c) = (a + d) \vee (a + c) \vee (c + d) \vee (2c).$$

Note that, due to monotonicity of addition, we have

$$\begin{aligned} a + d &\leq e(\varphi) + (e(\varphi) \to e(\psi)) = b \\ (a + c) \vee (c + d) \vee (2c) &\leq (0 + c) \vee (c + 0) \vee (c + 0) \leq c. \end{aligned}$$

Therefore, we obtain $0 \leq b \vee c = e(\psi) \vee e(\xi \wedge \mathtt{t}) \leq e(\psi \vee \xi)$ as required. ☐

The following corollary is a direct consequence of Theorems 1.12, 1.8, and 1.11.

Corollary 1.13. *Let* L *be a superabelian logic expanding* Ab *by a set $\mathcal{S}$ of consecutions such that $R^{\vee}$ is valid in* L *for each $R \in \mathcal{S}$. Then, $\vee$ is a strong disjunction in* L. *If, furthermore,* L *has a countable presentation, then* L *is semilinear.*

In order to obtain completeness results for the logics studied in this paper, we will make use of the following two theorems and corollary which offer algebraic characterizations adapted from the general framework of the monograph [28] to the setting of superabelian logics. For clarity, we first recall the definitions of the standard class operators from universal algebra.

Definition 1.14. *Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras. We define the following:*

- $\mathbf{I}(\mathbb{K})$, $\mathbf{S}(\mathbb{K})$, $\mathbf{H}(\mathbb{K})$, $\mathbf{P}(\mathbb{K})$, $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$, *and* $\mathbf{P}_{\omega}(\mathbb{K})$ *are the classes of all isomorphic images, subalgebras, homomorphic images, direct products, ultraproducts, and countably-filtered products of algebras in $\mathbb{K}$, respectively.*
- $\mathbb{K}_{\mathrm{RFSI}}$ *is the class of relatively finitely subdirectly irreducible members of $\mathbb{K}$.*
- *We use $\boldsymbol{1}$ to denote an arbitrary fixed trivial one-element algebra.*

Since superabelian logics are algebraically implicative, we can recall the following two useful characterizations of (finite) strong completeness proved as [28, Corollary 3.8.3 and Corollary 3.8.7] (in the latter case we also use the fact that for semilinear logics $\mathbf{Alg}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{Alg}^{\ell}(\mathrm{L})$, see [28, Theorem 6.1.7]).

Theorem 1.15. *Let* L *be a superabelian logic and* $\mathbb{K} \subseteq \mathbf{Alg}^*(\mathrm{L})$. *Then, the following are equivalent:*

1. L *is strongly complete w.r.t.* $\mathbb{K}$.
2. $\mathbf{Alg}^*(\mathrm{L})$ *is generated by* $\mathbb{K}$ *as a generalized quasivariety, i.e.* $\mathbf{Alg}^*(\mathrm{L}) = \mathbf{ISP}_\omega(\mathbb{K})$.

Theorem 1.16. *Let* L *be a superabelian logic and* $\mathbb{K} \subseteq \mathbf{Alg}^*(\mathrm{L})$. *Then, the following are equivalent:*

1. L *is finitely strongly complete w.r.t.* $\mathbb{K}$.
2. $\mathbf{Alg}^*(\mathrm{L})$ *is generated by* $\mathbb{K}$ *as a quasivariety, i.e.* $\mathbf{Alg}^*(\mathrm{L}) = \mathbf{ISPP}_{\mathrm{U}}(\mathbb{K})$.

If furthermore L *is finitary and semilinear, then we can add*

3. $\mathbf{Alg}^\ell(\mathrm{L}) \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K}, \boldsymbol{1})$.

Finally, we add a simple yet important corollary of the previous which follows from the fact that $\mathbf{ISP}_{\mathrm{U}}(\mathbb{K}_1, \mathbb{K}_2, \dots, \mathbb{K}_n) = \bigcup_{i \leq n} \mathbf{ISP}_{\mathrm{U}}(\mathbb{K}_i)$ [9, Theorem 5.6].

Corollary 1.17. *Let* L *be a finitary semilinear superabelian logic. Consider finitely many classes of* L*-algebras* $\mathbb{K}_1, \mathbb{K}_2, \dots, \mathbb{K}_n \subseteq \mathbf{Alg}^*(\mathrm{L})$. *Then, the following are equivalent:*

1. L *is finitely strongly complete w.r.t.* $\bigcup_{i=1}^n \mathbb{K}_i$.
2. $\mathbf{Alg}^\ell(\mathrm{L}) \subseteq \mathbf{I}(\boldsymbol{1}) \cup \bigcup_{i \leq n} \mathbf{ISP}_{\mathrm{U}}(\mathbb{K}_i)$.

Thanks to [64], we know the quasivariety of Abelian ℓ-groups is generated (as a quasivariety) by $\boldsymbol{Z}$ and, since $\boldsymbol{Z}$ can be embedded into any non-trivial Abelian ℓ-group, we obtain the following corollary.

Corollary 1.18. *The variety* $\mathbb{AL}$ *of Abelian* ℓ*-groups is generated (as a quasivariety) by any of its non-trivial members, that is,* $\mathbf{ISPP}_{\mathrm{U}}(\boldsymbol{A}) = \mathbb{AL}$ *for any non-trivial* $\boldsymbol{A} \in \mathbb{AL}$. *Therefore,* $\mathbb{AL}$ *has no non-trivial subquasivarieties.*

As we will see in the next section, $\mathbb{AL}$ is not generated as a *generalized* quasivariety by any of $\boldsymbol{R}$, $\boldsymbol{Q}$, or $\boldsymbol{Z}$, and it admits numerous non-trivial generalized subquasivarieties.

In logical terminology, we know that Ab is finitely strongly complete w.r.t. any non-trivial Abelian ℓ-group and has no consistent finitary extension, though it has numerous proper consistent infinitary extensions. Moreover, it is not strongly complete w.r.t. any of $\boldsymbol{R}$, $\boldsymbol{Q}$ or $\boldsymbol{Z}$. In the next section, we will axiomatize its proper extension strongly complete w.r.t. $\boldsymbol{R}$.

1.3 Infinitary extensions of Abelian logic

Having established that Abelian logic (Ab) lacks non-trivial finitary extensions, a well-known result that motivates our next step, we now turn our attention to the much richer landscape of its infinitary extensions. This section explores this domain by introducing infinitary rules designed to capture specific semantic properties.

Clearly, the most prominent candidates for such infinitary logics are $\vDash_{\boldsymbol{R}}$, $\vDash_{\boldsymbol{Q}}$, and $\vDash_{\boldsymbol{Z}}$. We know that:

$$\mathrm{Ab} \subseteq \vDash_{\boldsymbol{R}} \subseteq \vDash_{\boldsymbol{Q}} \subseteq \vDash_{\boldsymbol{Z}}$$

and that above $\vDash_{\boldsymbol{Z}}$ there is only the inconsistent logic (because $\boldsymbol{Z}$ obviously generates the smallest non-trivial generalized subquasivariety of Abelian ℓ-groups).

In this section, we obtain the following new results:

- strictness of all three inclusions, and thus these three logics are indeed not finitary (Example 1.3, Corollary 1.23 and Theorem 1.24)
- there are $2^{2^{\omega}}$ distinct infinitary logics between $\vDash_{\boldsymbol{R}}$ and $\vDash_{\boldsymbol{Q}}$ (Corollary 1.23)
- an axiomatization of the logic $\vDash_{\boldsymbol{R}}$ (Theorem 1.21).

Moreover, we give a conjecture for an axiomatization of the logic $\vDash_{\boldsymbol{Z}}$, give strong arguments to support it, and point to the impossibility of proving it by usual methods (Conjecture 1.25).

From now on, we will use the following notation: for $n \in \mathbb{N}$ and a formula φ, we write $n \cdot \varphi$ instead of $\underbrace{\varphi \,\&\, \cdots \,\&\, \varphi}_{n \text{ times}}$ (we also set $0 \cdot \varphi = \mathtt{t}$). We use the analogous notation for $+$ in the algebraic setting.

Let us first recall the infinitary rule used to axiomatize the infinitary Łukasiewicz logic (see [59]):

$$\{\neg\varphi \to n \cdot \varphi \mid n \in \mathbb{N}\} \blacktriangleright \varphi \tag{Hay}$$

Since the language of Ab does not include a negation with similar properties to that of Łukasiewicz logic, we have to replace $\neg\varphi$ by ψ and define:

$$\{\psi \to n \cdot \varphi \mid n \in \mathbb{N}\} \blacktriangleright \varphi \tag{Arch}$$

As its name suggests, the rule (Arch) is intended as a syntactic counterpart to the well-known algebraic property of being Archimedean. We introduce the standard definition of this property for Abelian ℓ-groups.

Definition 1.19. *Let $\boldsymbol{A}$ be an Abelian ℓ-group. We say $\boldsymbol{A}$ is Archimedean if and only if for each $a, b \in A$ it holds:*

$$(\forall n \in \mathbb{N})\; na \leq b \implies a \leq 0.$$

The following lemma makes the connection between the rule and the property precise, showing that the (Arch) rule is valid in an Abelian ℓ-group if and only if that group is Archimedean.

Lemma 1.20. *Let $\boldsymbol{A}$ be an Abelian ℓ-group. Then, the following conditions are equivalent:*

1. $\boldsymbol{A}$ *is Archimedean.*

2. (Arch) *is valid in* $\boldsymbol{A}$.

Furthermore, if $\boldsymbol{A}$ *is linearly ordered, we can add the following condition:*

3. $(\text{Arch})^{\vee}$ *is valid in* $\boldsymbol{A}$.

Proof. We first show the implication from 1 to 2. Let e be an evaluation such that we have $e(\varphi) \ngeq 0$. We set $a = -e(\varphi)$ and $b = -e(\psi)$. Since $\boldsymbol{A}$ is Archimedean and $a \nleq 0$, there is an $n \in \mathbb{N}$ such that $n \cdot a \nleq b$. Consequently, there is an $n \in \mathbb{N}$ such that $n \cdot e(\varphi) \ngeq e(\psi)$; thus $e(\psi \to n \cdot \varphi) \ngeq 0$. This shows that (Arch) holds.

To show the implication from 2 to 1, we assume that $\boldsymbol{A}$ is not Archimedean. Therefore, there exist $a, b \in A$ such that $a \nleq 0$ and $n \cdot a \leq b$ for each $n \in \mathbb{N}$. Consider an evaluation e such that $e(\varphi) = -a$ and $e(\psi) = -b$. Then, for each $n \in \mathbb{N}$, we have $n \cdot e(\varphi) \geq e(\psi)$ and thus $e(\psi \to n \cdot \varphi) \geq 0$ for each $n \in \mathbb{N}$. However, by assumption, $e(\varphi) = -a \ngeq 0$ and thus (Arch) fails in $\boldsymbol{A}$.

The rest of the statement follows from Lemma 1.10. □

By the next example, we know that (Arch) is not valid in Ab and so Ab is strictly contained in $\models_{\boldsymbol{R}}$ and thus $\models_{\boldsymbol{R}}$ cannot be finitary.

Example. Clearly the algebras $\boldsymbol{R}$, $\boldsymbol{Q}$, and $\boldsymbol{Z}$ are Archimedean. As an example of a non-Archimedean linearly ordered ℓ-group, consider the lexicographic product of two copies of the integers $\boldsymbol{Z} \times_{\text{lex}} \boldsymbol{Z}$, where $\boldsymbol{Z} \times_{\text{lex}} \boldsymbol{Z}$ has the same group structure as $\boldsymbol{Z} \times \boldsymbol{Z}$, with the ordering $\langle a, b\rangle > \langle c, d\rangle$ iff $a > c$ or $a = c$ and $b > d$. This ℓ-group is clearly not Archimedean, since by definition $n \cdot \langle 0, 1\rangle < \langle 1, 0\rangle$ for each $n \in \mathbb{N}$.

Let us call the logic $\models_{\boldsymbol{R}}$, the *real Abelian logic* and denote it rAb. The next theorem shows how to axiomatize it.

Theorem 1.21. rAb *is the extension of* Ab *by the rule* $(\text{Arch})^{\vee}$.

Proof. Let us denote the logic Ab+$(\text{Arch})^{\vee}$ by L. First note that, by Corollary 1.13, we know that L is a semilinear logic and so it is strongly complete with respect to the class $\mathbf{Alg}^{\ell}(\mathrm{L})$, which, due to the previous lemma, consists of linearly ordered Archimedean ℓ-groups. By Hölder's Theorem [60], each such ℓ-group is embeddable into $\boldsymbol{R}$ and so the proof follows. □

Note that this proof relies on the $\vee$-form of the rule (Arch) which is necessary to ensure that the resulting logic is semilinear. Therefore, it remains an open question whether the rule (Arch) itself would suffice on its own to axiomatize rAb, which is equivalent to the question whether $(\text{Arch})^{\vee}$ is valid in Ab + (Arch).

Next, we show that $\models_{\boldsymbol{Q}}$ is strictly stronger than $\models_{\boldsymbol{R}}$ (in other words, rAb is not strongly complete w.r.t. $\boldsymbol{Q}$).

We use the following convention. For an ℓ-group $\boldsymbol{A}$, elements $a, b \in A$ and $\frac{k}{l} \in \mathbb{Q} \setminus \{0\}$ we write $\frac{k}{l} \cdot a \leq b$ or $a \leq \frac{l}{k} \cdot b$ when $k \cdot a \leq l \cdot b$. Similarly, we write $\frac{k}{l} \cdot \varphi \to \psi$ or $\varphi \to \frac{l}{k} \cdot \psi$ instead of $k \cdot \varphi \to l \cdot \psi$.

For each $0 < \gamma \in \boldsymbol{R} \setminus \boldsymbol{Q}$, we introduce the following rule, in which $\langle q_j\rangle_{j \in \mathbb{N}}$ is a strictly increasing sequence of rationals converging to γ and $\langle r_j\rangle_{j \in \mathbb{N}}$ is a strictly decreasing sequence of rationals converging to γ:

$$\{q_j \cdot \varphi \to \psi, \psi \to r_j \cdot \varphi \mid j \in \mathbb{N}\} \blacktriangleright \varphi \to \mathtt{t}. \qquad (\text{str}_{\gamma})$$

The next proposition shows that in our context (str_γ) does not depend on the choice of the sequences.

Proposition 1.22. *Given $0 < \gamma \in \boldsymbol{R} \setminus \boldsymbol{Q}$, a sequence of rationals $\langle q_j \rangle_{j\in\mathbb{N}}$ strictly increasing and converging to γ, and a sequence of rationals $\langle r_j \rangle_{j\in\mathbb{N}}$ strictly decreasing and converging to γ, the logic* $\mathrm{rAb} + (\mathrm{str}_\gamma)^\vee$ *is strongly complete w.r.t.*

$$\mathbb{K} = \{\boldsymbol{A} \subseteq \boldsymbol{R} \mid (\forall a, b \in A)\ (b = \gamma \cdot a \implies a = 0)\}.$$

Moreover, for any $\boldsymbol{A} \in \mathbf{IS}(\boldsymbol{R})$ we have $\boldsymbol{A} \vDash (\mathrm{str}_\gamma)^\vee$ if and only if $\boldsymbol{A} \in \mathbb{K}$.

Proof. Since the logic $\mathrm{rAb} + (\mathrm{str}_\gamma)^\vee$ satisfies the conditions from Corollary 1.13, it is semilinear. Since $\mathrm{rAb} + (\mathrm{str}_\gamma)^\vee$ is an extension of rAb, we obtain that $\mathrm{rAb} + (\mathrm{str}_\gamma)^\vee$ is strongly complete w.r.t. some subclass of $\mathbf{S}(\boldsymbol{R})$.

We need the following technical observation:

Observation. Let $\boldsymbol{A} \subseteq \boldsymbol{R}$ and $a, b \in A$. Then we have:

$$q_j \cdot a \leq b \text{ and } b \leq r_j \cdot a \text{ for each } j \in \mathbb{N} \iff b = \gamma \cdot a \text{ and } a \geq 0.$$

Proof. First, since $q_0 \cdot a \leq b \leq r_0 \cdot a$ and $q_0 < r_0$, we can conclude that $0 \leq a$. We have $\lim q_j = \gamma$ and $\lim r_j = \gamma$. Consequently, $\gamma \cdot a = \lim q_j \cdot a \leq b \leq \lim r_j \cdot a = \gamma \cdot a$. Therefore, we obtain $\gamma \cdot a = b$. The other direction follows as well: if $\gamma \cdot a = b$ and $a \geq 0$, then $q_j \cdot a \leq \gamma \cdot a = b$ for each $j \in \mathbb{N}$ (since $a \geq 0$ and q_j is strictly increasing) and $b = \gamma \cdot a \leq r_j \cdot a$ for each $j \in \mathbb{N}$ (since $a \geq 0$ and r_j is strictly decreasing). △

First, let us take an arbitrary $\boldsymbol{A} \in \mathbb{K}$. We show that $\boldsymbol{A}$ satisfies $(\mathrm{str}_\gamma)^\vee$. By the observation and the definition of $\mathbb{K}$, if, for $a, b \in A$, we have $q_j \cdot a \leq b$ and $b \leq r_j \cdot a$ for each $j \in \mathbb{N}$, then it must be that $a = 0$. Thus, any evaluation in $\boldsymbol{A}$ that satisfies the premise of (str_γ) must map φ to 0, thereby satisfying the conclusion $\varphi \to \mathtt{t}$. Therefore, (str_γ) is valid in $\boldsymbol{A}$. Since $\boldsymbol{A}$ is a chain, by Lemma 1.10 $(\mathrm{str}_\gamma)^\vee$ is also valid in $\boldsymbol{A}$, and thus $\boldsymbol{A}$ is a model of $\mathrm{rAb} + (\mathrm{str}_\gamma)^\vee$.

To prove the other direction, assume for an algebra $\boldsymbol{1} \neq \boldsymbol{A} \subseteq \boldsymbol{R}$ that there exist $a, b \in A$ such that $b = \gamma \cdot a$ and $a \neq 0$. Clearly, $\gamma \cdot (-a) = -b$ holds as well. Since $\boldsymbol{A}$ is totally ordered, we can assume without loss of generality that $a > 0$. By the observation, this implies that for each $j \in \mathbb{N}$, $q_j \cdot a \leq b$ and $b \leq r_j \cdot a$. But since $a, b \in A$ and $a > 0$, this is a counterexample to the validity of rule (str_γ) in $\boldsymbol{A}$. By Lemma 1.10, $(\mathrm{str}_\gamma)^\vee$ fails in $\boldsymbol{A}$ as well. This shows $\boldsymbol{A}$ is not a model of $\mathrm{rAb} + (\mathrm{str}_\gamma)^\vee$. □

Clearly, for $\gamma_1, \gamma_2 \in \boldsymbol{R} \setminus \boldsymbol{Q}$, the rules $(\mathrm{str}_{\gamma_1})^\vee$ and $(\mathrm{str}_{\gamma_2})^\vee$ do not necessarily define different classes of models. For example, one can easily show that $(\mathrm{str}_{\sqrt{2}})^\vee$ is valid in an algebra $\boldsymbol{A}$ if and only if $(\mathrm{str}_{2\sqrt{2}})^\vee$ is valid in $\boldsymbol{A}$. Nevertheless, one can show the following.

Corollary 1.23. *There are 2^{2^ω} infinitary extensions of Abelian logic between $\vDash_{\boldsymbol{R}}$ and $\vDash_{\boldsymbol{Q}}$.*

Proof. It is a well-known result that the transcendence degree of $\boldsymbol{R}$ over $\boldsymbol{Q}$ is 2^ω. From this, it follows that there are 2^{2^ω} distinct subfields of $\boldsymbol{R}$ that are field extensions of $\boldsymbol{Q}$. We argue that the ℓ-group reduct of each such field is a model of a different infinitary extension of Ab. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be the ℓ-group reducts of two distinct subfields of $\boldsymbol{R}$ that are field extensions of $\boldsymbol{Q}$. Without loss of generality, assume that there is a $\gamma \in A \setminus B$ such that $0 < \gamma$. By Proposition 1.22, an ℓ-subgroup of $\boldsymbol{R}$ validates the rule $(\mathrm{str}_\gamma)^\vee$ if and only if $b = \gamma \cdot a \implies a = 0$ for all a, b in the corresponding ℓ-subgroup. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are the ℓ-group reducts of subfields of $\boldsymbol{R}$, this condition holds if and only if γ is not an element of their respective universes. Hence we obtain that the rule $(\mathrm{str}_\gamma)^\vee$ is valid in $\boldsymbol{B}$ but is not valid in $\boldsymbol{A}$. This shows that $\vDash_{\boldsymbol{A}}$ and $\vDash_{\boldsymbol{B}}$ are different logics. $\square$

Note that Proposition 1.22 also hints at a possible axiomatization of the logic $\vDash_{\boldsymbol{Q}}$. Indeed we can easily observe that $\vDash_{\boldsymbol{Q}}$ is axiomatized as $\mathrm{rAb} + \{(\mathrm{str}_\gamma)^\vee \mid \gamma \in \mathbb{R} \setminus \mathbb{Q}\}$ iff this logic is semilinear (which we unfortunately do not know, as Corollary 1.13 works for countably axiomatized logics only). Therefore, the axiomatization of $\vDash_{\boldsymbol{Q}}$ remains as an open problem.

Finally, we show that $\vDash_{\boldsymbol{Z}}$ is strictly stronger than $\vDash_{\boldsymbol{Q}}$. To this end, we propose the following infinitary rule IDC (standing for "infinite decreasing chain"),

$$\{2 \cdot \varphi_{n+1} \to \varphi_n, \varphi_n \to 4 \cdot \varphi_{n+1} \mid n \in \mathbb{N}\} \blacktriangleright \varphi_0 \to \mathtt{t}. \tag{IDC}$$

and prove the following theorem:

Theorem 1.24. *Let $\boldsymbol{A}$ be a non-trivial linearly ordered Archimedean Abelian ℓ-group. Then* (IDC) *is valid in $\boldsymbol{A}$ iff $\boldsymbol{A}$ is isomorphic to $\boldsymbol{Z}$.*

Before we prove this theorem, let us state a conjecture based on this result.[5]

Conjecture 1.25. *The logic $\vDash_{\boldsymbol{Z}}$ is the extension of* Ab *by the rules* $(\mathrm{Arch})^\vee$ *and* $(\mathrm{IDC})^\vee$*.*

Observe that the conjecture holds if and only if $\mathrm{Ab} + (\mathrm{IDC})^\vee + (\mathrm{Arch})^\vee$ is a semilinear logic. Unfortunately, the rule (IDC) uses infinitely many variables and thus, while we know that $\vee$ is a strong disjunction in $\mathrm{Ab} + (\mathrm{IDC})^\vee + (\mathrm{Arch})^\vee$ by Corollary 1.13, we do not know whether this logic is actually a semilinear logic.

This is, however, a systematic problem as *any* rule valid in $\vDash_{\boldsymbol{Z}}$ but not in $\vDash_{\boldsymbol{Q}}$ *has to* use infinitely many variables. Indeed, since any finitely generated subalgebra of $\boldsymbol{Q}$ is isomorphic to $\boldsymbol{Z}$, any rule with finitely many variables valid in $\vDash_{\boldsymbol{Z}}$ must also be valid in $\vDash_{\boldsymbol{Q}}$.

Let us now proceed towards the proof of Theorem 1.24. First, we need to introduce a machinery of decreasing sequences (which justifies the name of the rule).

Definition 1.26. *Let $\boldsymbol{A}$ be an Abelian ℓ-group. A sequence $\langle a_i \rangle_{i \in \mathbb{N}}$ of elements of A is* strongly decreasing *if, for each $i \in \mathbb{N}$, there is an $n_i \geq 4$ such that:*

$$n_i \cdot a_{i+1} \geq a_i \geq 2 \cdot a_{i+1}.$$

We call the sequence trivial *if $a_i = 0$ for each i.*

[5]Note that, by Example 1.3 we know that the rule $(\mathrm{Arch})^\vee$ is not redundant.

Let us note that any non-trivial strongly decreasing sequence in $\boldsymbol{A}$ is indeed decreasing and $a_i \geq 0$ for each $i \geq 1$. The latter claim follows from the fact that for each i there is an $n_i \geq 4$ such that $n_i \cdot a_i \geq 2n_i \cdot a_{i+1} \geq 2 \cdot a_i$, thus $(n_i - 2) \cdot a_i \geq 0$, from which one can derive $a_i \geq 0$ (since in $\boldsymbol{Z}$ for any $n \geq 1$ the quasiequation $n \cdot x \geq 0 \implies x \geq 0$ is valid, it holds in $\boldsymbol{A}$). The former claim then follows from the validity of $x \geq 0 \implies 2 \cdot x \geq x$ in $\boldsymbol{Z}$.

Lemma 1.27. *Let $\langle a_i \rangle_{i\in\mathbb{N}}$ be a non-trivial sequence of elements of an Abelian ℓ-group $\boldsymbol{A}$. If $\langle a_i \rangle_{i\in\mathbb{N}}$ is strongly decreasing, then it is strictly decreasing. If $\boldsymbol{A}$ is an Archimedean chain and it has a strictly decreasing sequence of positive elements, then it also has a non-trivial strongly decreasing sequence.*

Proof. To prove the first claim, it suffices to show the strictness. Assume that $a_i = a_{i+1}$ for some $i \in \mathbb{N}$. Then the inequality $a_i \geq 2 \cdot a_{i+1}$ becomes $a_i \geq 2 \cdot a_i$ and thus $0 \geq a_i$. Since also $a_i \geq 0$, we obtain $a_i = 0$. For each $j > i$ we have $a_i \geq a_j \geq 0$, thus we get $a_j = 0$ as well. Similarly, for $j < i$, there is an $n \in \mathbb{N}$ such that $n \cdot a_i \geq a_j \geq 0$ and thus again $a_j = 0$. Therefore, the sequence $\langle a_i \rangle_{i\in\mathbb{N}}$ has to be trivial, a contradiction.

We will prove the second claim. Let us assume that $\boldsymbol{A}$ is an Archimedean chain and $\langle a_i \rangle_{i\in\mathbb{N}}$ is a strictly decreasing sequence of positive elements. Without loss of generality, as $\boldsymbol{A}$ is Archimedean, we can assume that $\boldsymbol{A} \subseteq \boldsymbol{R}$, by Hölder's Theorem. We construct a non-trivial strongly decreasing sequence $\langle b_i \rangle_{i\in\mathbb{N}}$. Set $b_0 > 0$ arbitrarily. We will define the rest of the sequence recursively. Assume that we have already constructed $\langle b_i \rangle_{i\leq n}$ satisfying the conditions from the definition of a strongly decreasing sequence. Now we want a b_{n+1} such that $0 < 2 \cdot b_{n+1} \leq b_n$.

Since $\langle a_i \rangle_{i\in\mathbb{N}}$ is a strictly decreasing sequence, it follows that there exists a $j \in \mathbb{N}$ such that $\frac{a_j}{b_n} \notin \mathbb{Z}$ (otherwise $\langle \frac{a_j}{b_n} \rangle_{j\in\mathbb{N}}$ would be a strictly decreasing sequence of positive elements in $\boldsymbol{Z}$, which is not possible).

Since $\boldsymbol{A}$ is Archimedean, there is a $k \in \mathbb{N}$ such that $0 \leq a_j - k \cdot b_n < b_n$. Since $\frac{a_j}{b_n} \notin \mathbb{Z}$, we have $0 \neq a_j - k \cdot b_n$ and thus $0 < a_j - k \cdot b_n < b_n$. Hence,

$$0 < a_j - k \cdot b_n \leq \frac{b_n}{2} \quad \text{or} \quad 0 < b_n - (a_j - k \cdot b_n) \leq \frac{b_n}{2}.$$

Therefore, we set

$$b_{n+1} = \min\{a_j - k \cdot b_n, b_n - (a_j - k \cdot b_n)\}.$$

Clearly, $b_{n+1} > 0$ and $2 \cdot b_{n+1} \leq b_n$. The second condition for strong decreasing sequences easily follows from the fact that $\boldsymbol{A}$ is Archimedean. Thus, $\langle b_i \rangle_{i\in\mathbb{N}}$ is a non-trivial strongly decreasing sequence. □

Example. We will show that there are non-Archimedean chains with strictly decreasing sequences of positive elements and with no non-trivial strongly decreasing sequence. Consider the ℓ-group $\boldsymbol{Z} \times_{\text{lex}} \boldsymbol{Z}$ from Example 1.3. First, let us consider the sequence $\langle \langle 1, -i \rangle \rangle_{i\in\mathbb{N}}$. Clearly, this sequence is strictly decreasing and $\langle 1, -i \rangle > 0$ for each $i \in \mathbb{N}$. We show that there is no non-trivial strongly decreasing sequence in $\boldsymbol{Z} \times_{\text{lex}} \boldsymbol{Z}$.

Let us assume that $\langle \langle a_i, b_i \rangle \rangle_{i\in\mathbb{N}}$ is a strongly decreasing sequence. Clearly $a_i \geq 0$ for all $i \in \mathbb{N}$ since $\langle a_i, b_i \rangle \geq 0$. Thus, there has to be a $c \geq 0$ such that $a_i = c$ for infinitely many $i \in \mathbb{N}$. Therefore, there has to be a strongly decreasing subsequence $\langle\langle c, d_i\rangle\rangle_{i\in\mathbb{N}}$, where $d_i \in \mathbb{Z}$ and $c \geq 0$. Fix an arbitrary $i \in \mathbb{N}$. We know that $2 \cdot \langle c, d_{i+1}\rangle \leq \langle c, d_i\rangle$, thus $2 \cdot c \leq c$. This implies $c = 0$.

Since $\langle\langle 0, d_i\rangle\rangle_{i\in\mathbb{N}}$ is a strongly decreasing sequence in $\boldsymbol{Z} \times_{\text{lex}} \boldsymbol{Z}$, the sequence $\langle d_i\rangle$ is strongly decreasing in $\boldsymbol{Z}$. Therefore, $d_i = 0$ for all $i \in \mathbb{N}$. This shows that $\langle\langle c, d_i\rangle\rangle_{i\in\mathbb{N}}$ has to be the trivial sequence and so is $\langle\langle a_i, b_i\rangle\rangle_{i\in\mathbb{N}}$ as well.

Proposition 1.28. *Let $\boldsymbol{A}$ be an Abelian ℓ-group. Then* (IDC) *is valid in $\boldsymbol{A}$ iff only the trivial sequence is strongly decreasing in $\boldsymbol{A}$.*

Proof. First, assume that $\boldsymbol{A}$ has a non-trivial strongly decreasing sequence $\langle a_i\rangle_{i\in\mathbb{N}}$ and note that w.l.o.g. we can assume that $n_i = 4$ for each i (if for some i we have $a_i > 4 \cdot a_{i+1}$, expand the sequence by putting $2 \cdot a_{i+1}$ in a proper place, and repeat ...). Let us consider the $\boldsymbol{A}$-evaluation e induced by setting $e(x_i) = a_i$. Clearly, all the premises of (IDC) are valid but we have $e(x_0 \to \mathtt{t}) = -a_0 < 0$ and thus the rule (IDC) is not valid in $\boldsymbol{A}$.

Conversely, assume that the rule (IDC) is not valid in $\boldsymbol{A}$, i.e., we have an evaluation e such that if we set $e(\varphi_i) = a_i$ the validity of premises of (IDC) tells us that $4 \cdot a_{i+1} \geq a_i \geq 2 \cdot a_{i+1}$, i.e., $\langle a_i\rangle_{i\in\mathbb{N}}$ is a strongly decreasing sequence and, by the failure of the conclusion of the rule, we have $a_0 > 0$, i.e., the sequence is non-trivial. $\square$

The previous proposition gives us a description of Abelian ℓ-groups satisfying (IDC) in terms of the existence of strongly decreasing sequences. Lemma 1.27 shows that there is a connection between strictly decreasing sequences of positive elements and strongly decreasing sequences. Therefore, it is natural to ask whether the previous proposition could be stated in terms of the existence of *strictly* decreasing sequences of positive elements. The following example shows that this is not the case.

Example. There exist ℓ-groups in $\mathbf{P}(\boldsymbol{Z})$ which have strictly decreasing sequences of positive elements but lack a non-trivial strongly decreasing sequence.

Consider the Abelian ℓ-group $\boldsymbol{Z}^\omega$ and note that the sequence of positive elements

$$\langle\langle \underbrace{0, \ldots, 0}_{\text{i-times}}, 1, 1, \ldots\rangle\rangle_{i\in\mathbb{N}}$$

is strictly decreasing. However, the previous proposition clearly states that only the trivial sequence is strongly decreasing in $\boldsymbol{Z}^\omega$, since $\boldsymbol{Z}^\omega \in \mathbf{P}(\boldsymbol{Z})$, $\mathbf{P}(\boldsymbol{Z}) \subseteq \mathbf{ISP}_\omega(\boldsymbol{Z})$, and (IDC) holds in $\mathbf{ISP}_\omega(\boldsymbol{Z})$.

Now we are finally ready to give the promised proof of Theorem 1.24.

Proof of Theorem 1.24. Let $\boldsymbol{A}$ be a non-trivial linearly ordered Archimedean Abelian ℓ-group. We want to show that (IDC) is valid in $\boldsymbol{A}$ if and only if $\boldsymbol{A}$ is isomorphic to $\boldsymbol{Z}$. By the previous proposition, (IDC) is valid in $\boldsymbol{A}$ if and only if $\boldsymbol{A}$ has only the trivial strongly decreasing sequence.

Since $\boldsymbol{A}$ is an Archimedean chain, by Lemma 1.27, having only the trivial strongly decreasing sequence is equivalent to having no strictly decreasing sequence of positive elements.

Thus, it suffices to show that a non-trivial linearly ordered Archimedean ℓ-group $\boldsymbol{A}$ has no strictly decreasing sequence of positive elements if and only if $\boldsymbol{A} \cong \boldsymbol{Z}$.

Clearly, $\boldsymbol{Z}$ has no decreasing sequences of positive elements. To show the other implication, one has to argue that $\boldsymbol{Z}$ is the only non-trivial linearly ordered Archimedean ℓ-group with no strictly decreasing sequences of positive elements. We can assume, by Hölder's Theorem, that without loss of generality, $\boldsymbol{Z} \subseteq \boldsymbol{A} \subseteq \boldsymbol{R}$ and $\boldsymbol{A}$ is not isomorphic to $\boldsymbol{Z}$. We have to distinguish two cases.

1. Let $\zeta \in \mathbb{R} \setminus \mathbb{Q}$ be an element of A. Without loss of generality we can assume $\zeta > 1$. Since $1 \in A$, we can generate recursively the following sequence. We start with $r_0 = \zeta$ and $r_1 = 1$. For $n > 0$, we define $k_n = \lfloor \frac{r_{n-1}}{r_n} \rfloor$ and let $r_{n+1} = r_{n-1} - k_n \cdot r_n$. By the properties of division, $0 \leq r_{n+1} < r_n$ for all $n \geq 1$. This means that the sequence $\langle r_n \rangle_{n \in \mathbb{N}}$ is strictly decreasing. Furthermore, because ζ is irrational, the ratio $\frac{r_n}{r_{n+1}}$ is irrational for each $n \in \mathbb{N}$. This shows that this algorithm never terminates with a zero remainder; thus, $r_n > 0$ for all n.

2. $\boldsymbol{A} \subseteq \boldsymbol{Q}$. Since $\boldsymbol{A}$ is not isomorphic to $\boldsymbol{Z}$, $\boldsymbol{A}$ is not generated by any single element. Thus, for each $0 < a_n \in A$, there is a $0 < b_n \in A$ such that a_n does not generate b_n. If $b_n < a_n$ we set $a_{n+1} = b_n$; otherwise, we set $k_n = \lfloor \frac{b_n}{a_n} \rfloor$ and $a_{n+1} = b_n - k_n \cdot a_n$. Thus, we have constructed $0 < a_{n+1} < a_n$. This allows us to build the desired sequence, and therefore we finish the proof. □

1.4 Pointed Abelian logic

In the previous sections, we have explored the landscape of Abelian logic and its infinitary extensions. We now turn our attention to expansions, that is, we allow not only new rules, but also enriched logical languages. This section introduces *pointed Abelian logic* (pAb), which extends Abelian logic with a new constant symbol `f`. This seemingly simple addition of an additional "point" to the algebraic semantics dramatically alters the character of the logic. Whereas Abelian logic has no non-trivial finitary extensions, we will show that this pointed version gives rise to a plentiful landscape of logical extensions.[6] The central result of this section is a completeness theorem demonstrating that this new logic is finitely strongly complete with respect to just three canonical pointed structures over the real numbers: $\boldsymbol{R}_{-1}, \boldsymbol{R}_0$, and $\boldsymbol{R}_1$.

Let us consider the expansion of the language $\mathcal{L}_{\mathrm{Ab}}$ of Abelian logic by a new constant symbol `f`. We denote this language by $\mathcal{L}_{\mathrm{pAb}}$. In this section, we study the minimal expansion of Ab in this language.

Definition 1.29. *The pointed Abelian logic is the least expansion of* Ab *in the language $\mathcal{L}_{\mathrm{pAb}}$, i.e., it is axiomatized by the same axiomatic system as* Ab *only closed under substitution of the expanded language.*

Recall that Full Lambek logic with exchange, $\mathrm{FL_e}$, is a prominent substructural logic in the language of $\mathcal{L}_{\mathrm{pAb}}$ which is axiomatized by omitting the axiom (rel) from the axiomatic system given in Table 1.1. Thus, pointed Abelian logic is the extension of $\mathrm{FL_e}$ by axiom (rel).

[6] All the semilinear extensions of pAb are characterized in the submitted article [61].

Interestingly, we can see Ab as an extension of pAb by the axiom which collapses both constants:

$$(\mathtt{f} \to \mathtt{t}) \wedge (\mathtt{t} \to \mathtt{f}).$$

Therefore, sometimes in the literature Ab is presented as the expansion of $\mathrm{FL_e}$ by axioms (rel) and $(\mathtt{f} \to \mathtt{t}) \wedge (\mathtt{t} \to \mathtt{f})$. But, as we have commented in footnote 3, we prefer to see pAb as an expansion of Ab.

Clearly, pAb and all its extensions have (sCng) and hence they are superabelian logics. Therefore, it is easy to see that $\mathbf{Alg}^*(\mathrm{pAb})$, i.e. the equivalent algebraic semantic of pAb, is the class of *pointed Abelian ℓ-groups*, i.e. Abelian ℓ-groups expanded to the language $\mathcal{L}_{\mathrm{pAb}}$ with a new constant symbol interpreted in an arbitrary way.

To be able to easily talk about particular pAb-algebras, we present the following definition.

Definition 1.30. *Let us have an algebra $\boldsymbol{A}$ in a language $\mathcal{L}$ such that $\mathtt{f} \notin \mathcal{L}$ and take an element $a \in A$. By $\boldsymbol{A}_a$ we denote the algebra in the language $\mathcal{L} \cup \{\mathtt{f}\}$, which is the expansion of $\boldsymbol{A}$ and where the constant symbol $\mathtt{f}$ is interpreted as a.*

If $\mathbb{K}$ is a class of algebras, by $\mathbf{p}\mathbb{K}$ we denote $\{\boldsymbol{A}_a \mid \boldsymbol{A} \in \mathbb{K}, a \in A\}$. For a singleton $\{\boldsymbol{A}\}$, we may write $\mathbf{p}\boldsymbol{A}$ instead of $\mathbf{p}\{\boldsymbol{A}\}$.

$\boldsymbol{R}_{-1}$, $\boldsymbol{R}_0$, and $\boldsymbol{R}_1$ are natural examples of pointed Abelian ℓ-groups. Note that obviously, $\mathbf{Alg}^*(\mathrm{pAb}) = \mathbf{p}\mathbb{AL}$. It is also a straightforward, albeit rather technical, matter to show that this pointing operator $\mathbf{p}$ commutes with the standard algebraic class operators for taking isomorphisms, subalgebras, products, and ultraproducts.

Theorem 1.31. *Let $\boldsymbol{A}$ be a non-trivial Abelian ℓ-group.* pAb *is finitely strongly complete w.r.t.* $\mathbf{p}\boldsymbol{A} = \{\boldsymbol{A}_a \mid a \in \boldsymbol{A}\}$.

Proof. Assume that $\Gamma \nvdash_{\mathrm{pAb}} \varphi$. Given an $\mathcal{L}_{\mathrm{pAb}}$-formula ψ, we define a $\mathcal{L}_{\mathrm{Ab}}$-formula ψ' by replacing all occurrences of $\mathtt{f}$ by an arbitrary fixed variable p not occurring in $\Gamma \cup \{\varphi\}$. Let us denote $\Gamma' = \{\psi' \mid \psi \in \Gamma\}$. Then, clearly, $\Gamma' \nvdash_{\mathrm{Ab}} \varphi'$ and so $\Gamma' \nvDash_{\boldsymbol{A}} \varphi'$ as witnessed by an evaluation e which entails $\Gamma \nvDash_{\boldsymbol{A}_{e(p)}} \varphi$. □

Let us formulate a trivial lemma which allows us to improve the previous result for $\boldsymbol{A}$ being either $\boldsymbol{R}$ or $\boldsymbol{Q}$ and will also be of use later.

Lemma 1.32. *Let $\boldsymbol{A}$ be either $\boldsymbol{R}$ or $\boldsymbol{Q}$ and $a, b \in A \setminus \{0\}$. Then, $\boldsymbol{A}_a$ and $\boldsymbol{A}_b$ are isomorphic iff $a \cdot b > 0$.*

Proof. If $\mathbf{A}_a$ and $\mathbf{A}_b$ are isomorphic, the isomorphism must preserve the underlying natural order, which trivially forces the designated units a and b to have the same sign; hence $a \cdot b > 0$. For the other direction, assume $a \cdot b > 0$. Then $\frac{b}{a} > 0$, and the mapping $h\colon x \mapsto \frac{b}{a} \cdot x$ is an order-preserving group automorphism mapping a to b, providing the needed isomorphism. □

Theorem 1.33. *The logic* pAb *is finitely strongly complete w.r.t.* $\{\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1\}$ *and w.r.t.* $\{\boldsymbol{Q}_{-1}, \boldsymbol{Q}_0, \boldsymbol{Q}_1\}$.

Observe that the same would not work e.g. for the Abelian ℓ-group of the integers, since $\boldsymbol{Z}_a$ and $\boldsymbol{Z}_b$ are not isomorphic unless $a = b$. In fact, one can show that $\boldsymbol{Z}_a$ generates a different variety for each $a \in \mathbb{Z}$.

Proposition 1.34. *For any $a, b \in \mathbb{Z}$, we have* $\mathbf{HSP}(\boldsymbol{Z}_a) = \mathbf{HSP}(\boldsymbol{Z}_b)$ *if and only if $a = b$.*

Proof. One direction is obvious. To prove the converse one, let us assume $a \neq b$. To show that $\mathbf{HSP}(\boldsymbol{Z}_a) \neq \mathbf{HSP}(\boldsymbol{Z}_b)$, it is sufficient to find an equation which is valid in one of the generating algebras, $\boldsymbol{Z}_a$ or $\boldsymbol{Z}_b$, but not in the other. If $a < 0$ and $b \geq 0$ take the equation $\mathtt{f} \geq 0$. This equation clearly holds in $\boldsymbol{Z}_b$ but it does not hold in $\boldsymbol{Z}_a$. The case $a \leq 0$ and $b > 0$ is similar. Therefore, it remains to check the cases where $a, b > 0$ or $a, b < 0$. We will check only the first case (the second one is analogous). Moreover, let us assume without loss of generality that $0 < a < b$.

Consider the following equation:

$$(a \cdot x - \mathtt{f}) \vee (-x) \geq 0. \tag{*}$$

This equation is not valid in $\boldsymbol{Z}_b$, since if we take an evaluation e such that $e(x) = 1$ we get $(a \cdot e(x) - b) \vee (-e(x)) = (a - b) \vee -1 < 0$. Thus the equation (*) does not hold in $\boldsymbol{Z}_b$.

We show the same equation holds in $\boldsymbol{Z}_a$. First let us consider an evaluation e such that $e(x) \leq 0$. Clearly, we get $-e(x) \geq 0$ and thus $(a \cdot e(x) - a) \vee e(-x) \geq 0$. It remains to consider an evaluation e such that $e(x) > 0$ (and thus $e(x) \geq 1$). Then $a \cdot e(x) - a \geq a \cdot 1 - a = 0$. Thus the equation (*) is valid in $\boldsymbol{Z}_a$. $\square$

1.5 Łukasiewicz unbound logic (and other logics of reals)

In this section, we formally introduce Łukasiewicz unbound logic within our framework, motivated by the observation that its original algebraic semantics is isomorphic to the pointed Abelian ℓ-group $\boldsymbol{R}_{-1}$. Our primary goal is to axiomatize this logic as an extension of pointed Abelian logic. We first define the finitary version, Lu, and establish its finite strong completeness with respect to $\boldsymbol{R}_{-1}$. Then, we introduce its infinitary version, Lu_∞, by adding the Archimedean rule to achieve strong completeness, and make the connection to standard Łukasiewicz logic precise by providing a formal translation. Generalizing the methods developed for this specific case, we broaden our scope to systematically axiomatize the logics of other prominent pointed groups, such as $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$. We establish a correspondence between algebraic properties of the point element $\mathtt{f}$ and simple syntactic rules, allowing us to provide a comprehensive overview of completeness results for this family of logics.

Let us start this section by observing that the algebra $\boldsymbol{R}_{-1}$ is isomorphic to the algebra $\boldsymbol{LU}$ that was used in [24] to introduce Łukasiewicz unbound logic. $\boldsymbol{LU}$ is a member of $\mathbf{p}\mathbb{AL}$ defined over the reals with lattice operations defined in the usual way and other operations/constants defined as:

$$x \to^{\boldsymbol{LU}} y = 1 - x + y \qquad x \mathbin{\&}^{\boldsymbol{LU}} y = x + y - 1 \qquad \mathtt{t}^{\boldsymbol{LU}} = 1 \qquad \mathtt{f}^{\boldsymbol{LU}} = 0$$

Clearly, $f\colon \boldsymbol{R}_{-1} \longrightarrow \boldsymbol{LU}$ defined as $f(x) = x + 1$ is the desired isomorphism. As already mentioned in the introduction, the operations on $\boldsymbol{LU}$ closely resemble those of the well-known standard MV-algebra **Ł** which provides semantics for the

Łukasiewicz logic (see more details at the end of this section). In the context of this paper we will see it as an algebra in the language of pAb (though it clearly is not a member of $\mathbf{Alg}^*(\mathrm{pAb})$)[7] defined on the real unit interval $[0,1]$ with lattice operations defined in the usual way and other operations/constants defined as:

$$x \to^{\text{Ł}} y = \min\{1, 1-x+y\} \qquad x \,\&^{\text{Ł}} y = \max\{0, x+y-1\} \qquad \mathtt{t}^{\text{Ł}} = 1 \qquad \mathtt{f}^{\text{Ł}} = 0$$

Recall that we know that pAb is *finitely* strongly complete with respect to $\{\boldsymbol{R}_{-1}, \boldsymbol{R}_1, \boldsymbol{R}_0\}$, thus our goal is to find a finitary rule which allows us to extend pAb in a way that preserves semilinearity and also ensures that the rule is not valid in $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$.

Definition 1.35. Łukasiewicz unbound logic Lu *is the extension of the logic* pAb *by the rule:*

$$\mathtt{f} \vee \varphi \blacktriangleright \varphi \tag{Lu}$$

Lemma 1.36. Lu *is a semilinear logic.*

Proof. By Corollary 1.13, it is enough to check that $(\mathtt{f} \vee \varphi) \vee \psi \blacktriangleright \varphi \vee \psi$ is valid in Lu. This clearly follows from the use of rules $(\mathrm{A}_\vee)$ and $\mathtt{f} \vee \varphi \blacktriangleright \varphi$. □

To obtain the desired completeness, let us characterize linearly ordered Abelian ℓ-groups satisfying the rule (Lu). We give a slightly more general result, which will allow us to show that we cannot replace (Lu) by a simpler, weaker rule that might initially seem sufficient.

Lemma 1.37. *Let* $\boldsymbol{A} \in \mathbf{Alg}^*(\mathrm{pAb})$ *and* $\boldsymbol{A} \neq \boldsymbol{1}$*. Then, the following conditions are equivalent:*

1. $\mathtt{f}^{\boldsymbol{A}} \not\geq 0$.
2. *The rule* $\mathtt{f} \blacktriangleright \varphi$ *is valid in* $\boldsymbol{A}$*.*

Furthermore, if $\boldsymbol{A}$ *is linearly ordered, we can add the following condition:*

3. *The rule* $\mathtt{f} \vee \varphi \blacktriangleright \varphi$ *is valid in* $\boldsymbol{A}$*.*

Proof. The implications from 1 to 2 and from 1 to 3 are clear (note that in the second case we need to use linearity of $\boldsymbol{A}$ to obtain that if $\mathtt{f}^{\boldsymbol{A}} \not\geq 0$ and $\mathtt{f} \vee \varphi \geq 0$, then $\varphi \geq 0$). We prove the converse ones at once by contradiction: assume that $\mathtt{f} \vDash_{\boldsymbol{A}} \varphi$ (resp. that $\mathtt{f} \vee \varphi \vDash_{\boldsymbol{A}} \varphi$) and $\mathtt{f}^{\boldsymbol{A}} \geq 0$. For an arbitrary element a and any evaluation $e(\varphi) = a$, the premises of both rules are trivially valid and so we obtain $a \geq 0$, a contradiction with the assumption that $\boldsymbol{A} \neq \boldsymbol{1}$. □

Thus, we know that in particular $\boldsymbol{R}_{-1} \in \mathbf{Alg}^\ell(\mathrm{Lu})$ but $\boldsymbol{R}_0, \boldsymbol{R}_1 \notin \mathbf{Alg}^\ell(\mathrm{Lu})$. The next example shows that the rules $\mathtt{f} \vee \varphi \blacktriangleright \varphi$ and $\mathtt{f} \blacktriangleright \varphi$ are not interderivable and thus the extension of pAb by the rule $\mathtt{f} \blacktriangleright \varphi$ is strictly weaker than Lu.

Example. Let us consider the Abelian ℓ-group $\boldsymbol{R}_{-1} \times \boldsymbol{R}_0$. Clearly, this Abelian ℓ-group satisfies the rule $\mathtt{f} \blacktriangleright \varphi$, but it does not satisfy $\mathtt{f} \vee \varphi \blacktriangleright \varphi$.

[7]Indeed, here we choose a nonstandard representation of MV-algebras. It is easy to see that this representation is equivalent to the most common one (found e.g. in [21]), the reader can check it as an exercise or check any basic literature related to MV-algebras (again see e.g. [21]).

Theorem 1.38. Lu *is finitely strongly complete with respect to any of the following sets of algebras:* $\{\boldsymbol{R}_{-1}\}$, $\{\boldsymbol{Q}_{-1}\}$, *and* $\{\boldsymbol{A}_a \mid a < 0\}$ *for any non-trivial Abelian* ℓ*-group* $\boldsymbol{A}$.

Proof. We prove the final part of the claim, the rest then follows from Lemma 1.32. Using Theorem 1.31 and Corollary 1.17, we know that

$$\mathbf{Alg}^{\ell}(\mathrm{Lu}) \subseteq \mathbf{Alg}^{\ell}(\mathrm{pAb}) \subseteq \mathbf{I}(\boldsymbol{1}) \cup \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{A}_a \mid a < 0\}) \cup \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{A}_a \mid a \geq 0\}).$$

Clearly, for any non-trivial element of $\mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{A}_a \mid a \geq 0\})$, we have $\mathtt{f}^{\boldsymbol{A}_a} \geq 0$ (an equation preserved by ultrapowers) and thus it is not a member of $\mathbf{Alg}^{\ell}(\mathrm{Lu})$ due to the previous lemma. Thus

$$\mathbf{Alg}^{\ell}(\mathrm{Lu}) \subseteq \mathbf{I}(\boldsymbol{1}) \cup \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{A}_a \mid a < 0\})$$

and so the proof is done by Corollary 1.17. □

It is easy to see that the logic Lu is not strongly complete with respect to $\boldsymbol{R}_{-1}$. For instance, the rule $(\mathrm{Arch})^{\vee}$, introduced in the previous section, is valid in $\boldsymbol{R}_{-1}$ but is not derivable in Lu (by Example 1.3). This shows that the full set of validities in $\boldsymbol{R}_{-1}$ cannot be captured by a finitary system extending Lu and will therefore require an infinitary axiomatization. To this end, we define now the following extension of Lu:

Definition 1.39. *By* infinitary Łukasiewicz unbound logic Lu_{∞} *we understand the extension of the logic* Lu *by the rule* $(\mathrm{Arch})^{\vee}$.

Equivalently, Lu_{∞} can be seen as the expansion of rAb to the language of pAb by the rule (Lu).

Theorem 1.40. *The logic* Lu_{∞} *is strongly complete with respect to* $\boldsymbol{R}_{-1}$.

Proof. By Corollary 1.13, the logic Lu_{∞} is semilinear. Since Lu_{∞} is an expansion of rAb, every linearly ordered Lu_{∞}-algebra is a pointed Archimedean ℓ-group. Since Lu_{∞} is an extension of Lu, by Lemma 1.37, we know that in any non-trivial $\boldsymbol{A}$ linearly ordered Lu_{∞}-algebra we have $\mathtt{f}^{\boldsymbol{A}} < 0$. Thus, $\boldsymbol{A}$ has to be a pointed Archimedean ℓ-group with a negative point a. Thus, by Hölder's Theorem [60], there is an embedding h of the $\mathtt{f}$-free reduct of $\boldsymbol{A}$ into $\boldsymbol{R}$. Clearly h can be seen as an embedding of $\boldsymbol{A}$ into $\boldsymbol{R}_{h(a)}$. As $h(a) < 0$, we know that $\boldsymbol{R}_{h(a)}$ is isomorphic to $\boldsymbol{R}_{-1}$ (Lemma 1.32). Therefore, $\mathbf{Alg}^{\ell}(\mathrm{Lu}_{\infty}) \subseteq \mathbf{IS}(\boldsymbol{R}_{-1})$. Since Lu_{∞} is semilinear and $\boldsymbol{R}_{-1}$ is an Lu_{∞}-algebra it follows that Lu_{∞} is strongly complete with respect to $\boldsymbol{R}_{-1}$. □

To conclude our analysis of these systems, we discuss the connection between (infinitary) Łukasiewicz unbound logic and (infinitary) Łukasiewicz logic. The Łukasiewicz logic Ł is a finitary logic axiomatized by four axioms and *modus ponens* and is finitely strongly complete w.r.t. the standard MV-algebra **Ł** introduced at the beginning of this section. The infinitary Łukasiewicz logic Ł_{∞} is its extension by the rule (Hay) introduced in Section 1.3, this logic indeed is not finitary but is strongly complete w.r.t. the standard MV-algebra **Ł**.

As we have seen in the beginning of the section, the algebra **Ł** is a restriction of the algebra $\boldsymbol{LU}$ which in turn is an isomorphic copy of $\boldsymbol{R}_{-1}$. We use this fact (together with the completeness results for the involved logics), to prove the following results linking (infinitary) Łukasiewicz logic and (infinitary) Łukasiewicz unbound logics via the mapping τ[8] defined recursively on the set of formulas:

- $\tau(p) = (p \vee \mathtt{f}) \wedge \mathtt{t}$ for all variables p.
- $\tau(\psi \to \chi) = (\tau(\psi) \to \tau(\chi)) \wedge \mathtt{t}$.
- $\tau(\psi \,\&\, \chi) = (\tau(\psi) \,\&\, \tau(\chi)) \vee \mathtt{f}$.
- $\tau(c) = c$ for $c \in \{\mathtt{f}, \mathtt{t}\}$.
- $\tau(\psi \circ \chi) = \tau(\psi) \circ \tau(\chi)$ for $\circ \in \{\wedge, \vee\}$.

Theorem 1.41. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas. Then, we have*

$$\Gamma \vdash_{\text{Ł}_\infty} \varphi \qquad \textit{iff} \qquad \tau[\Gamma] \vdash_{\text{Lu}_\infty} \tau(\varphi).$$

Moreover, if Γ is finite we have

$$\Gamma \vdash_{\text{Ł}} \varphi \qquad \textit{iff} \qquad \tau[\Gamma] \vdash_{\text{Lu}} \tau(\varphi).$$

Proof. Recall that, thanks to the known completeness properties, we can replace the syntactical consequence relations by semantical ones w.r.t. algebras **Ł** and **Lu** (because it is an isomorphic copy of $\boldsymbol{R}_{-1}$).

Let us consider any mapping e of propositional variables into reals and define the mapping $\bar{e}$ as its restriction to $[0, 1]$, i.e.,

$$\bar{e}(p) = \begin{cases} e(p) & \text{if } e(p) \in [0,1] \\ 1 & \text{if } e(p) \geq 1 \\ 0 & \text{if } e(p) \leq 0. \end{cases}$$

We denote by $(\bar{e})^{\boldsymbol{Ł}}$ and $e^{\boldsymbol{Lu}}$ the corresponding evaluations. By a simple proof by induction over the complexity of formula χ, we obtain:

$$(\bar{e})^{\boldsymbol{Ł}}(\chi) = e^{\boldsymbol{Lu}}(\tau(\chi)).$$

Clearly, it holds for variables and constants by definition of $\bar{e}$ and τ. Now assume $(\bar{e})^{\boldsymbol{Ł}}(\alpha) = e^{\boldsymbol{Lu}}(\tau(\alpha))$ and $(\bar{e})^{\boldsymbol{Ł}}(\beta) = e^{\boldsymbol{Lu}}(\tau(\beta))$ for formulas α, β. We want to show that $(\bar{e})^{\boldsymbol{Ł}}(\alpha \circ \beta) = e^{\boldsymbol{Lu}}(\tau(\alpha \circ \beta))$ for $\circ \in \{\wedge, \vee, \&, \to\}$. We show this step only for $\circ = \&$. We write:

$$e^{\boldsymbol{Lu}}(\tau(\alpha \,\&\, \beta)) = e^{\boldsymbol{Lu}}((\tau(\alpha) \,\&\, \tau(\beta)) \vee \mathtt{f}) = (e^{\boldsymbol{Lu}}(\tau(\alpha)) + e^{\boldsymbol{Lu}}(\tau(\beta)) - 1) \vee 0 = \\ ((\bar{e})^{\boldsymbol{Ł}}(\alpha) + (\bar{e})^{\boldsymbol{Ł}}(\beta) - 1) \vee 0 = (\bar{e})^{\boldsymbol{Ł}}(\alpha \,\&\, \beta).$$

[8] A related mapping is presented in [78, Definition 23] as a translation of Łukasiewicz logic into Abelian logic. The only difference between that translation and the present one is that the former employs an arbitrary fixed propositional variable $q^{\perp}$ in precisely those places where our construction uses the constant symbol $\mathtt{f}$. In fact, more can be said about the connection between Ł and Lu: In [78, Theorem 20], it is shown that Ł is the $\{\supset, \mathtt{f}\}$-fragment of Lu, where $\varphi \supset \psi$ is defined as $(\varphi \wedge \mathtt{t}) \to (\psi \vee \mathtt{f})$.

Now we can prove the left-to-right direction of the first equivalence. Assume that $\Gamma \models_{\mathbf{Ł}} \varphi$ and that e is a mapping such that for each $\gamma \in \Gamma$ we have (recall that $\mathtt{t}^{\boldsymbol{LU}} = 1$)

$$e^{\boldsymbol{Lu}}(\tau(\gamma)) \geq 1.$$

As $e^{\boldsymbol{Lu}}(\tau(\gamma)) = (\bar{e})^{\mathbf{Ł}}(\gamma)$ we have $(\bar{e})^{\mathbf{Ł}}(\gamma) = 1$ for $\gamma \in \Gamma$ and so, by the assumption, also $(\bar{e})^{\mathbf{Ł}}(\varphi) = 1$, which entails $e^{\mathbf{Lu}}(\tau(\varphi)) = 1$.

To prove the converse direction, it suffices to note that if the co-domain of e is a subset of $[0, 1]$, then $e = \bar{e}$.

The second equivalence can either be proved in exactly the same way or actually follows from the first one and the fact that on finite sets of premises the Łukasiewicz (unbound) logic coincides with its infinitary variant. $\square$

The translation provides some information regarding computational complexity. In Łukasiewicz logic, the problem of deciding the validity of finitary consecutions is known to be coNP-complete [55]. As Theorem 1.41 provides a polynomial-time reduction of finitary consecutions from Ł to Lu, it immediately follows that the corresponding problem for Lu is coNP-hard. The more involved question of whether the set of valid finitary consecutions of Lu is also in coNP, and thus coNP-complete, has been answered affirmatively in the paper [56].

Having established the axiomatization for the logic of $\boldsymbol{R}_{-1}$, we now turn to the complementary task of characterizing the logics of the other prominent pointed algebras, $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$. As we will show, a similar approach allows us to axiomatize these systems in a straightforward manner. Let us define a new connective $-\varphi$ as $\varphi \to \mathtt{t}$. By simple checking we prove the following lemma (note that we could prove a more complex variant akin to Lemma 1.37 to demonstrate that seemingly simpler rules would not do).

Lemma 1.42. *Let $\boldsymbol{A} \in \mathbf{Alg}^{\ell}(\mathrm{pAb})$ and $\boldsymbol{A} \neq \boldsymbol{1}$. Then:*

- $\mathtt{f}^{\boldsymbol{A}} < 0$ *iff the rule* $\mathtt{f} \vee \varphi \blacktriangleright \varphi$ *is valid in* $\boldsymbol{A}$*.*
- $\mathtt{f}^{\boldsymbol{A}} \leq 0$ *iff the axiom* $\blacktriangleright -\mathtt{f}$ *is valid in* $\boldsymbol{A}$*.*
- $\mathtt{f}^{\boldsymbol{A}} = 0$ *iff the axiom* $\blacktriangleright \mathtt{f} \wedge -\mathtt{f}$ *is valid in* $\boldsymbol{A}$*.*
- $\mathtt{f}^{\boldsymbol{A}} \neq 0$ *iff the rule* $(\mathtt{f} \wedge -\mathtt{f}) \vee \varphi \blacktriangleright \varphi$ *is valid in* $\boldsymbol{A}$*.*
- $\mathtt{f}^{\boldsymbol{A}} \geq 0$ *iff the axiom* $\blacktriangleright \mathtt{f}$ *is valid in* $\boldsymbol{A}$*.*
- $\mathtt{f}^{\boldsymbol{A}} > 0$ *iff the rule* $-\mathtt{f} \vee \varphi \blacktriangleright \varphi$ *is valid in* $\boldsymbol{A}$*.*

Theorem 1.43. *Let $\boldsymbol{A}$ be any non-trivial* rAb*-algebra. Then the extension of* pAb *by the rule/axiom mentioned in a given row of Table 1.2 is finitely strongly complete w.r.t. the sets of algebras mentioned in the corresponding columns Set 1–3. Its extension by the rule* $(\mathrm{Arch})^{\vee}$ *is strongly complete w.r.t. the set of algebras mentioned in the corresponding column Set 3.*

As mentioned before, the expansion of pAb by $\blacktriangleright \mathtt{f} \wedge -\mathtt{f}$ can be seen as the Abelian logic itself (formally speaking, these two logics are termwise equivalent by setting $\mathtt{f} = \mathtt{t}$).

Extension of pAb by	Set 1	Set 2	Set 3
$\emptyset$	$\{\boldsymbol{A}_a \mid a \in A\}$	$\{\boldsymbol{Q}_{-1}, \boldsymbol{Q}_0, \boldsymbol{Q}_1\}$	$\{\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1\}$
$\blacktriangleright \mathtt{f}$	$\{\boldsymbol{A}_a \mid a \geq 0\}$	$\{\boldsymbol{Q}_0, \boldsymbol{Q}_1\}$	$\{\boldsymbol{R}_0, \boldsymbol{R}_1\}$
$\blacktriangleright -\mathtt{f}$	$\{\boldsymbol{A}_a \mid a \leq 0\}$	$\{\boldsymbol{Q}_{-1}, \boldsymbol{Q}_0\}$	$\{\boldsymbol{R}_{-1}, \boldsymbol{R}_0\}$
$(\mathtt{f} \wedge -\mathtt{f}) \vee \varphi \blacktriangleright \varphi$	$\{\boldsymbol{A}_a \mid a \neq 0\}$	$\{\boldsymbol{Q}_{-1}, \boldsymbol{Q}_1\}$	$\{\boldsymbol{R}_{-1}, \boldsymbol{R}_1\}$
$\mathtt{f} \vee \varphi \blacktriangleright \varphi$	$\{\boldsymbol{A}_a \mid a < 0\}$	$\{\boldsymbol{Q}_{-1}\}$	$\{\boldsymbol{R}_{-1}\}$
$-\mathtt{f} \vee \varphi \blacktriangleright \varphi$	$\{\boldsymbol{A}_a \mid a > 0\}$	$\{\boldsymbol{Q}_1\}$	$\{\boldsymbol{R}_1\}$
$\blacktriangleright \mathtt{f} \wedge -\mathtt{f}$	$\{\boldsymbol{A}_0\}$	$\{\boldsymbol{Q}_0\}$	$\{\boldsymbol{R}_0\}$

Table 1.2 Completeness properties of prominent extensions of pAb

Let us conclude this section by showing a natural and simple translation between Lu and the expansion of pAb by $-\mathtt{f} \vee \varphi \blacktriangleright \varphi$, denoted here by Lu*.

Theorem 1.44. *Let us consider the mapping $\tau\colon Fm_{\mathrm{L_{pAb}}} \longrightarrow Fm_{\mathrm{L_{pAb}}}$ defined recursively as follows:*

- $\tau\colon \mathtt{f} \mapsto -\mathtt{f}$.
- $\tau\colon \mathtt{t} \mapsto \mathtt{t}$.
- $\tau\colon x \mapsto x$ *for each variable* x.
- $\tau\colon (\psi \circ \xi) \mapsto (\tau(\psi) \circ \tau(\xi))$ *for all binary connectives* $\circ \in \mathcal{L}_{\mathrm{pAb}}$.

For any consecution $\Gamma \blacktriangleright \varphi$ and for each Abelian ℓ-group $\boldsymbol{A}$, and $a \in A$ we have

$$\Gamma \vDash_{\boldsymbol{A}_a} \varphi \qquad \textit{iff} \qquad \tau[\Gamma] \vDash_{\boldsymbol{A}_{-a}} \tau(\varphi).$$

In particular, for any consecution $\Gamma \blacktriangleright \varphi$ we have

1. $\Gamma \vdash_{\mathrm{Lu}} \varphi$ *iff* $\tau[\Gamma] \vdash_{\mathrm{Lu}^*} \tau(\varphi)$.
2. $\Gamma \vdash_{\mathrm{Lu}^*} \varphi$ *iff* $\tau[\Gamma] \vdash_{\mathrm{Lu}} \tau(\varphi)$.

Proof. Consider an arbitrary $\boldsymbol{A}_a$-evaluation $e_1\colon Fm_{\mathcal{L}_{\mathrm{pAb}}} \longrightarrow \boldsymbol{A}_a$. Clearly, e_1 extends the evaluation $e_0\colon Fm_{\mathcal{L}_{\mathrm{Ab}}} \longrightarrow \boldsymbol{A}$. Clearly there is a unique evaluation $e_2\colon Fm_{\mathcal{L}_{\mathrm{pAb}}} \longrightarrow \boldsymbol{A}_{-a}$ extending e_0 such that $e_1(\mathtt{f}) = -e_2(\mathtt{f}) = e_2(-\mathtt{f})$ and $e_1(x) = e_2(x)$ for all variables x. By induction one can show that $e_1(\psi) = e_2(\tau(\psi))$ for each formula $\psi \in Fm_{\mathcal{L}_{\mathrm{pAb}}}$. Therefore, $e_1(\psi) \geq 0$ iff $e_2(\tau(\psi)) \geq 0$. From this, one can easily derive the statement of this theorem. □

Clearly, the same translation would link the infinitary versions of these two logics (i.e., their extensions of $(\mathrm{Arch})^{\vee}$); the expansions of pAb by (1) by axiom $\mathtt{f}$ and (2) axiom $-\mathtt{f}$; and their extensions of $(\mathrm{Arch})^{\vee}$.

1.6 Future Work

In this section, we outline possible directions for future research on the topics covered in this paper.

- Study of the lattice of infinitary extensions of Abelian logic: While the present paper provides an infinitary rule for the extension corresponding to the generalized quasivariety generated by $\boldsymbol{R}$ and establishes the existence of 2^{2^ω} distinct infinitary extensions, it is clear that we have only started the topic and the structure of the full lattice of infinitary extensions of Ab remains largely unexplored. One of the goals for future research is a systematic classification of this lattice. In particular, a major open challenge is explicitly axiomatizing other fundamental logics, namely the logics of $\boldsymbol{Z}$ and $\boldsymbol{Q}$.

- Infinitary logics of distinct total orders on $\boldsymbol{Z}^n$: It is a well-established result [29] that the space of total orders on $\boldsymbol{Z}^n$ is homeomorphic to a Cantor space for $n > 1$. A natural open problem is to determine whether each distinct ordering within this space generates a unique infinitary logic.

- Study of finitary extensions of pointed Abelian logic: In the case of pAb, one should first focus on *finitary* extensions, as they are plentiful. In this paper we have axiomatized several prominent logics of pointed Abelian ℓ-groups, but a systematic study remains an interesting topic for the future. On this matter, it is relevant to mention that the relation between Łukasiewicz logic and the Łukasiewicz unbound logic proved in Theorem 1.41 also holds between the n-valued Łukasiewicz logic and the logic of $\boldsymbol{Z}_{n-1}$ (and analogously between other axiomatic extensions of Łukasiewicz logic and properly chosen extensions of Lu). Algebraically speaking, the lattice of subvarieties of Łukasiewicz logic (described by Komori in [66]) can be embedded into the lattice of subvarieties of Lu (first shown in [98]). The submitted paper [61] describes all quasivarieties generated by linearly ordered pointed Abelian ℓ-groups. Nevertheless, the case of non-semilinear finitary extensions of pointed Abelian logic remains unexplored.

- Semilinearity and completeness: Another area for further investigation is to determine whether certain extensions of Abelian logic, specifically the logic Ab + $(\mathrm{IDC})^\vee$ + $(\mathrm{Arch})^\vee$, are semilinear, which would imply strong completeness with respect to $\boldsymbol{Z}$. We have conjectured that this logic may be strongly complete with respect to $\boldsymbol{Z}$, but this remains to be proven.

- Study of other superabelian logics: In this paper we have focused on Łukasiewicz unbound logic and infinitary extensions of Abelian logic, but the framework from Section 1.2 applies to any superabelian logic, e.g. certain modal logics or the expansions of Ab with connectives corresponding to additional arithmetical operations on real numbers, such as multiplication or division.

2 Subvarieties of pointed Abelian ℓ-groups

2.1 Introduction

Lattice-ordered groups (ℓ-groups) represent a foundational class of algebraic structures with a rich history of celebrated results [51, 47, 48, 10, 32]. The field remains an active area of research, largely because the lattice of subvarieties of ℓ-groups is notoriously complex and not yet fully understood [67]. Even the simplest non-trivial case, the variety of Abelian ℓ-groups, presents a deep mathematical challenge.

MV-algebras form another extremely important class of algebraic structures. Originally introduced as the algebraic semantics for Łukasiewicz logic [71], these algebras have been the subject of extensive study for decades [21, 44, 68, 84]. One of the most essential tools in their study is the celebrated Mundici functor, a categorical equivalence between MV-algebras and Abelian ℓ-groups with a strong unit. This functor provides a powerful bridge between the two fields, allowing many fundamental results in the theory of MV-algebras to be obtained by translating problems into the more established context of ℓ-groups.

However, the theory of MV-algebras is in some respects richer than that of Abelian ℓ-groups, primarily because its language contains an additional designated constant. A similar level of expressive power can be achieved in the theory of Abelian ℓ-groups by augmenting their language with an additional constant, which gives rise to the class of pointed Abelian ℓ-groups. As this article demonstrates, this class allows one to reason about MV-algebras while remaining entirely within the realm of Abelian ℓ-groups. Furthermore, this class forms a variety, making it a natural and well-behaved algebraic framework. This contrasts with the class of Abelian ℓ-groups with a strong unit, which is not even first-order definable.

In addition to its algebraic significance, our work is also strongly motivated by logic. Pointed Abelian ℓ-groups serve as the algebraic models for pointed Abelian logic [27], a system that can be viewed as a meeting-point of Łukasiewicz logic [71] and Abelian logic [18, 81]. This paper provides a complete classification of all axiomatic and semilinear finitary extensions of pointed Abelian logic. Further details on the logical perspective can be found in [27].

The starting point for our investigation is Komori's foundational classification of the varieties of MV-algebras [66]. This was later connected to the ℓ-group setting by Young [98], who used the Mundici functor to establish a correspondence between these varieties and varieties of positively pointed Abelian ℓ-groups. This paper generalizes and extends Young's work in several key aspects:

1. First, we broaden the scope from positively pointed Abelian ℓ-groups to the entire class of pointed Abelian ℓ-groups.

2. Second, we develop our theory of varieties semantically from first principles within the theory of Abelian ℓ-groups, rather than relying on the Mundici functor and the theory of MV-algebras. This contrasts with the more syntactic approach of [98] and allows us to prove more general theorems

(e.g., compare our Theorem 2.23 with [98, Lemma 4.6]). Moreover, this strategy provides us with more structural insight, and we believe readers will find our reasoning more structurally transparent than that presented in [98]. As a consequence of our framework, we derive Komori's classification of subvarieties of MV-algebras in Theorem 2.47.

3. Third, while the work in [98] established the correspondence between Abelian ℓ-groups and MV-algebras, it did not provide axiomatizations for the corresponding equational classes. This paper fills that gap by providing such axiomatizations, analogous to Komori's original description of the varieties of MV-algebras in [66].

4. Finally, we will use Gispert's classification of universal classes of totally ordered MV-algebras and quasivarieties of MV-algebras generated by their totally ordered members from [46] to provide the corresponding classification for pointed Abelian ℓ-groups.

The paper is structured as follows. Section 2.2 introduces the necessary preliminaries from universal algebra and the theory of Abelian ℓ-groups. Section 2.3 develops the core technical results concerning lexicographic decompositions of finitely generated totally ordered ℓ-groups, culminating in Corollary 2.19 and Theorem 2.23. Section 2.4 provides a semantic characterization of the varieties of pointed Abelian ℓ-groups, presenting novel and more elegant proofs for some known results (e.g., Lemma 2.33). The central contribution of the paper is presented in Section 2.5, which gives a complete classification and axiomatization of all subvarieties of pointed Abelian ℓ-groups. Finally, Section 2.6 applies our results to the theory of MV-algebras, providing an alternative proof of Komori's classification. Moreover, by using Gispert's classification of universal classes of totally ordered MV-algebras, we provide a similar classification for totally ordered pointed Abelian ℓ-groups and consequently obtain a classification of all quasivarieties generated by totally ordered pointed Abelian ℓ-groups.

2.2 Preliminaries

In this section, we introduce some basic terminology and well-known results about Abelian ℓ-groups. We assume the reader has a basic background in universal algebra, but no advanced prerequisites are required to follow the rest of the paper. We will use the standard notations $\mathbf{H}, \mathbf{I}, \mathbf{S}, \mathbf{P}, \mathbf{P}_{\mathrm{U}}$ to denote closure under homomorphisms, isomorphisms, subalgebras, products, and ultraproducts, respectively. We will write $\boldsymbol{A} \in \mathbb{K}_{\mathrm{FSI}}$ iff the algebra $\boldsymbol{A} \in \mathbb{K}$ and $\boldsymbol{A}$ is finitely subdirectly irreducible.

First let us recall Birkhoff's Theorem and its variants.

Theorem 2.1 (Birkhoff [11, 12]). *Let $\mathbb{K}$ be a class of algebras of the same signature. The following conditions are equivalent:*

1. $\mathbb{K} = \mathbf{HSP}(\mathbb{K})$.

2. $\mathbb{K}$ *is the class of all models of some theory, all of whose axioms are equations.*

Moreover, if algebras in $\mathbb{K}$ have a group reduct, we can add the following:

3. *$\mathbb{K}$ is the class of all models of some theory, all of whose axioms are equations with the right-hand side equal to* 0.

Furthermore, if $\mathbb{K}$ satisfies any of the equivalent conditions above (i.e., if $\mathbb{K}$ is a variety), then:

$$\mathbb{K} = \mathbf{ISP}(\mathbb{K}_{\mathrm{FSI}}).$$

Let $\boldsymbol{A}$ be an algebra, (eq) an equation, and e an evaluation. We write $\boldsymbol{A} \vDash_e$ (eq) to denote that (eq) *is satisfied* in $\boldsymbol{A}$ under the evaluation e. Conversely, $\boldsymbol{A} \nvDash_e$ (eq) indicates that (eq) *is not satisfied* under e.

An equation (eq) is said to *hold* in an algebra $\boldsymbol{A}$, denoted $\boldsymbol{A} \vDash$ (eq), if it is satisfied for all possible evaluations. Conversely, if there is at least one evaluation for which (eq) is not satisfied, it does *not hold* in $\boldsymbol{A}$, and we write $\boldsymbol{A} \nvDash$ (eq).

We now provide the definition of an essential notion used in this paper.

Definition 2.2. *Let $\mathbb{K} \cup \{\boldsymbol{A}\}$ be a class of algebras of the same language $\mathcal{L}$. Let F be a finite subset of A and $\boldsymbol{B} \in \mathbb{K}$. We say that a mapping $f_F : F \to B$ is a* partial embedding *of the set F from $\boldsymbol{A}$ into $\boldsymbol{B}$ if f_F is a one-to-one mapping and for every operation symbol $\lambda \in \mathcal{L}$ of arity n, and for all $a_1, \ldots, a_n \in F$ such that $\lambda^{\boldsymbol{A}}(a_1, \ldots, a_n) \in F$ we have*

$$f_F(\lambda^{\boldsymbol{A}}(a_1, \ldots, a_n)) = \lambda^{\boldsymbol{B}}(f_F(a_1), \ldots, f_F(a_n)).$$

We say that $\boldsymbol{A}$ is partially embeddable into the class $\mathbb{K}$ if for each finite set $F \subseteq A$ there is a $\boldsymbol{B} \in \mathbb{K}$ and there is a partial embedding f_F of F into $\boldsymbol{B}$. If $\boldsymbol{A}$ is partially embeddable into the singleton class $\{\boldsymbol{B}\}$, we simply say $\boldsymbol{A}$ is partially embeddable into $\boldsymbol{B}$.

Theorem 2.3 (Mal'tcev [73])**.** *Let $\mathbb{K}$ be a class of algebras of the same signature. The following conditions are equivalent:*

1. *$\mathbb{K}$ is the class of all models of some theory, all of whose axioms are universal formulas.*
2. *$\mathbb{K} = \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$.*
3. *$\mathbb{K}$ is closed under partial embeddings (i.e. for any algebra $\boldsymbol{A}$ that is partially embeddable into $\mathbb{K}$ we have $\boldsymbol{A} \in \mathbb{K}$).*

Now we will focus on Abelian ℓ-groups.

Definition 2.4. *An algebra $\boldsymbol{A} = \langle A, +, -, \vee, \wedge, 0\rangle$ is an* Abelian ℓ-group *if $\langle A, +, -, 0\rangle$ is an Abelian group, $\langle A, \vee, \wedge\rangle$ is a lattice and $\boldsymbol{A}$ satisfies the monotonicity condition: for all $x, y, z \in A$, if $x \leq y$, then $x + z \leq y + z$. A subalgebra of an Abelian ℓ-group is referred to as an* ℓ-subgroup.

It is well-known (see, e.g. [43, Chapter V]), that the defining conditions of Abelian ℓ-groups can be expressed by means of equations, so they form a variety which we denote by $\mathbb{AL}$. Also, it is well-known that all Abelian ℓ-groups are torsion-free.

We will denote by $\boldsymbol{Z}$ and $\boldsymbol{R}$ the ℓ-groups of integers and reals with the underlying universes $\mathbb{Z}$ and $\mathbb{R}$, taken under their natural order.

For $a \in A$, we define the *absolute value* $|a| := a \vee -a$ and we let $n \cdot a := a + \cdots + a$ denote the n-fold sum for every $n \in \mathbb{N}$ (with $0 \cdot a := 0$). For $z \in \mathbb{Z} \backslash \mathbb{N}$ we set $z \cdot a := (-z) \cdot (-a)$.

Definition 2.5. *An ℓ-subgroup $\boldsymbol{B}$ of an Abelian ℓ-group $\boldsymbol{A}$ is called* convex *if it satisfies the following:*

$$\forall b \in B, \forall a \in A, \quad \Big(|a| \leq |b| \implies a \in B\Big).$$

It is well known that congruences on Abelian ℓ-groups are in one-to-one correspondence with their convex ℓ-subgroups (often referred to as *ℓ-ideals*), see e.g; [43].

The following results are also well-known.

Theorem 2.6 (Clifford [30])**.** *An Abelian ℓ-group is finitely subdirectly irreducible if and only if it is totally ordered.*

Theorem 2.7 (Gurevich, Kokorin [50])**.** *All non-trivial totally ordered Abelian ℓ-groups are universally equivalent. Equivalently, for every non-trivial totally ordered Abelian ℓ-group $\boldsymbol{A}$ we have $\mathbf{ISP}_{\mathrm{U}}(\boldsymbol{A}) = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z})$.*

From Theorems 2.6 and 2.7 one can easily derive the following:

Theorem 2.8 (Khisamiev [64])**.** *The quasivariety of all Abelian ℓ-groups is generated by $\boldsymbol{Z}$. Equivalently, $\mathbf{ISPP}_{\mathrm{U}}(\boldsymbol{Z}) = \mathbb{AL}$.*

Theorem 2.9 (Hölder's Theorem [60])**.** *The following are equivalent for any totally ordered Abelian ℓ-group $\boldsymbol{A}$:*

1. *$\boldsymbol{A}$ embeds into $\boldsymbol{R}$.*
2. *$\boldsymbol{A}$ is Archimedean.*
3. *$\boldsymbol{A}$ is simple.*

Lemma 2.10. *Let $\boldsymbol{A}$ be a non-trivial ℓ-subgroup of $\boldsymbol{R}$. Then A is a dense subset of $\mathbb{R}$ (with respect to the standard topology on $\mathbb{R}$) or $\boldsymbol{A} \cong \boldsymbol{Z}$.*

Proof. Since multiplication by any strictly positive element of $\mathbb{R}$ is an ℓ-group automorphism on $\boldsymbol{R}$ we can without loss of generality assume $1 \in A$. If A also contains some $\xi \in \mathbb{R} \setminus \mathbb{Q}$, we can recursively generate a strictly decreasing sequence of elements of A converging to 0 as follows. We start with $r_0 = \xi$ and $r_1 = 1$. For $n > 0$, we define $k_n = \lfloor \frac{r_{n-1}}{r_n} \rfloor$ and let $r_{n+1} = r_{n-1} - k_n \cdot r_n$. Hence, A must be a dense subset of $\mathbb{R}$.

Otherwise, $\boldsymbol{A} \in \mathbf{S}(\boldsymbol{Q})$. By [8, Theorem 2] and [8, Corollary 2] for every non-cyclic $\boldsymbol{A} \in \mathbf{S}(\boldsymbol{Q})$ and for each $0 < \epsilon \in \mathbb{R}$ there is an $a \in A$ such that $0 < a \leq \epsilon$. Since $\boldsymbol{A}$ is also closed under addition, A must be a dense subset of $\mathbb{R}$. If $\boldsymbol{A}$ is cyclic, then $\boldsymbol{A} \cong \boldsymbol{Z}$. $\square$

Definition 2.11. *For an Abelian ℓ-group $\boldsymbol{A} = \langle A, +, -, \vee, \wedge, 0\rangle$ and $a \in A$ we define a* pointed *Abelian ℓ-group $\boldsymbol{A}_a = \langle A, +, -, \vee, \wedge, 0, \mathtt{f}\rangle$,*[1] *where $\mathtt{f}^{\boldsymbol{A}_a} = a$. We denote the class of pointed Abelian ℓ-groups by $p\mathbb{AL}$. We say that $\boldsymbol{A}_a \in p\mathbb{AL}$ is* positively *pointed if $a \geq 0$,* negatively *pointed if $a \leq 0$, and* 0-pointed *if $a = 0$. We denote these classes by $p\mathbb{AL}^+, p\mathbb{AL}^-$ and $p\mathbb{AL}^0$.*

Clearly, the classes $p\mathbb{AL}, p\mathbb{AL}^+, p\mathbb{AL}^-$ and $p\mathbb{AL}^0$ are varieties. Observe that for an Abelian ℓ-group $\boldsymbol{A}$ and $a \in A$ we have that $\boldsymbol{A}$ is finitely subdirectly irreducible iff $\boldsymbol{A}_a$ is finitely subdirectly irreducible. Therefore, using Theorem 2.6 we obtain that a pointed Abelian ℓ-group is finitely subdirectly irreducible iff it is totally ordered.

As a corollary of Theorems 2.6 and 2.1, together with the fact that every algebra embeds into an ultraproduct of its finitely generated subalgebras (which follows directly from Theorem 2.3), we obtain the following.

Corollary 2.12. *Any subvariety of $p\mathbb{AL}$ is generated by its finitely generated totally ordered members.*

Furthermore, the variety $p\mathbb{AL}^0$ is clearly term equivalent to $\mathbb{AL}$. We now establish the following result.

Lemma 2.13.
1. *$p\mathbb{AL}_{\mathrm{FSI}} = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1)$.*
2. *$p\mathbb{AL}^+_{\mathrm{FSI}} = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_0, \boldsymbol{R}_1)$.*
3. *$p\mathbb{AL}^-_{\mathrm{FSI}} = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1}, \boldsymbol{R}_0)$.*
4. *$p\mathbb{AL}^0_{\mathrm{FSI}} = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_0) = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_0)$.*
5. *$p\mathbb{AL} = \mathbf{HSP}(\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1)$.*
6. *$p\mathbb{AL}^+ = \mathbf{HSP}(\boldsymbol{R}_0, \boldsymbol{R}_1)$.*
7. *$p\mathbb{AL}^- = \mathbf{HSP}(\boldsymbol{R}_{-1}, \boldsymbol{R}_0)$.*
8. *$p\mathbb{AL}^0 = \mathbf{HSP}(\boldsymbol{R}_0) = \mathbf{HSP}(\boldsymbol{Z}_0)$.*

Proof. Let us recall that pointed Abelian ℓ-groups are finitely subdirectly irreducible iff they are totally ordered.

The inclusions $\supseteq$ are trivial. We prove the remaining inclusions in a convenient order.

4. Follows, since the class $p\mathbb{AL}^0_{\mathrm{FSI}}$ is term equivalent to $\mathbb{AL}_{\mathrm{FSI}}$.

1. Let us fix an arbitrary $\boldsymbol{A}_a \in p\mathbb{AL}_{\mathrm{FSI}}$. Using Theorem 2.7 and [27, Lemma 4.3] we obtain $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{R}_b \mid b \in \mathbb{R}\})$. Since for $b \in \mathbb{R}$ we have $\boldsymbol{R}_b \cong \boldsymbol{R}_{-1}$, $\boldsymbol{R}_b \cong \boldsymbol{R}_0$, or $\boldsymbol{R}_b \cong \boldsymbol{R}_1$ we obtain $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1)$.

[1] It should be stressed that the choice of the symbol f has no algebraic motivation; its origin is purely logical. The symbol is chosen to align with its use as a 'falsum' constant (representing falsehood) in logical systems. This becomes relevant when we consider pointed Abelian ℓ-groups as algebraic counterparts to expansions of Abelian logic, as detailed in [81, 27].

2. For $\boldsymbol{A}_a \in p\mathbb{AL}^0_{\mathrm{FSI}}$ we already know $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_0, \boldsymbol{R}_1)$. Therefore, we fix arbitrary $\boldsymbol{A}_a \in p\mathbb{AL}^+_{\mathrm{FSI}} \setminus p\mathbb{AL}^0_{\mathrm{FSI}}$. By the previous point we know that $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1)$. By [9, Theorem 5.6] we have

$$\mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1) = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1}) \cup \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_0) \cup \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_1).$$

Since $\boldsymbol{A}_a \in p\mathbb{AL}^+_{\mathrm{FSI}} \setminus p\mathbb{AL}^0_{\mathrm{FSI}}$, it follows $a > 0$. Therefore, $\boldsymbol{A}_a \notin \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1})$ since $\boldsymbol{A}_a \not\models \mathtt{f} \leq 0$. Thus $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_0, \boldsymbol{R}_1)$.

3. Can be proved similarly as the previous point.

The other four points follow directly by applying Theorem 2.1. □

We will later prove a strengthening of the second part of Lemma 2.13 in Lemma 2.20.

2.3 The lexicographic decompositions

This section focuses on understanding the structure of the lexicographic product of ℓ-groups. The lexicographic product is a key operation on ℓ-groups. It allows for the creation of otherwise unintuitive ℓ-groups that play an essential role in various classifications (see e.g. the famous Hahn Theorem from [51]). Since we will need to use this tool frequently in the following sections, here we establish several basic properties of this construction. The main result of this section is Theorem 2.23, which is a stronger version of [98, Lemma 4.6], which demonstrates that when classifying universal classes of totally ordered pointed Abelian ℓ-groups, we can restrict our focus to those that are strongly pointed or 0-pointed. We will proceed to the definition of the lexicographic product.

Definition 2.14. *Let $\boldsymbol{A}$ be a totally ordered Abelian ℓ-group and $\boldsymbol{B}$ be an Abelian ℓ-group. We define the Abelian ℓ-group $\boldsymbol{A} \ltimes \boldsymbol{B}$ as the Abelian ℓ-group $\boldsymbol{A} \times \boldsymbol{B}$ with modified lattice operations:*

$$\langle a_1, b_1\rangle \vee \langle a_2, b_2\rangle = \begin{cases} \langle a_1, b_1\rangle & a_1 > a_2 \\ \langle a_2, b_2\rangle & a_1 < a_2 \\ \langle a_1, b_1 \vee b_2\rangle & a_1 = a_2 \end{cases}$$

and

$$\langle a_1, b_1\rangle \wedge \langle a_2, b_2\rangle = \begin{cases} \langle a_1, b_1\rangle & a_1 < a_2 \\ \langle a_2, b_2\rangle & a_1 > a_2 \\ \langle a_1, b_1 \wedge b_2\rangle & a_1 = a_2 \end{cases}$$

Note that for totally ordered Abelian ℓ-groups $\boldsymbol{A}, \boldsymbol{B}$ and an Abelian ℓ-group $\boldsymbol{C}$ it holds that $\boldsymbol{A} \ltimes \boldsymbol{B}$ is totally ordered and moreover $(\boldsymbol{A} \ltimes \boldsymbol{B}) \ltimes \boldsymbol{C} \cong \boldsymbol{A} \ltimes (\boldsymbol{B} \ltimes \boldsymbol{C})$. Therefore, we will commonly omit parentheses and we will just write $\boldsymbol{A} \ltimes \boldsymbol{B} \ltimes \boldsymbol{C}$.

Lemma 2.15. *Let $\boldsymbol{A}$ be a finitely generated totally ordered Abelian ℓ-group and $\boldsymbol{B}$ be a convex ℓ-subgroup of $\boldsymbol{A}$. Then $\boldsymbol{A} \cong (\boldsymbol{A}/\boldsymbol{B}) \ltimes \boldsymbol{B}$.*

Proof. Since $\boldsymbol{A}$ is finitely generated, $\boldsymbol{A}/\boldsymbol{B}$ is also finitely generated. Since $\boldsymbol{A}/\boldsymbol{B}$ is a finitely generated torsion-free Abelian group, it is group-isomorphic (but not necessarily ℓ-group isomorphic) to a free Abelian group (see [93, Theorem 10.19]). Thus the group exact sequence $0 \to \boldsymbol{B} \to^{\iota} \boldsymbol{A} \to^{\pi} \boldsymbol{A}/\boldsymbol{B} \to 0$ splits and there is a group homomorphism $p : \boldsymbol{A}/\boldsymbol{B} \to \boldsymbol{A}$ such that $\pi \circ p = id_{\boldsymbol{A}/\boldsymbol{B}}$. It is well-known (see [93, Lemma 10.3]) that the mapping $\varphi : (\boldsymbol{A}/\boldsymbol{B}) \times \boldsymbol{B} \to \boldsymbol{A}$ defined as $\langle x, y\rangle \mapsto p(x) + \iota(y)$ is a group isomorphism. We want to show that φ is an ℓ-group isomorphism between $(\boldsymbol{A}/\boldsymbol{B}) \ltimes \boldsymbol{B}$ and $\boldsymbol{A}$ as well. We will show that φ is order-preserving. Let us pick $\langle a, b\rangle, \langle c, d\rangle \in (A/B) \times B$ such that $\langle a, b\rangle \leq \langle c, d\rangle$. We distinguish two cases:

1. If $a < c$, we obtain $\pi(p(a) + \iota(b)) < \pi(p(c) + \iota(d))$ since

   $$\pi(p(a)+\iota(b)) = \pi(p(a))+\pi(\iota(b)) = a < c = \pi(p(c))+\pi(\iota(d)) = \pi(p(c)+\iota(d)).$$

   Since π is order preserving and $\boldsymbol{A}$ is totally ordered we derive from $\pi(p(a) + \iota(b)) < \pi(p(c) + \iota(d))$ that $p(a) + \iota(b) < p(c) + \iota(d)$.

2. If $a = c$, then $b \leq d$, so clearly $p(a) + \iota(b) \leq p(c) + \iota(d)$.

This proves that φ is an ℓ-group homomorphism. It remains to prove that φ^{-1} preserves ordering as well. This follows easily from the fact that $\boldsymbol{A}$ is a totally ordered algebra and φ is an order preserving isomorphism. Thus φ is an ℓ-group isomorphism between $(\boldsymbol{A}/\boldsymbol{B}) \ltimes \boldsymbol{B}$ and $\boldsymbol{A}$. □

Generally, it is well known that one can define a lexicographic ordering on a totally ordered Abelian group $\boldsymbol{A}$, provided one has a short exact sequence of Abelian groups $0 \to \boldsymbol{B} \to \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{B} \to 0$ and $\boldsymbol{B}, \boldsymbol{A}/\boldsymbol{B}$ are totally ordered [29, Problem 1.8]. However, one still needs to assume that $\boldsymbol{A}/\boldsymbol{B}$ is finitely generated to prove that the sequence $0 \to \boldsymbol{B} \to \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{B} \to 0$ splits. Let us also remark that one can easily prove a pointed version of this lemma by just verifying that the isomorphism in the proof of Lemma 2.15 preserves the point structure.

Lemma 2.16. *Let $\boldsymbol{A}_b$ be a finitely generated totally ordered pointed Abelian ℓ-group and $\boldsymbol{B}_b$ be a pointed convex ℓ-subgroup of $\boldsymbol{A}_b$. Then $\boldsymbol{A}_b \cong (\boldsymbol{A}/\boldsymbol{B})_0 \ltimes \boldsymbol{B}_b$.*

Definition 2.17. *Let $\boldsymbol{A}_a \in p\mathbb{AL}$. We say a is a* strong unit *of $\boldsymbol{A}$ if for each $b \in A$ there is $z \in \mathbb{Z}$ such that $z \cdot a \geq b$. We say that a pointed Abelian ℓ-group $\boldsymbol{A}_a$ is* strongly pointed*, whenever a is a strong unit of $\boldsymbol{A}$.*

Let us note that for any pointed Abelian ℓ-group $\boldsymbol{A}_b$ with $b \neq 0$, there is a unique strongly pointed convex ℓ-subgroup $\boldsymbol{B}_b$, where $\boldsymbol{B}$ is the ℓ-subgroup of $\boldsymbol{A}$ generated (as a convex ℓ-subgroup) by b, expanded by b as its designated element.

Lemma 2.18. *Let $\boldsymbol{A}_a$ be a non-trivial pointed Abelian ℓ-group. Moreover, assume there exists a strong unit of $\boldsymbol{A}$. Then $\boldsymbol{Z}_0 \ltimes \boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{A}_a)$.*

Proof. Let us denote a strong unit of $\boldsymbol{A}$ by b. Consider an ultrapower of $\boldsymbol{A}_a$ defined as $\boldsymbol{C} = \prod_{i\in\omega} \boldsymbol{A}_a/\mathfrak{U}$, where $\mathfrak{U}$ is a non-principal ultrafilter on ω and let $\iota : d \mapsto \langle d, \ldots, d\rangle$ be the canonical embedding of $\boldsymbol{A}$ into $\boldsymbol{C}$. Consider the element $c \in C$ defined as $c = \langle |b|, 2|b|, 3|b|, \ldots\rangle$. Since the set $\{n \cdot |b| \geq |d|\}$ is cofinite for any $d \in \boldsymbol{A}$, we have $\{n \mid n \cdot |b| \geq |d|\} \in \mathfrak{U}$ and thus $c > |\iota(d)|$ for any

$d \in \boldsymbol{A}$. Let us denote by $\boldsymbol{B}$ the ℓ-subgroup of $\boldsymbol{C}$ generated by $\iota[A] \cup \{c\}$. For any $d \in A$ we have $-c < \iota(d) < c$ thus we can uniquely express any element of B as $n \cdot c + \iota(d)$ for some $n \in \mathbb{Z}$ and $d \in A$. For any $n_1, n_2 \in \mathbb{Z}$ and $d_1, d_2 \in A$ we have $n_1 \cdot c + \iota(d_1) \geq n_2 \cdot c + \iota(d_2)$ iff $n_1 > n_2$ or $n_1 = n_2$ and $d_1 \geq d_2$. This proves $\boldsymbol{B}_{\iota(a)} \cong \boldsymbol{Z}_0 \ltimes \boldsymbol{A}_a$. Since $\boldsymbol{B}_{\iota(a)} \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{A}_a)$, we conclude $\boldsymbol{Z}_0 \ltimes \boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{A}_a)$. $\square$

Corollary 2.19. *$p\mathbb{AL}^0$ is the smallest nontrivial subvariety of $p\mathbb{AL}$. Alternatively, we can say that any non-trivial proper subvariety of $p\mathbb{AL}$ contains $p\mathbb{AL}^0$ as a subvariety.*

Proof. Let $\mathbb{K}$ be a nontrivial subvariety of $p\mathbb{AL}$. Take any non-trivial totally ordered $\boldsymbol{A}_a \in \mathbb{K}$. Then $\boldsymbol{A}_a$ contains a pointed ℓ-subgroup isomorphic to $\boldsymbol{Z}_1$, $\boldsymbol{Z}_0$ or $\boldsymbol{Z}_{-1}$. If we get $\boldsymbol{Z}_1 \in \mathbb{K}$ or $\boldsymbol{Z}_{-1} \in \mathbb{K}$, by Lemma 2.18 we get $\boldsymbol{Z}_0 \ltimes \boldsymbol{Z}_1 \in \mathbb{K}$ or $\boldsymbol{Z}_0 \ltimes \boldsymbol{Z}_{-1} \in \mathbb{K}$. Since $\mathbb{K}$ is closed under $\mathbf{H}$, we get $\boldsymbol{Z}_0 \in \mathbb{K}$ in all cases. Therefore, $\boldsymbol{Z}_0 \in \mathbb{K}$ and by Lemma 2.13 we conclude that $p\mathbb{AL}^0 \subseteq \mathbb{K}$. $\square$

Using this corollary we get the stronger version of Lemma 2.13.

Lemma 2.20. *1. $p\mathbb{AL} = \mathbf{HSP}(\boldsymbol{R}_{-1}, \boldsymbol{R}_1)$.*

2. $p\mathbb{AL}^+ = \mathbf{HSP}(\boldsymbol{R}_1)$.

3. $p\mathbb{AL}^- = \mathbf{HSP}(\boldsymbol{R}_{-1})$.

4. $p\mathbb{AL}^0 = \mathbf{HSP}(\boldsymbol{R}_0) = \mathbf{HSP}(\boldsymbol{Z}_0)$.

We will need to understand how partial embedding interacts with lexicographic products.

Lemma 2.21. *Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ be totally ordered Abelian ℓ-groups and assume $\boldsymbol{A}$ partially embeds into $\boldsymbol{B}$. Then $\boldsymbol{C} \ltimes \boldsymbol{A} \ltimes \boldsymbol{D}$ partially embeds into $\boldsymbol{C} \ltimes \boldsymbol{B} \ltimes \boldsymbol{D}$.*

Equivalently, $\boldsymbol{A} \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{B})$ implies $\boldsymbol{C} \ltimes \boldsymbol{A} \ltimes \boldsymbol{D} \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{C} \ltimes \boldsymbol{B} \ltimes \boldsymbol{D})$.

Proof. Let $\{\varphi_F\}_{F \subseteq A, |F| < \omega}$ be a family of partial embeddings from $\boldsymbol{A}$ to $\boldsymbol{B}$. Denote by π_A the projection from $\boldsymbol{C} \ltimes \boldsymbol{A} \ltimes \boldsymbol{D}$ to $\boldsymbol{A}$. We claim that $\{\psi_G\}_{G \subseteq C \times A \times D, |G| < \omega}$, where $\psi_G : \langle c, a, d \rangle \mapsto \langle c, \varphi_{\pi_A[G]}(a), d \rangle$, is a family of partial embeddings from $\boldsymbol{C} \ltimes \boldsymbol{A} \ltimes \boldsymbol{D}$ to $\boldsymbol{C} \ltimes \boldsymbol{B} \ltimes \boldsymbol{D}$.

Let us fix G a finite subset of $C \times A \times D$. We show ψ_G is a partial embedding. Assume $\langle c_1, a_1, d_1 \rangle, \langle c_2, a_2, d_2 \rangle, \langle c_1 + c_2, a_1 + a_2, d_1 + d_2 \rangle \in G$. Consequently $a_1, a_2, a_1 + a_2 \in \pi_A[G]$ and we have

$$\psi_G(\langle c_1, a_1, d_1 \rangle) + \psi_G(\langle c_2, a_2, d_2 \rangle) = \langle c_1, \varphi_{\pi_A[G]}(a_1), d_1 \rangle + \langle c_2, \varphi_{\pi_A[G]}(a_2), d_2 \rangle =$$
$$\langle c_1 + c_2, \varphi_{\pi_A[G]}(a_1) + \varphi_{\pi_A[G]}(a_2), d_1 + d_2 \rangle = \langle c_1 + c_2, \varphi_{\pi_A[G]}(a_1 + a_2), d_1 + d_2 \rangle =$$
$$= \psi_G(\langle c_1 + c_2, a_1 + a_2, d_1 + d_2 \rangle).$$

In a similar fashion we can check that if $\langle c, a, d \rangle, \langle -c, -a, -d \rangle \in G$ we have $-\psi_G \langle c, a, d \rangle = \psi_G \langle -c, -a, -d \rangle$ and always we have $\psi_G \langle 0, 0, 0 \rangle = \langle 0, 0, 0 \rangle$.

Therefore, ψ_G preserves addition, subtraction and zero constant. It remains to check ψ_G preserves operations $\vee, \wedge$. Since $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are totally ordered, it is enough to show that ψ_G preserves the lattice ordering. Let us assume that $\langle c_1, a_1, d_1 \rangle, \langle c_2, a_2, d_2 \rangle \in G$ and $\langle c_1, a_1, d_1 \rangle < \langle c_2, a_2, d_2 \rangle$. Let us distinguish three cases:

1. If $c_1 < c_2$ we get
$$\psi_G(\langle c_1, a_1, d_1\rangle) = \langle c_1, \varphi_{\pi_A[G]}(a_1), d_1\rangle < \langle c_2, \varphi_{\pi_A[G]}(a_2), d_2\rangle = \psi_G(\langle c_2, a_2, d_2\rangle).$$
2. If $c_1 = c_2$ and $a_1 < a_2$ we get $\varphi_{\pi_A[G]}(a_1) < \varphi_{\pi_A[G]}(a_2)$ (by injectivity and order-preservation of $\varphi_{\pi_A[G]}$) and thus we get
$$\psi_G(\langle c_1, a_1, d_1\rangle) = \langle c_1, \varphi_{\pi_A[G]}(a_1), d_1\rangle < \langle c_1, \varphi_{\pi_A[G]}(a_2), d_2\rangle = \psi_G(\langle c_2, a_2, d_2\rangle).$$
3. If $c_1 = c_2, a_1 = a_2$ and $d_1 < d_2$ we get
$$\psi_G(\langle c_1, a_1, d_1\rangle) = \langle c_1, \varphi_{\pi_A[G]}(a_1), d_1\rangle < \langle c_1, \varphi_{\pi_A[G]}(a_1), d_2\rangle = \psi_G(\langle c_2, a_2, d_2\rangle).$$

This shows ψ_G is indeed a partial embedding of G into $\boldsymbol{C} \ltimes \boldsymbol{B} \ltimes \boldsymbol{D}$. Since the finite set G was arbitrary, we get that $\boldsymbol{C} \ltimes \boldsymbol{A} \ltimes \boldsymbol{D}$ partially embeds into $\boldsymbol{C} \ltimes \boldsymbol{B} \ltimes \boldsymbol{D}$. ☐

We can easily observe that the lemma also holds for pointed Abelian ℓ-groups.

Lemma 2.22. *Let $\boldsymbol{A}_a, \boldsymbol{B}_b, \boldsymbol{C}_c, \boldsymbol{D}_d$ be totally ordered pointed Abelian ℓ-groups and assume $\boldsymbol{A}_a$ partially embeds into $\boldsymbol{B}_b$. Then $\boldsymbol{C}_c \ltimes \boldsymbol{A}_a \ltimes \boldsymbol{D}_d$ partially embeds into $\boldsymbol{C}_c \ltimes \boldsymbol{B}_b \ltimes \boldsymbol{D}_d$.*

Equivalently, $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{B}_b)$ implies $\boldsymbol{C}_c \ltimes \boldsymbol{A}_a \ltimes \boldsymbol{D}_d \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{C}_c \ltimes \boldsymbol{B}_b \ltimes \boldsymbol{D}_d)$.

Finally, we conclude this section with the following theorem which is a strengthening of [98, Lemma 4.6].

Theorem 2.23. *Let $\boldsymbol{A}_b$ be a finitely generated totally ordered pointed Abelian ℓ-group with $b \neq 0$ and $\boldsymbol{B}_b$ be its strongly pointed convex ℓ-subgroup. Then $\mathbf{ISP}_{\mathrm{U}}(\boldsymbol{A}_b) = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{B}_b)$.*

Proof. Clearly, $\boldsymbol{B}_b \in \mathbf{S}(\boldsymbol{A}_b)$ and thus $\boldsymbol{B}_b \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{A}_b)$.

We prove the other implication. By Lemma 2.18 we have $\boldsymbol{Z}_0 \ltimes \boldsymbol{B}_b \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{B}_b)$. Also, by Lemma 2.13 we have $(\boldsymbol{A}/\boldsymbol{B})_0 \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_0)$ and by Lemma 2.22 we have $(\boldsymbol{A}/\boldsymbol{B})_0 \ltimes \boldsymbol{B}_b \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_0 \ltimes \boldsymbol{B}_b)$. By Lemma 2.16 we get $(\boldsymbol{A}/\boldsymbol{B})_0 \ltimes \boldsymbol{B}_b \cong \boldsymbol{A}_b$. Therefore we showed $\boldsymbol{A}_b \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{B}_b)$, which completes the proof. ☐

Theorem 2.23 tells us that the universal theory of any totally ordered pointed Abelian ℓ-group is equal to the universal theory generated by its strongly pointed convex ℓ-subgroup or the universal theory generated by $\boldsymbol{Z}_0$. In other words, when classifying universal classes of totally ordered pointed Abelian ℓ-groups, we can restrict our focus to those which are strongly pointed or 0-pointed.

From Theorem 2.23 and Corollary 2.19 we obtain the following two additional corollaries.

Corollary 2.24. *Every non-trivial subvariety of $\mathrm{p}\mathbb{AL}$ that is not equal to $\mathrm{p}\mathbb{AL}^0$ is generated by its finitely generated totally ordered strongly pointed members.*

Corollary 2.25. *Every non-trivial subquasivariety of $\mathrm{p}\mathbb{AL}$ generated by its totally ordered members is also generated by those of its finitely generated totally ordered members that are either strongly pointed or 0-pointed.*

2.4 Characterization of varieties generated by a single finitely generated totally ordered pointed Abelian ℓ-group

In this section we describe all join-irreducible subvarieties of pointed Abelian ℓ-groups. Although one could obtain the result of this section using Theorem 2.23 from Section 2.3, the Mundici functor and the Komori classification, we have chosen a different, more self-contained approach that does not rely on the theory of MV-algebras. However, a reader familiar with Komori classification of MV-algebras will find some of the proofs here possibly familiar, since they often use similar techniques (for comparison see [21]).

Lemma 2.26. *Let $\boldsymbol{A}_a, \boldsymbol{B}_b, \boldsymbol{C}_c$ be totally ordered pointed Abelian ℓ-groups and $\boldsymbol{D}_d$ be a pointed Abelian ℓ-group. Let $\psi : \boldsymbol{A}_a \to \boldsymbol{B}_b$ be an injective homomorphism. Then $\varphi : \boldsymbol{C}_c \ltimes \boldsymbol{A}_a \ltimes \boldsymbol{D}_d \to \boldsymbol{C}_c \ltimes \boldsymbol{B}_b \ltimes \boldsymbol{D}_d$ defined as $\langle x, y, z\rangle \mapsto \langle x, \psi(y), z\rangle$ is an injective homomorphism as well.*

Moreover, if ψ is an isomorphism then also φ is an isomorphism.

Proof. First, $\langle x, y, z\rangle \mapsto \langle x, \psi(y), z\rangle$ is clearly an injective homomorphism of groups $\boldsymbol{C}_c \times \boldsymbol{A}_a \times \boldsymbol{D}_d$ and $\boldsymbol{C}_c \times \boldsymbol{B}_b \times \boldsymbol{D}_d$.

We check that φ preserves the lattice operations of the lexicographic product. For each $x_1, x_2 \in C, y_1, y_2 \in A$ and $z_1, z_2 \in D$ we have

$$\begin{aligned}
\varphi(\langle x_1, y_1, z_1\rangle) \vee \varphi(\langle x_2, y_2, z_2\rangle) &= \langle x_1, \psi(y_1), z_1\rangle \vee \langle x_2, \psi(y_2), z_2\rangle = \\
&= \begin{cases} \langle x_1, \psi(y_1), z_1\rangle & \text{if } x_1 > x_2 \\ \langle x_2, \psi(y_2), z_2\rangle & \text{if } x_1 < x_2 \\ \langle x_1, \psi(y_1), z_1\rangle & \text{if } x_1 = x_2, y_1 > y_2 \\ \langle x_2, \psi(y_2), z_2\rangle & \text{if } x_1 = x_2, y_1 < y_2 \\ \langle x_1, \psi(y_1), z_1 \vee z_2\rangle & \text{if } x_1 = x_2, y_1 = y_2 \end{cases} \\
&= \begin{cases} \varphi(\langle x_1, y_1, z_1\rangle) & \text{if } x_1 > x_2 \\ \varphi(\langle x_2, y_2, z_2\rangle) & \text{if } x_1 < x_2 \\ \varphi(\langle x_1, y_1, z_1\rangle) & \text{if } x_1 = x_2, y_1 > y_2 \\ \varphi(\langle x_2, y_2, z_2\rangle) & \text{if } x_1 = x_2, y_1 < y_2 \\ \varphi(\langle x_1, y_1, z_1 \vee z_2\rangle) & \text{if } x_1 = x_2, y_1 = y_2 \end{cases} \\
&= \varphi(\langle x_1, y_1, z_1\rangle \vee \langle x_2, y_2, z_2\rangle).
\end{aligned}$$

This shows φ preserves $\vee$. Similarly, we can show that φ preserves $\wedge$. Therefore, φ is an injective homomorphism.

In the case when ψ is an isomorphism, there exists the inverse isomorphism $\psi^{-1} : \boldsymbol{B}_b \to \boldsymbol{A}_a$. By the previous part of the proof $\varphi^{-1} : \boldsymbol{C}_c \ltimes \boldsymbol{B}_b \ltimes \boldsymbol{D}_d \to \boldsymbol{C}_c \ltimes \boldsymbol{A}_a \ltimes \boldsymbol{D}_d$ defined as $\langle x, y, z\rangle \mapsto \langle x, \psi^{-1}(y), z\rangle$ is an injective homomorphism. Clearly, φ^{-1} is also inverse to φ, which shows φ must be an isomorphism. □

Lemma 2.27. *Let $a_1, \ldots, a_m \in \mathbb{R}$ and let $f : (\boldsymbol{R} \ltimes \boldsymbol{R})^m \to \boldsymbol{R} \ltimes \boldsymbol{R}$ be a pointed Abelian ℓ-group term function. Let $\pi_2 : \boldsymbol{R} \ltimes \boldsymbol{R} \to \boldsymbol{R}$ be a projection to the second coordinate. Then the m-ary function*

$$g(x_1, \ldots, x_m) := \pi_2(f(\langle a_1, x_1\rangle, \ldots, \langle a_m, x_m\rangle))$$

is a continuous function from $\boldsymbol{R}^m$ to $\boldsymbol{R}$. In particular, every pointed Abelian ℓ-group term function on $\boldsymbol{R}$ is continuous.

Proof. We show that g is a composition of continuous functions. We proceed by induction on the complexity of the term f. Clearly, constants and projections are continuous functions. Moreover, addition and subtraction are defined component-wise, thus they are continuous functions.

It remains to check maximum and minimum. For any constants $b_1, b_2 \in \mathbb{R}$ we define

$$x_1 \vee_{b_1,b_2} x_2 = \begin{cases} x_1 & \text{if } b_1 > b_2 \\ x_2 & \text{if } b_1 < b_2 \\ x_1 \vee x_2 & \text{if } b_1 = b_2 \end{cases}$$

and

$$x_1 \wedge_{b_1,b_2} x_2 = \begin{cases} x_2 & \text{if } b_1 > b_2 \\ x_1 & \text{if } b_1 < b_2 \\ x_1 \wedge x_2 & \text{if } b_1 = b_2 \end{cases}.$$

Thus for all $b_1, b_2 \in \mathbb{R}$ we obtain that $\vee_{b_1,b_2}$ and $\wedge_{b_1,b_2}$ are continuous functions. Since g is a composition of addition, subtraction, constants and functions $\vee_{b_i,b_j}$ and $\wedge_{b_i,b_j}$ for some $b_i, b_j \in \mathbb{R}$, we obtain that g is continuous as well. □

The following lemma is the generalization of [21, Propositions 8.1.1 and 8.1.2]

Lemma 2.28. *1. Let $\boldsymbol{A}_a$ be a pointed ℓ-subgroup of $\boldsymbol{R}_a$ and A be a dense subset of $\mathbb{R}$. Then $\mathbf{HSP}(\boldsymbol{R}_a) = \mathbf{HSP}(\boldsymbol{A}_a)$.*

2. Let I be an infinite set of positive integers, $b > 0$ and $\boldsymbol{A}_a$ be a pointed ℓ-subgroup of $\boldsymbol{R}_a$. Then $\mathbf{HSP}(\boldsymbol{A}_a \ltimes \boldsymbol{R}_b) = \mathbf{HSP}(\{\boldsymbol{A}_a \ltimes \boldsymbol{Z}_n \mid n \in I\})$.

3. Let I be an infinite set of negative integers, $b < 0$ and $\boldsymbol{A}_a$ be a pointed ℓ-subgroup of $\boldsymbol{R}_a$. Then $\mathbf{HSP}(\boldsymbol{A}_a \ltimes \boldsymbol{R}_b) = \mathbf{HSP}(\{\boldsymbol{A}_a \ltimes \boldsymbol{Z}_n \mid n \in I\})$.

Proof. 1. Clearly, $\boldsymbol{A}_a \in \mathbf{HSP}(\boldsymbol{R}_a)$.

For the other inclusion assume that some equation $\alpha(\vec{x}) = 0$ does not hold in $\boldsymbol{R}_a$. Thus there is $\vec{r} \in \mathbb{R}^m$, where m is the number of variables in α, such that $\alpha(\vec{r}) \neq 0$. By Lemma 2.27 the function $\vec{x} \mapsto \alpha(\vec{x})$ is continuous. Since the set $\{x \in \mathbb{R} \mid x \neq 0\}$ is open in $\mathbb{R}$, the set $O := \{\vec{x} \in \mathbb{R}^m \mid \alpha(\vec{x}) \neq 0\}$ is open in $\mathbb{R}^m$. We know that $\vec{r} \in O$ thus the set O is nonempty. Since A is a dense subset of $\mathbb{R}$, by [85, Theorem 19.5] A^m is a dense subset of $\mathbb{R}^m$ and therefore $A^m \cap O \neq \emptyset$. Thus there exists $\vec{a} \in A^m \cap O$ such that $\alpha(\vec{a}) \neq 0$. This shows that the equation $\alpha(\vec{x}) = 0$ is not valid in $\boldsymbol{A}_a$. Since $\alpha(\vec{x}) = 0$ was an arbitrary equation we derive that $\boldsymbol{R}_a \in \mathbf{HSP}(\boldsymbol{A}_a)$. Thus $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{R}_a)$.

2. Without loss of generality we can assume $b = 1$, since $\boldsymbol{R}_1 \cong \boldsymbol{R}_b$ for any $b > 0$ and by Lemma 2.26 $\boldsymbol{A}_a \ltimes \boldsymbol{R}_1 \cong \boldsymbol{A}_a \ltimes \boldsymbol{R}_b$ for any such b. Since $\boldsymbol{Z}_n \in \mathbf{IS}(\boldsymbol{R}_1)$ for each $n \in I$, by Lemma 2.26 we get $\boldsymbol{A}_a \ltimes \boldsymbol{Z}_n \in \mathbf{IS}(\boldsymbol{A}_a \ltimes \boldsymbol{R}_1)$ for each $n \in I$.

For the other inclusion assume that some equation $\alpha(\langle\vec{y},\vec{x}\rangle) = 0$ does not hold in $\boldsymbol{A}_a \ltimes \boldsymbol{R}_1$. Let m be the number of variables in α. There exist $\langle\vec{s},\vec{r}\rangle \in A^m \times \mathbb{R}^m$ such that $\alpha(\langle\vec{s},\vec{r}\rangle) \neq 0$ in $\boldsymbol{A}_a \ltimes \boldsymbol{R}_1$. By Lemma 2.27 the function $\vec{x} \to \alpha(\vec{s},\vec{x})$ is continuous and thus the set $O := \{\vec{x} \in \boldsymbol{R}^m \mid \alpha(\langle\vec{s},\vec{x}\rangle) \neq 0\}$ is open in $\boldsymbol{R}^m$ and contains $\vec{r}$. Therefore, there is $0 < \epsilon \in \mathbb{R}$ such that $V := \{\vec{x} \mid |\vec{x}-\vec{r}| < m{\cdot}\epsilon\} \subseteq O$. We fix $n \in I$ such that $n > \frac{\sqrt{m}}{2\epsilon}$. Let us denote $\boldsymbol{B}$ the pointed ℓ-subgroup of $\boldsymbol{R}$ generated by $\frac{1}{n}$. We have $\epsilon > \frac{\sqrt{m}}{2n}$ and by [31, Section 5, Chapter 4] it follows that $\boldsymbol{B}_1^m \cap V \neq \emptyset$. Since $V \subseteq O$ the equation $\alpha(\vec{s},\vec{x}) = 0$ is not valid in $\boldsymbol{A}_a \ltimes \boldsymbol{B}_1$. Since $\boldsymbol{B}_1 \cong \boldsymbol{Z}_n$ by Lemma 2.26 we get $\boldsymbol{A}_a \ltimes \boldsymbol{B}_1 \cong \boldsymbol{A}_a \ltimes \boldsymbol{Z}_n$, which proves $\boldsymbol{A}_a \ltimes \boldsymbol{R}_1 \in \mathbf{HSP}(\{\boldsymbol{A}_a \ltimes \boldsymbol{Z}_n \mid n \in I\})$.

3. This can be proved similarly as the previous point. □

Lemma 2.29. *Let $a, k \in \mathbb{Z}$. Then $\boldsymbol{Z}_a \ltimes \boldsymbol{Z}_k \cong \boldsymbol{Z}_a \ltimes \boldsymbol{Z}_{k+a}$.*

Proof. Consider the mapping $\varphi : \boldsymbol{Z}_a \ltimes \boldsymbol{Z}_k \to \boldsymbol{Z}_a \ltimes \boldsymbol{Z}_{k+a}$ defined as $\langle x, y\rangle \mapsto \langle x, y + x\rangle$. We show φ is an isomorphism of ℓ-groups. Clearly, this is a group isomorphism with inverse mapping φ^{-1} defined as $\langle x, y\rangle \mapsto \langle x, y - x\rangle$. It remains to check φ and φ^{-1} are order preserving. Assume $\langle a_1, b_1\rangle \leq \langle a_2, b_2\rangle$ for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$.

1. If $a_1 < a_2$ we have

$$\varphi(\langle a_1, b_1\rangle) = \langle a_1, a_1 + b_1\rangle < \langle a_2, a_2 + b_2\rangle = \varphi(\langle a_2, b_2\rangle).$$

2. If $a_1 = a_2$ we obtain $b_1 \leq b_2$ and $a_1 + b_1 \leq a_2 + b_2$. Therefore,

$$\varphi(\langle a_1, b_1\rangle) = \langle a_1, a_1 + b_1\rangle \leq \langle a_1, a_2 + b_2\rangle = \langle a_2, a_2 + b_2\rangle = \varphi(\langle a_2, b_2\rangle).$$

This shows that φ preserves the ordering. In a similar way one can show that φ^{-1} preserves the ordering as well. Thus φ is indeed an isomorphism. □

Definition 2.30. *Let $\boldsymbol{A}_b$ be a pointed Abelian ℓ-group with $b \neq 0$. A convex ℓ-subgroup of $\boldsymbol{A}$ that is maximal with respect to not containing b is called a* value *of b. If $\boldsymbol{A}_b$ is totally ordered, the value of b is unique and we denote it by $(\boldsymbol{A}_b)$. Furthermore, we say that $\boldsymbol{A}_b$ is p-simple if $(\boldsymbol{A}_b) = \{0\}$.*

For any pointed Abelian ℓ-group $\boldsymbol{A}_b$, $(\boldsymbol{A}_b)$ is not a *pointed* Abelian ℓ-group. To treat $(\boldsymbol{A}_b)$ itself as an algebra in $p\mathbb{AL}$, we must assign it a designated element. When $\boldsymbol{A}$ is finitely generated, Lemma 2.16 applies to establish $\boldsymbol{A} \cong \boldsymbol{A}/(\boldsymbol{A}_b) \ltimes (\boldsymbol{A}_b)$. Under this isomorphism, the original designated element $b \in A$ must map to a specific pair. The first coordinate is its projection $b + (\boldsymbol{A}_b)$ in the quotient, while the second coordinate is a uniquely determined element $c \in (\boldsymbol{A}_b)$. We use this point c as the designated element for $(\boldsymbol{A}_b)$. Henceforth, we adopt the convention that whenever $(\boldsymbol{A}_b)$ is treated as a pointed Abelian ℓ-group, its designated element $\mathtt{f}^{(\boldsymbol{A}_b)}$ is precisely this element c, yielding:

$$\boldsymbol{A}_b \cong (\boldsymbol{A}/(\boldsymbol{A}_b))_{b+(\boldsymbol{A}_b)} \ltimes ((\boldsymbol{A}_b))_c.$$

Let $\boldsymbol{A}_b$ be a totally ordered pointed Abelian ℓ-group with $b \neq 0$ and let $\boldsymbol{B}_b$ be the totally ordered strongly pointed convex ℓ-subgroup of $\boldsymbol{A}_b$. It can be shown that

$(\boldsymbol{A}_b) = (\boldsymbol{B}_b)$. Since $\boldsymbol{B}_b$ is strongly pointed, $(\boldsymbol{B}_b)$ is a maximal convex ℓ-subgroup of $\boldsymbol{B}$ and hence the quotient $\boldsymbol{B}/(\boldsymbol{B}_b)$ must be a simple Abelian ℓ-group. Therefore, according to Hölder's Theorem (see Theorem 2.9), $\boldsymbol{B}_b/(\boldsymbol{B}_b)$ is isomorphic to a pointed ℓ-subgroup of the real numbers. Since any pointed Abelian ℓ-group $\boldsymbol{A}_b$ with $b \neq 0$ has a unique convex strongly pointed ℓ-subgroup, we can state the following definition.

Definition 2.31. *Let $\boldsymbol{A}_b$ be a totally ordered pointed Abelian ℓ-group with $b \neq 0$ and $\boldsymbol{B}_b$ be its strongly pointed convex ℓ-subgroup. We define the rank of $\boldsymbol{A}_b$ as*

$$\mathbf{rank}(\boldsymbol{A}_b) = \begin{cases} n & \text{if } \boldsymbol{B}_b/(\boldsymbol{B}_b) \cong \boldsymbol{Z}_n, \\ \infty & \text{if } \boldsymbol{B}_b/(\boldsymbol{B}_b) \ncong \boldsymbol{Z}_n \text{ for any } n \in \mathbb{Z} \;\&\; b > 0, \\ -\infty & \text{if } \boldsymbol{B}_b/(\boldsymbol{B}_b) \ncong \boldsymbol{Z}_n \text{ for any } n \in \mathbb{Z} \;\&\; b < 0. \end{cases}$$

Moreover, for any totally ordered 0-pointed Abelian ℓ-group $\boldsymbol{A}_0$ we set $\mathbf{rank}(\boldsymbol{A}_0) = 0$.

Lemma 2.32. *Let $\boldsymbol{A}_a$ be a totally ordered strongly pointed Abelian ℓ-group and $\mathbf{rank}(\boldsymbol{A}_a) = n \in \mathbb{Z} \setminus \{0\}$. Moreover, assume $\boldsymbol{A}_a$ is p-simple. Then $\boldsymbol{A}_a \cong \boldsymbol{Z}_n$.*

Proof. Follows from the definitions of rank and p-simple Abelian ℓ-group. □

Lemma 2.33. *Let $\boldsymbol{A}_a$ be a finitely generated totally ordered pointed Abelian ℓ-group which is not p-simple and has rank $n \in \mathbb{Z}\setminus\{0\}$. Then $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0)$.*

Proof. By Theorem 2.23 we can assume without loss of generality that $\boldsymbol{A}_a$ is strongly pointed. Since $\mathbf{rank}(\boldsymbol{A}_a) = n$, by Lemma 2.15 and Lemma 2.26 we obtain $\boldsymbol{A}_a/(\boldsymbol{A}_a) \cong \boldsymbol{Z}_n$ and thus $\boldsymbol{A}_a \cong \boldsymbol{A}_a/(\boldsymbol{A}_a) \ltimes (\boldsymbol{A}_a) \cong \boldsymbol{Z}_n \ltimes (\boldsymbol{A}_a)$. We first show $\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0 \in \mathbf{HSP}(\boldsymbol{Z}_n \ltimes (\boldsymbol{A}_a))$.

Since $\boldsymbol{A}_a$ is finitely generated and totally ordered, $(\boldsymbol{A}_a)$ has a strong unit and thus Lemma 2.18 gives us $\boldsymbol{Z}_0 \ltimes (\boldsymbol{A}_a) \in \mathbf{ISP}_{\mathrm{U}}((\boldsymbol{A}_a))$ and thus by Lemma 2.22 we obtain $\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0 \ltimes (\boldsymbol{A}_a) \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_n \ltimes (\boldsymbol{A}_a))$. Finally, we observe $\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0 \in \mathbf{H}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0 \ltimes (\boldsymbol{A}_a))$ and thus $\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0 \in \mathbf{HSP}(\boldsymbol{Z}_n \ltimes (\boldsymbol{A}_a))$.

We have to show the other inclusion, i.e. that $\boldsymbol{Z}_n \ltimes (\boldsymbol{A}_a) \in \mathbf{HSP}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0)$. By Lemma 2.29 we have $\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_{kn} \in \mathbf{I}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0)$ for each $k \in \mathbb{Z}$. Thus by Lemma 2.28 we obtain $\boldsymbol{Z}_n \ltimes \boldsymbol{R}_1, \boldsymbol{Z}_n \ltimes \boldsymbol{R}_{-1} \in \mathbf{HSP}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0)$. Also, by Lemma 2.13 we have $\boldsymbol{R}_0 \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_0)$ and thus by Lemma 2.22 and Lemma 2.26 we have $\boldsymbol{Z}_n \ltimes \boldsymbol{R}_0 \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0)$.

Since $(\boldsymbol{A}_a)$ is totally ordered, by Lemma 2.13 and [9, Theorem 5.6] we have

$$(\boldsymbol{A}_a) \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1}, \boldsymbol{R}_0, \boldsymbol{R}_1) = \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_{-1}) \cup \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_0) \cup \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{R}_1).$$

By Lemma 2.22 we obtain

$$\boldsymbol{Z}_n \ltimes (\boldsymbol{A}_a) \in \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_n \ltimes \boldsymbol{R}_{-1}) \cup \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_n \ltimes \boldsymbol{R}_0) \cup \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_n \ltimes \boldsymbol{R}_1).$$

Since

$$\boldsymbol{Z}_n \ltimes \boldsymbol{R}_{-1}, \boldsymbol{Z}_n \ltimes \boldsymbol{R}_0, \boldsymbol{Z}_n \ltimes \boldsymbol{R}_1 \in \mathbf{HSP}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0),$$

we get $\boldsymbol{Z}_n \ltimes (\boldsymbol{A}_a) \in \mathbf{HSP}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0)$. This proves the claim. □

Lemma 2.34. *Let $\boldsymbol{A}_a$ be a finitely generated totally ordered pointed Abelian ℓ-group.*

1. *If* $\mathbf{rank}(\boldsymbol{A}_a) = \infty$ *then* $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{R}_1)$.

2. *If* $\mathbf{rank}(\boldsymbol{A}_a) = -\infty$ *then* $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{R}_{-1})$.

Proof. By Theorem 2.23 we can assume without loss of generality that $\boldsymbol{A}_a$ is strongly pointed. Let us assume $\mathbf{rank}(\boldsymbol{A}_a) = \infty$. Since $(\boldsymbol{A}_a)$ is a maximal convex ℓ-subgroup of $\boldsymbol{A}_a$, the Abelian ℓ-group $\boldsymbol{A}_a/(\boldsymbol{A}_a)$ must be p-simple and thus by Theorem 2.9 $\boldsymbol{A}_a/(\boldsymbol{A}_a) \in \mathbf{IS}(\boldsymbol{R}_b)$ for some $0 < b \in \mathbb{R}$. Let us without loss of generality assume $\boldsymbol{A}_a/(\boldsymbol{A}_a) \in \mathbf{S}(\boldsymbol{R}_1)$. By Lemma 2.10 we obtain $\boldsymbol{A}_a/(\boldsymbol{A}_a) \cong \boldsymbol{Z}_n$ for some $n \in \mathbb{Z}$ or the universe of $\boldsymbol{A}_a/(\boldsymbol{A}_a)$ is a dense subset of $\mathbb{R}$. Since $\mathbf{rank}(\boldsymbol{A}_a) = \infty$, the universe of $\boldsymbol{A}_a/(\boldsymbol{A}_a)$ is dense in $\mathbb{R}$. Thus by Lemma 2.28 we get $\mathbf{HSP}(\boldsymbol{A}_a/(\boldsymbol{A}_a)) = \mathbf{HSP}(\boldsymbol{R}_1)$. Since $\boldsymbol{A}_a/(\boldsymbol{A}_a) \in \mathbf{H}(\boldsymbol{A}_a)$ and by Lemma 2.20 $\boldsymbol{A}_a \in \mathbf{HSP}(\boldsymbol{R}_1)$ we obtain $\mathbf{HSP}(\boldsymbol{R}_1) = \mathbf{HSP}(\boldsymbol{A}_a)$.

The case $\mathbf{rank}(\boldsymbol{A}_a) = -\infty$ is analogous. □

Altogether, we obtained the following characterization.

Theorem 2.35. *Let $\boldsymbol{A}_a$ be a finitely generated non-trivial totally ordered pointed Abelian ℓ-group. Then the following holds.*

1. $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{Z}_0) = \mathbf{HSP}(\boldsymbol{R}_0)$ *iff* $a = 0$.

2. $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{Z}_n)$ *iff* $\boldsymbol{A}_a$ *is p-simple and* $\mathbf{rank}(\boldsymbol{A}_a) = n$.

3. $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0)$ *iff* $\boldsymbol{A}_a$ *is not p-simple and* $\mathbf{rank}(\boldsymbol{A}_a) = n$.

4. $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{R}_1)$ *iff* $\mathbf{rank}(\boldsymbol{A}_a) = \infty$.

5. $\mathbf{HSP}(\boldsymbol{A}_a) = \mathbf{HSP}(\boldsymbol{R}_{-1})$ *iff* $\mathbf{rank}(\boldsymbol{A}_a) = -\infty$.

2.5 Axiomatization of subvarieties of $p\mathbb{AL}$

In this section we focus on axiomatizing all the subvarieties of $p\mathbb{AL}^+$. To achieve this we define the following equations: $(\mathrm{s\text{-}rank_n})$,$(\mathrm{rank_n})$, $(\mathrm{div_{p,n}})$ and $(\mathrm{mix_{p,n}})$. Observe that all these four equations are valid for all parameters in all negatively pointed Abelian ℓ-groups. Consequently, we introduce another four dual equations $(\mathrm{s\text{-}rank^d_n})$,$(\mathrm{rank^d_n})$, $(\mathrm{div^d_{p,n}})$ and $(\mathrm{mix^d_{p,n}})$, which we use to characterize all subvarieties of $p\mathbb{AL}^-$. Finally, we combine these strategies to provide a characterization of all subvarieties of $p\mathbb{AL}$.

Lemma 2.36. *Let $\boldsymbol{A}_a$ be a finitely generated totally ordered pointed Abelian ℓ-group and $n \geq 0$. Let us consider the following equation.*

$$(n \cdot x - \mathtt{f}) \vee (-x) \geq 0. \qquad (\mathrm{s\text{-}rank}_n)$$

This equation is satisfied in $\boldsymbol{A}_a$ iff $a \leq 0$ or $\boldsymbol{A}_a$ is p-simple with $\mathbf{rank}(\boldsymbol{A}_a) \leq n$.

Proof. First, if $a = 0$, the equation is satisfied by Corollary 2.19. Now, we can assume $a \neq 0$. By Theorem 2.23 we can assume $\boldsymbol{A}_a$ is strongly pointed. Thus by Lemma 2.32 we need to check that

$$\boldsymbol{A}_a \models (\mathrm{s\text{-}rank_n}) \text{ iff } a \leq 0 \text{ or } \boldsymbol{A} \cong \boldsymbol{Z}_m \text{ for some } m \leq n.$$

Let e be a fixed evaluation on $\boldsymbol{A}_a$. We have $\boldsymbol{A}_a \models_e$ (s-rank$_\mathrm{n}$) iff $\boldsymbol{A}_a \models_e x \leq 0$ or $\boldsymbol{A}_a \models_e n \cdot x \geq a$. Therefore $\boldsymbol{A}_a \models$ (s-rank$_\mathrm{n}$) iff $\boldsymbol{A}_a \models_e n \cdot x \geq a$ for all evaluations e, such that $e(x) > 0$.

We need to distinguish several cases.

1. For $a \leq 0$ and $e(x) > 0$ we have $n \cdot e(x) > 0 \geq a$. Thus $\boldsymbol{A}_a \models$ (s-rank$_\mathrm{n}$) for $a \leq 0$.

2. For $\boldsymbol{A}_a = \boldsymbol{Z}_m$, $0 < m \leq n$ and $0 < e(x)$ we have $n \cdot e(x) \geq n \cdot 1 \geq m$. Therefore $\boldsymbol{Z}_m \models$ (s-rank$_\mathrm{n}$) for $m \leq n$.

3. For $\boldsymbol{A}_a = \boldsymbol{Z}_m$, $0 \leq n < m$ set $e(x) = 1$. We have $n \cdot e(x) = n < m$, thus $\boldsymbol{Z}_m \not\models$ (s-rank$_\mathrm{n}$) for $m > n$.

4. For $a > 0$ and $\boldsymbol{A}_a$ not p-simple we have $(\boldsymbol{A}_a) \neq 0$ and by Lemma 2.15 there is an isomorphism $\iota : \boldsymbol{A}_a/(\boldsymbol{A}_a) \ltimes (\boldsymbol{A}_a) \to \boldsymbol{A}_a$. Pick $0 < v \in (\boldsymbol{A}_a)$ and we consider an evaluation e on $\boldsymbol{A}$, where $e(x) = \iota(\langle 0, v\rangle)$. Since $n \cdot v \in (\boldsymbol{A}_a)$, we have $n \cdot e(x) = \iota(\langle 0, n \cdot v\rangle) < a$, thus showing $\boldsymbol{A}_a \not\models$ (s-rank$_\mathrm{n}$) for not p-simple $\boldsymbol{A}_a$ with $a > 0$. □

Let us note that (s-rank$_0$) is equivalent to the equation $f \leq 0$. For $m \in \mathbb{Z} \setminus \{0\}$ we denote div(m) to be the set of all divisors of m.

Lemma 2.37. *Let $m \in \mathbb{Z}$, $n, p \in \mathbb{N}$ and $m \leq n$.*

Let us consider the following equation.

$$((n+1) \cdot ((p \cdot x - \mathtt{f}) \vee (\mathtt{f} - p \cdot x)) - \mathtt{f}) \vee -x \geq 0 \qquad (\mathrm{div}_{p,n})$$

The following conditions are equivalent.

1. $\boldsymbol{Z}_m \models (\mathrm{div}_{\mathrm{p,n}})$,

2. $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{div}_{\mathrm{p,n}})$,

3. $m \leq 0$ *or* $p \notin \mathrm{div}(m)$.

Proof. Since $\boldsymbol{Z}_m \in \mathbf{H}(\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0)$, it is enough to check that $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{div}_{\mathrm{p,n}})$ if $m \leq 0$ or $p \notin \mathrm{div}(m)$ and that $\boldsymbol{Z}_m \not\models (\mathrm{div}_{\mathrm{p,n}})$ if $0 < m$ and $p \in \mathrm{div}(m)$.

1. Let e be a fixed evaluation on $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0$. We have

$$\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models_e (\mathrm{div}_{\mathrm{p,n}}) \text{ iff } e(x) \leq 0 \text{ or } (n+1) \cdot |p \cdot e(x) - \langle m, 0\rangle| \geq \langle m, 0\rangle.$$

Therefore $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{div}_{\mathrm{p,n}})$ iff $(n+1) \cdot |p \cdot e(x) - \langle m, 0\rangle| \geq \langle m, 0\rangle$ for all evaluations e, such that $e(x) > 0$. For an evaluation $e(x) > 0$ and $m \leq 0$, we get

$$(n+1) \cdot |p \cdot e(x) - \langle m, 0\rangle| \geq \langle 0, 0\rangle \geq \langle m, 0\rangle.$$

This tells us that $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{div}_{\mathrm{p,n}})$ for $m \leq 0$.

Now assume $0 < m \leq n, p \notin \mathrm{div}(m)$, $e(x) = \langle a_1, a_2\rangle$ and $a_1 \geq 0$. Since $p \notin \mathrm{div}(m)$ we get $p \cdot a_1 \neq m$, and thus $|p \cdot a_1 - m| \geq 1$. Consequently,

$$(n+1) \cdot |p \cdot a_1 - m| > m \cdot |p \cdot a_1 - m| \geq m.$$

Thus

$$(n+1)\cdot|p\cdot\langle a_1,a_2\rangle-\langle m,0\rangle| = \\ |\langle(n+1)\cdot(p\cdot a_1-m),((n+1)\cdot p\cdot a_2)\rangle|\geq\langle m,0\rangle.$$

Therefore, $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{div}_{\mathrm{p,n}})$ for $m>0$ such that $p\notin \mathrm{div}(m)$.

2. Now assume $0<m$ and $p \mid m$. Since $p\mid m$, there is $r\in\mathbb{N}$ such that $r\cdot p=m$. Let us consider an evaluation e on $\boldsymbol{Z}_m$, where $e(x)=r$. We have

$$(n+1)\cdot|p\cdot r-m|=(n+1)\cdot 0=0<m.$$

This shows $\boldsymbol{Z}_m \not\models (\mathrm{div}_{\mathrm{p,n}})$ if $0<m$ and $p\in\mathrm{div}(m)$.

This completes the proof. □

Now we have enough tools to axiomatize the variety $\mathbf{HSP}(\boldsymbol{Z}_n)$.

Theorem 2.38. *Let $n\in\mathbb{N}$. The variety $\mathbf{HSP}(\boldsymbol{Z}_n)$ can be axiomatized as a subvariety of $p\mathbb{AL}^+$ by the following set of formulas: $\{(s\text{-}rank_{\mathrm{n}})\}\cup\{(div_{\mathrm{p,n}})\mid p\notin \mathrm{div}(n), p<n\}$.*

Proof. By Lemma 2.36 we have $\boldsymbol{Z}_n \models (\text{s-rank}_{\mathrm{n}})$ and by Lemma 2.37 we have $\boldsymbol{Z}_n \models (\mathrm{div}_{\mathrm{p,n}})$ for all $p\notin\mathrm{div}(n)$.

Now assume there is a finitely generated $\boldsymbol{A}_a\in p\mathbb{AL}^+_{\mathrm{FSI}}$, such that $\boldsymbol{A}_a\models(\text{s-rank}_{\mathrm{n}})$ and $\boldsymbol{A}_a\models(\mathrm{div}_{\mathrm{p,n}})$ for all $p\notin\mathrm{div}(n)$. We will show $\boldsymbol{A}_a\in\mathbf{HSP}(\boldsymbol{Z}_n)$.

By Theorem 2.6, $\boldsymbol{A}_a$ is totally ordered. If $a=0$ we obtain the desired result by applying Corollary 2.19. Let us now assume that $a\neq 0$. By Theorem 2.23 we can assume without loss of generality that $\boldsymbol{A}_a$ is strongly pointed. By Lemma 2.36 we obtain $\boldsymbol{A}_a$ must be p-simple with rank less or equal to n and thus by Lemma 2.32 $\boldsymbol{A}_a\cong\boldsymbol{Z}_k$ for some $k\leq n$. By Lemma 2.37 we obtain k divides n. That means $\boldsymbol{A}_a\cong\boldsymbol{Z}_k\in\mathbf{IS}(\boldsymbol{Z}_n)$ and thus $\boldsymbol{A}_a\in\mathbf{HSP}(\boldsymbol{Z}_n)$. □

Since Theorem 2.42 also provides an axiomatization for the variety generated by $\boldsymbol{Z}_n$, it might seem that this would render Theorem 2.38 redundant. However, this is not the case. The varieties generated by $\boldsymbol{Z}_n$ are structurally simpler than the general case, as they are not generated by any ℓ-groups of the form $\boldsymbol{Z}_m\ltimes\boldsymbol{Z}_0$. Consequently, Theorem 2.38 provides a much simpler and more direct axiomatization than the one required by the general framework of Theorem 2.42. This is particularly appealing because the varieties generated by $\boldsymbol{Z}_n$ correspond via the Mundici functor to the varieties generated by single finite simple MV-algebra, which are among the most fundamental structures in the theory of MV-algebras.

Lemma 2.39. *Let $n\geq 0$ and $m\in\mathbb{Z}$. Let us consider the following equation.*

$$((2n+1)\cdot x-2\cdot\mathtt{f})\vee(\mathtt{f}-(2n+2)\cdot x)\vee -x\geq 0. \qquad (\mathrm{rank}_n)$$

The following conditions are equivalent.

1. $\boldsymbol{Z}_m\models(rank_{\mathrm{n}})$,
2. $\boldsymbol{Z}_m\ltimes\boldsymbol{Z}_0\models(rank_{\mathrm{n}})$,

3. $m \le n$.

Proof. Since $\boldsymbol{Z}_n \in \mathbf{H}(\boldsymbol{Z}_n \ltimes \boldsymbol{Z}_0)$, we only have to show $\boldsymbol{Z}_m \not\models (\mathrm{rank_n})$ for $m > n$ and $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{rank_n})$ for $m \le n$.

1. First we show $\boldsymbol{Z}_m \not\models (\mathrm{rank_n})$ for $n < m$. Since $\boldsymbol{Z}_m$ is Archimedean, there is a maximal $k \ge 1$ such that $k \cdot (2n+1) < 2m$. Let us note that from the maximality of k it follows $2k \cdot (2n+1) \ge 2m$ and thus $k \cdot (2n+1) \ge m$. Let e be an evaluation on $\boldsymbol{Z}_m$ such that $e(x) = k$. Now we have

   $$(2n+1) \cdot e(x) - 2 \cdot e(\mathtt{f}) = (2n+1) \cdot k - 2 \cdot m < 0$$

   by the definition of k. Moreover, we have

   $$e(\mathtt{f}) - (2n+2) \cdot e(x) = m - k(2n+2) \le k(2n+1) - k(2n+2) = -k < 0.$$

   Clearly, also $-e(x) = -k < 0$. This shows $\boldsymbol{Z}_m \not\models_e (\mathrm{rank_n})$ for $n < m$.

2. It remains to show that $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{rank_n})$ for $m \le n$. Trivially, we have $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models_e (\mathrm{rank_n})$ for any evaluation $e(x) \le 0$. Therefore, from now on we can focus only on evaluations such that $e(x) > 0$. For $m \le 0$ and evaluation e such that $e(x) > 0$ we have $(2n+1) \cdot e(x) - 2e(\mathtt{f}) \ge 0$. Thus $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models_e (\mathrm{rank_n})$ for $m \le 0$.

   Now assume $m > 0$. First, let us consider the case $e(x) = \langle 0, b\rangle$ for some $b \in \mathbb{Z}$. We have

   $$e(\mathtt{f}) - (2n+2) \cdot e(x) = \langle m, 0\rangle - (2n+2) \cdot \langle 0, b\rangle = \langle m, -(2n+2) \cdot b\rangle > 0.$$

   Finally we have to check the case when $e(x) = \langle a, b\rangle$ for some $a, b \in \mathbb{Z}$ and $a > 0$. We have

   $$\begin{aligned}(2n+1)\cdot e(x) - 2\cdot e(\mathtt{f}) = (2n+1)\cdot\langle a, b\rangle - 2\cdot\langle m, 0\rangle \ge (2n+1)\cdot\langle 1, b\rangle - 2\cdot\langle n, 0\rangle = \\ = \langle (2n+1) - 2n, (2n+1)\cdot b\rangle = \langle 1, (2n+1)\cdot b\rangle \ge 0.\end{aligned}$$

   This shows that indeed $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{rank_n})$ for $m \le n$. □

Now we can introduce the final equation, which we need to axiomatize the subvarieties of pAb. This equation is equivalent to the disjunction of the equations (s-rank$_\mathrm{n}$) and (div$_\mathrm{p,n}$).

Lemma 2.40. *Let $\boldsymbol{A}$ be a finitely generated totally ordered pointed Abelian ℓ-group and $0 \le p \le n$. Let us consider the following equation.*

$$((n+1) \cdot ((p \cdot x - \mathtt{f}) \vee (\mathtt{f} - p \cdot x)) - \mathtt{f}) \vee (-x) \vee (n \cdot y - \mathtt{f}) \vee (-y) \ge 0 \quad (\mathrm{mix}_{p,n})$$

We have $\boldsymbol{A} \models (\mathrm{mix_{p,n}})$ if and only if $\boldsymbol{A} \models (\mathrm{s\text{-}rank_n})$ or $\boldsymbol{A} \models (\mathrm{div_{p,n}})$.

Proof. Denote

$$(n \cdot y - \mathtt{f}) \vee (-y) \ge 0. \qquad (\mathrm{s\text{-}rank}_n(y/x))$$

Since $\boldsymbol{A}$ is totally ordered we know that for each evaluation e of $\boldsymbol{A}$ we have $\boldsymbol{A} \models_e (\mathrm{mix_{p,n}})$ iff $\boldsymbol{A} \models_e (\mathrm{div_{p,n}})$ or $\boldsymbol{A} \models_e (\mathrm{s\text{-}rank_n(y/x)})$. Therefore if $\boldsymbol{A} \models (\mathrm{div_{p,n}})$

or $\boldsymbol{A} \models (\text{s-rank}_\text{n}(\text{y/x}))$ then clearly $\boldsymbol{A} \models (\text{mix}_{\text{p,n}})$. For the other implication assume $\boldsymbol{A} \not\models (\text{div}_{\text{p,n}})$ and $\boldsymbol{A} \not\models (\text{s-rank}_\text{n}(\text{y/x}))$. Then there are evaluations e_1, e_2 such that $\boldsymbol{A} \not\models_{e_1} (\text{div}_{\text{p,n}})$ and $\boldsymbol{A} \not\models_{e_2} (\text{s-rank}_\text{n}(\text{y/x}))$. We will define an evaluation e_3 on $\boldsymbol{A}$, such that $e_3(x) = e_1(x)$ and $e_3(y) = e_2(y)$. Then we get $\boldsymbol{A} \not\models_{e_3} (\text{mix}_{\text{p,n}})$. Therefore we have $\boldsymbol{A} \models (\text{mix}_{\text{p,n}})$ iff $\boldsymbol{A} \models (\text{div}_{\text{p,n}})$ or $\boldsymbol{A} \models (\text{s-rank}_\text{n}(\text{y/x}))$. Since $\boldsymbol{A} \models (\text{s-rank}_\text{n}(\text{y/x}))$ if and only if $\boldsymbol{A} \models (\text{s-rank}_\text{n})$, we obtain that $\boldsymbol{A} \models (\text{mix}_{\text{p,n}})$ if and only if $\boldsymbol{A} \models (\text{div}_{\text{p,n}})$ or $\boldsymbol{A} \models (\text{s-rank}_\text{n})$, which completes the proof. □

Corollary 2.41. *Let $0 \leq p \leq n$ and $m \leq n$. We have $\boldsymbol{Z}_m \models (\text{mix}_{\text{p,n}})$. Moreover, $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\text{mix}_{\text{p,n}})$ iff $p \notin \text{div}(m)$ or $m \leq 0$.*

At this point we have all we need to provide the axiomatization of subvarieties of $p\mathbb{AL}$. First we will discuss subvarieties of positively and negatively pointed Abelian ℓ-groups.

Theorem 2.42. *Any non-trivial proper subvariety of $p\mathbb{AL}^+$ is of the form*

$$\mathcal{V}_{I,J} = \mathbf{HSP}(\boldsymbol{Z}_i, \boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \mid i \in I, j \in J)$$

for some finite sets $\{0\} \subseteq J \subseteq I \subsetneq \mathbb{N}$, where both $I \setminus \{0\}$ and $J \setminus \{0\}$ are closed under divisors.

Moreover, $\mathcal{V}_{I,J}$ is axiomatized by the following set of equations:

$$S_{I,J} = \{(\text{rank}_\text{n})\} \cup \{(\text{div}_{\text{p,n}}) \mid p \notin I\} \cup \{(\text{mix}_{\text{p,n}}) \mid p \in I \setminus J\},$$

where $n = \max I$.

Proof. First we show that any proper subvariety of $p\mathbb{AL}^+$ is equal to $\mathcal{V}_{I,J}$ for some finite sets $\{0\} \subseteq J \subseteq I$. Let $\mathcal{V}$ be an arbitrary subvariety of $p\mathbb{AL}^+$. Let us denote $I = \{i \mid \boldsymbol{Z}_i \in \mathcal{V}\}$ and $J = \{j \mid \boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \in \mathcal{V}\}$. By Corollary 2.19 we have $\{0\} \in I \cap J$. We show $\mathcal{V} = \mathcal{V}_{I,J}$. Since $\mathcal{V}$ is a *proper* subvariety of $p\mathbb{AL}^+$, by Lemma 2.20 we know that $\boldsymbol{R}_1 \notin \mathcal{V}$. By Theorem 2.35 we have that $\mathcal{V}$ is equal to

$$\mathbf{HSP}(\{\boldsymbol{A} \in \mathcal{V}_{\text{FSI}} \mid \boldsymbol{A} \text{ is fin. gen.}\}) = \mathbf{HSP}(\{\boldsymbol{Z}_i, \boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \mid i \in I, j \in J\}) = \mathcal{V}_{I,J}.$$

Clearly, both $I \setminus \{0\}$ and $J \setminus \{0\}$ are closed under divisors, since for any $d, k \in \mathbb{N} \setminus \{0\}$ such that $d \in \text{div}(k)$ we have $\boldsymbol{Z}_d \in \mathbf{IS}(\boldsymbol{Z}_k)$ and $\boldsymbol{Z}_d \ltimes \boldsymbol{Z}_0 \in \mathbf{IS}(\boldsymbol{Z}_k \ltimes \boldsymbol{Z}_0)$. Since for each $j \in J$ we have $\boldsymbol{Z}_j \in \mathbf{H}(\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0)$, we get $J \subseteq I$. The set I (and consequently J) must be finite, otherwise we would get by Lemma 2.28 that $\boldsymbol{R}_1 \in \mathcal{V}$. This proves $\mathcal{V} = \mathcal{V}_{I,J}$ for some finite sets $J \subseteq I$.

Now we have to show the axiomatization of $\mathcal{V}_{I,J}$. First, we will argue that all the equations from $S_{I,J}$ hold in $\mathcal{V}_{I,J}$. Since $n = \max I \geq \max J$, by Lemma 2.39 we know that $\boldsymbol{Z}_i \models (\text{rank}_\text{n})$ for all $i \in I$ and $\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \models (\text{rank}_\text{n})$ for all $j \in J$. Thus $\mathcal{V}_{I,J} \models (\text{rank}_\text{n})$.

Let us fix $p \notin I$. Since I is closed under divisors, we get that $p \notin \text{div}(i)$ for all $i \in I$. Using $J \subseteq I$ and by Lemma 2.37 we obtain that $\boldsymbol{Z}_i \models (\text{div}_{\text{p,n}})$ for each $i \in I$ and $\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \models (\text{div}_{\text{p,n}})$ for each $j \in J$. Therefore, $\mathcal{V}_{I,J} \models (\text{div}_{\text{p,n}})$ for each $p \notin I$.

Now let us fix $p \in I \setminus J$. Since $p \notin J$ by the same argument as in the previous paragraph we obtain $\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \models (\text{div}_{\text{p,n}})$ for each $j \in J$. By Lemma 2.36 we have $\boldsymbol{Z}_i \models (\text{s-rank}_\text{n})$ for each $i \in I$ and thus by Lemma 2.40 we get $\boldsymbol{Z}_i \models (\text{mix}_{\text{p,n}})$

and $\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \models (\mathrm{mix_{p,n}})$ for each $i \in I$ and $j \in J$. Thus $\mathcal{V}_{I,J} \models (\mathrm{mix_{p,n}})$ for each $p \in I \setminus J$.

This concludes that $\mathcal{V}_{I,J} \models$ (eq) for each (eq) $\in S_{I,J}$. It remains to show $\mathcal{V}_{I,J}$ is uniquely determined by the equations from $S_{I,J}$.

Let $\boldsymbol{A}_a \in p\mathbb{AL}^+_{\mathrm{FSI}}$ be a finitely generated satisfying all the equations from $S_{I,J}$. We will show $\boldsymbol{A}_a \in \mathcal{V}_{I,J}$. First, let us note that clearly $\mathbf{HSP}(\boldsymbol{A}_a) \neq \mathbf{HSP}(\boldsymbol{R}_1)$ since from Lemma 2.20 we know $\mathbf{HSP}(\boldsymbol{R}_1) = p\mathbb{AL}^+$ and $\boldsymbol{A}$ satisfies equations from $S_{I,J}$, which are not generally valid in $p\mathbb{AL}^+$. By Corollary 2.19 and Theorems 2.23 and 2.35 it is enough to consider the cases $\boldsymbol{A}_a \in \{\boldsymbol{Z}_i, \boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \mid i, j \in \mathbb{N}\}$.

For $\boldsymbol{A}_a = \boldsymbol{Z}_m$ we get $m \leq n$, by applying Lemma 2.39 using $\boldsymbol{Z}_m \models (\mathrm{rank_n})$. Since $\boldsymbol{Z}_m \models (\mathrm{div_{p,n}})$ for all $p \notin I$, we get by Lemma 2.37 that $m \in I$.

For $\boldsymbol{A}_a = \boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0$ we similarly get $m \leq n$ and $m \in I$. Since for every $p \in I \setminus J$ we have $\boldsymbol{Z}_m \ltimes \boldsymbol{Z}_0 \models (\mathrm{mix_{p,n}})$, by Corollary 2.41 we get $m \in J$.

Thus $\boldsymbol{A}_a \in \mathcal{V}_{I,J}$, which completes the proof. □

The structural part of Theorem 2.42 was proved in [98]. This result can be obtained by applying the Mundici functor, Corollary 2.19, Theorem 2.23, and Komori classification, an approach we will discuss later in Section 2.6 and demonstrate in Theorems 2.47 and 2.50.

Despite the fact that the statement of Theorem 2.42 was about proper varieties, we can also claim that any variety of $p\mathbb{AL}^+$ is equal to $\mathcal{V}_{I,J}$ for some sets $J \subseteq I$, since $p\mathbb{AL}^+ = \mathcal{V}_{\mathbb{N},\mathbb{N}}$.

In this chapter, we have so far been describing the subvarieties of $p\mathbb{AL}^+$. However, we could state all the results similarly for $p\mathbb{AL}^-$ with accordingly modified equations:

$$(\mathtt{f} - n \cdot x) \vee x \geq 0. \tag{s-rank^d_n}$$

$$(2 \cdot \mathtt{f} - (2n+1) \cdot x) \vee ((2n+2) \cdot x - \mathtt{f}) \vee x \geq 0. \tag{rank^d_n}$$

$$(\mathtt{f} - (n+1) \cdot ((p \cdot x - \mathtt{f}) \wedge (\mathtt{f} - p \cdot x))) \vee x \geq 0. \tag{div$^d_{p,n}$}$$

$$(\mathtt{f} - n \cdot y) \vee y \vee (\mathtt{f} - ((n+1) \cdot ((p \cdot x - \mathtt{f}) \wedge (\mathtt{f} - p \cdot x)))) \vee x \geq 0 \tag{mix$^d_{p,n}$}$$

Using these equations we can state a dual result to Theorem 2.42.

Theorem 2.43. *Any non-trivial proper subvariety of $p\mathbb{AL}^-$ is of the form*

$$\mathcal{V}^d_{I,J} = \mathbf{HSP}(\boldsymbol{Z}_{-i}, \boldsymbol{Z}_{-j} \ltimes \boldsymbol{Z}_0 \mid i \in I, j \in J)$$

for some finite sets $\{0\} \subseteq J \subseteq I \subsetneq \mathbb{N}$, where both $I \setminus \{0\}$ and $J \setminus \{0\}$ are closed under divisors.

Moreover, $\mathcal{V}^d_{I,J}$ is axiomatized by the following set $S^d_{I,J}$ of equations:

$$S^d_{I,J} = \{(\mathrm{rank^d_n})\} \cup \{(\mathrm{div^d_{p,n}}) \mid p \notin I\} \cup \{(\mathrm{mix^d_{p,n}}) \mid p \in I \setminus J\},$$

where $n = \max I$.

Now we can combine the Theorems 2.43 and 2.42 into one theorem covering all subvarieties of $p\mathbb{AL}$.

Theorem 2.44. *Any non-trivial subvariety of* $p\mathbb{AL}$ *can be written as* $\mathcal{V} = \mathcal{V}_{I_1,J_1} \vee \mathcal{V}^d_{I_2,J_2}$ *for some (not necessarily finite) sets* I_1, I_2, J_1, J_2*. Such a variety is axiomatized by the equations from* $S_{I_1,J_1} \cup S^d_{I_2,J_2}$*, where* S_{I_1,J_1} *and* $S^d_{I_2,J_2}$ *are defined in Theorems 2.42 and 2.43.*

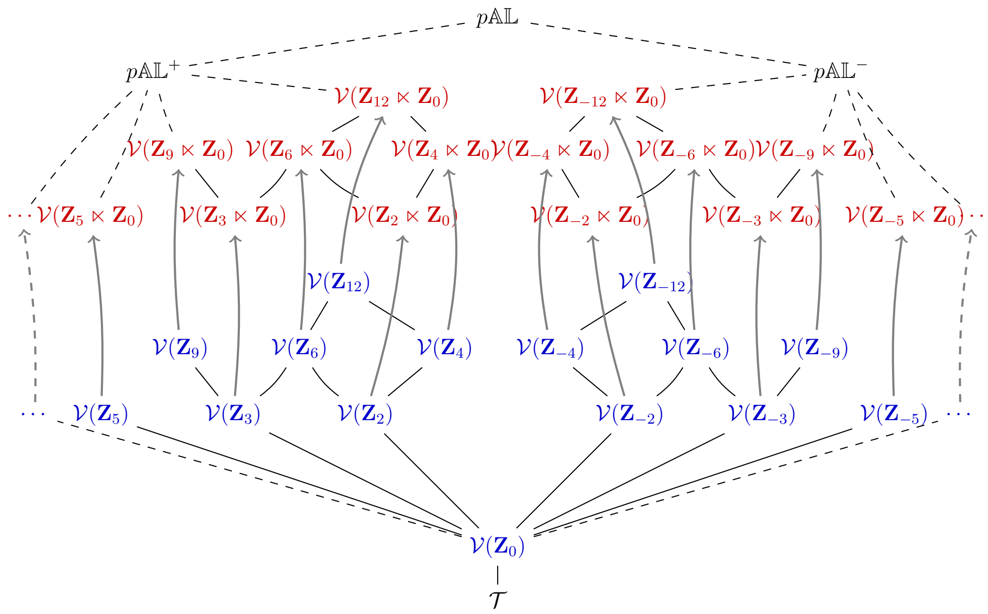

Figure 2.1 The lattice of join irreducible subvarieties of $p\mathbb{AL}$.

Proof. Let $\mathcal{V}$ be an arbitrary subvariety of $p\mathbb{AL}$. Using Theorem 2.6 we can derive that finitely subdirectly irreducible pointed Abelian ℓ-groups are totally ordered. Let us denote $\mathcal{V}^+_{\mathrm{FSI}} = \mathcal{V}_{\mathrm{FSI}} \cap p\mathbb{AL}^+$ and $\mathcal{V}^-_{\mathrm{FSI}} = \mathcal{V}_{\mathrm{FSI}} \cap p\mathbb{AL}^-$. Thus we have

$$\mathbf{HSP}(\mathcal{V}^+_{\mathrm{FSI}} \cup \mathcal{V}^-_{\mathrm{FSI}}) = \mathbf{HSP}(\mathcal{V}_{\mathrm{FSI}}) = \mathcal{V}.$$

This shows that $\mathcal{V}$ is the join of $\mathbf{HSP}(\mathcal{V}^+_{\mathrm{FSI}})$ and $\mathbf{HSP}(\mathcal{V}^-_{\mathrm{FSI}})$. By Theorem 2.42 $\mathbf{HSP}(\mathcal{V}^+_{\mathrm{FSI}}) = \mathcal{V}_{I_1,J_1}$ for some sets I_1, J_1 and by Theorem 2.43 $\mathbf{HSP}(\mathcal{V}^-_{\mathrm{FSI}}) = \mathcal{V}^d_{I_2,J_2}$ for some sets I_2, J_2. Thus we have $\mathcal{V} = \mathcal{V}_{I_1,J_1} \vee \mathcal{V}^d_{I_2,J_2}$.

We show that the equations from $S_{I_1,J_1} \cup S^d_{I_2,J_2}$ axiomatize the variety $\mathcal{V}$. First all the equations from S_{I_1,J_1} are valid in all negatively pointed Abelian ℓ-groups by Lemmas 2.37, 2.39 and 2.40. Similarly, all the equations from $S^d_{I_2,J_2}$ are valid in all positively pointed Abelian ℓ-groups. Since all the equations from S_{I_1,J_1} are valid in $\mathcal{V}_{I_1,J_1}$ and all the equations from $S^d_{I_2,J_2}$ are valid in $\mathcal{V}^d_{I_2,J_2}$, we conclude that all equations from $S_{I_1,J_1} \cup S^d_{I_2,J_2}$ are valid in $\mathcal{V}$.

Now let us take arbitrary $\boldsymbol{A}_a \in p\mathbb{AL}_{\mathrm{FSI}}$ satisfying all the equations from $S_{I_1,J_1} \cup S^d_{I_2,J_2}$. By Theorem 2.6 $\boldsymbol{A}_a$ is totally ordered. We distinguish three cases:

1. If $a = 0$ then $\boldsymbol{A}_a \in \mathcal{V}$ by Corollary 2.19.
2. If $a > 0$ then $\boldsymbol{A}_a \in \mathcal{V}^+_{\mathrm{FSI}}$ and thus $\boldsymbol{A}_a \in \mathcal{V}_{I_1,J_1}$. Consequently $\boldsymbol{A}_a \in \mathcal{V}$.
3. If $a < 0$ then $\boldsymbol{A}_a \in \mathcal{V}^-_{\mathrm{FSI}}$ and thus $\boldsymbol{A}_a \in \mathcal{V}^d_{I_2,J_2}$. This implies $\boldsymbol{A}_a \in \mathcal{V}$.

This proves that the equations from $S_{I_1,J_1} \cup S^d_{I_2,J_2}$ axiomatize $\mathcal{V}$. □

2.6 Applying the Mundici functor

In the previous sections, we developed a self-contained theory using the language of pointed Abelian ℓ-groups. In this section we will establish several connections between the theory of MV-algebras and the theory of pointed Abelian ℓ-groups. We will show that the semantical description of subvarieties of pAb^+ is analogous to the Komori classification of subvarieties of MV-algebras. We will conclude the section by deriving a complete classification of quasivarieties of pAb generated by totally ordered elements from the similar result about MV-algebras [46].

Let us recall that MV-algebras can be understood as bounded intervals of Abelian ℓ-groups. For us, an MV-algebra will be a structure of the following signature $\langle \oplus, \otimes, \vee, \wedge, \neg, 0, 1\rangle$. For the basics of MV-algebras we refer the reader to any classical bibliography e.g. [21].

We shall note that we are adding the symbols $\vee, \wedge$ and $\otimes$ into our language purely for our convenience. It is well-known that $\vee, \wedge$ and $\otimes$ are term definable using the remaining operations.

The fundamental tool for working with MV-algebras is the Mundici functor, which is a categorical equivalence from the category of strongly positively pointed Abelian ℓ-groups into the category of MV-algebras. As discussed in [98], one can generalize the Mundici functor to the whole category of positively pointed Abelian ℓ-groups. We will denote this extended functor as Γ and define it as follows:

Definition 2.45. *Let $\boldsymbol{A}_u = \langle A, +, -, \vee, \wedge, 0, u\rangle$ be a positively pointed Abelian ℓ-group and $u \geq 0$. Define MV-algebra $\Gamma(\boldsymbol{A}_u) = \langle [0, u], \oplus, \otimes, \vee, \wedge, \neg, 0, 1\rangle$, where the operations are defined as follows:*

1. $a \oplus b := (a + b) \wedge u$,
2. $a \otimes b := 0 \vee (a + b - u)$,
3. $\neg a := u - a$,
4. $1 := u$.

For positively pointed Abelian ℓ-groups $\boldsymbol{A}_u$, $\boldsymbol{B}_v$ and a homomorphism $f : \boldsymbol{A}_u \to \boldsymbol{B}_v$ we define $\Gamma(f) : \Gamma(\boldsymbol{A}_u) \to \Gamma(\boldsymbol{B}_v)$ as $\Gamma(f) := f \restriction [0, u]$.

It is well-known [21] that the Mundici functor (Γ restricted to the subcategory of strongly positively pointed Abelian ℓ-groups) is a categorical equivalence. By Γ^{-1} we denote the corresponding inverse functor. In particular, we have $\Gamma(\Gamma^{-1}(\boldsymbol{B})) \cong \boldsymbol{B}$ for each MV-algebra $\boldsymbol{B}$. It is well known [98] that for positively pointed Abelian ℓ-group $\boldsymbol{A}_a$ with $a \neq 0$ we have that $\Gamma^{-1}(\Gamma(\boldsymbol{A}_a))$ is equal to its strongly pointed convex ℓ-subgroup. Specially, if $\boldsymbol{A}_a$ is itself strongly pointed, we have $\Gamma^{-1}(\Gamma(\boldsymbol{A}_a)) \cong \boldsymbol{A}_a$.

Lemma 2.46. *Let $\mathbb{K} \cup \{\boldsymbol{A}_u\}$ be a class of positively pointed Abelian ℓ-groups. Then it holds $\boldsymbol{A}_u \in \mathbf{HSP}(\mathbb{K})$ implies $\Gamma(\boldsymbol{A}_u) \in \mathbf{HSP}(\Gamma[\mathbb{K}])$.*

Proof. One has to show the following:

1. $\Gamma(\mathbf{P}(\mathbb{K})) = \mathbf{P}(\Gamma[\mathbb{K}])$ for any class $\mathbb{K} \subseteq p\mathbb{AL}^+$,

2. $\boldsymbol{A}_u \in \mathbf{S}(\boldsymbol{B}_v)$ implies $\Gamma(\boldsymbol{A}_u) \in \mathbf{S}(\Gamma(\boldsymbol{B}_v))$ for any $\boldsymbol{A}_u, \boldsymbol{B}_v \in p\mathbb{AL}^+$,

3. $\boldsymbol{A}_u \in \mathbf{H}(\boldsymbol{B}_v)$ implies $\Gamma(\boldsymbol{A}_u) \in \mathbf{H}(\Gamma(\boldsymbol{B}_v))$ for any $\boldsymbol{A}_u, \boldsymbol{B}_v \in p\mathbb{AL}^+$.

For detailed discussion, see [98]. $\square$

Using this observation we can easily provide the syntactical classification of all varieties of MV-algebras.

Theorem 2.47 (Komori)**.** *Every non-trivial proper variety of MV-algebras is generated by*

$$\{\Gamma(\boldsymbol{Z}_i), \Gamma(\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0) \mid i \in I, j \in J\} \text{ for some finite sets } J \subseteq I.$$

Proof. Let $\mathcal{V}$ be a proper variety of MV-algebras. Since $\mathcal{V}$ is proper, there is $\boldsymbol{A} \notin \mathcal{V}$. Let us consider the variety of positively pointed Abelian ℓ-groups $\mathbb{K} = \mathbf{HSP}(\Gamma^{-1}[\mathcal{V}])$. By Lemma 2.46 we have

$$\Gamma[\mathbb{K}] = \Gamma(\mathbf{HSP}(\Gamma^{-1}[\mathcal{V}])) \subseteq \mathbf{HSP}(\Gamma[\Gamma^{-1}[\mathcal{V}]]) = \mathbf{HSP}(\mathcal{V}) = \mathcal{V}.$$

Since $\boldsymbol{A} \notin \mathcal{V}$, also $\Gamma(\Gamma^{-1}(\boldsymbol{A})) \notin \mathcal{V}$ and by Lemma 2.46 we obtain $\Gamma^{-1}(\boldsymbol{A}) \notin \mathbb{K}$. Therefore, $\mathbb{K}$ is a proper subvariety of $p\mathbb{AL}^+$ and thus by Theorem 2.42 we have $\mathbf{HSP}(\Gamma^{-1}[\mathcal{V}]) = \mathcal{V}_{I,J}$ for some finite sets $J \subseteq I$. Since $\boldsymbol{Z}_i, \boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \in \mathcal{V}_{I,J}$ for $i \in I, j \in J$, by Lemma 2.46 we obtain $\Gamma(\boldsymbol{Z}_i), \Gamma(\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0) \in \Gamma[\mathcal{V}_{I,J}] \subseteq \mathcal{V}$ for $i \in I$ and $j \in J$.

Now it remains to check $\mathcal{V}$ is generated by $\Gamma(\boldsymbol{Z}_i), \Gamma(\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0)$ for every $i \in I, j \in J$. Let $\boldsymbol{B} \in \mathcal{V}$. We obtain $\Gamma^{-1}(\boldsymbol{B}) \in \Gamma^{-1}[\mathcal{V}] \subseteq \mathbf{HSP}(\Gamma^{-1}[\mathcal{V}]) = \mathcal{V}_{I,J} = \mathbf{HSP}(\boldsymbol{Z}_i, \boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0 \mid i \in I, j \in J)$. By applying Γ it follows $\boldsymbol{B} \in \mathbf{HSP}(\Gamma(\boldsymbol{Z}_i), \Gamma(\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_0) \mid i \in I, j \in J)$. This completes the proof. $\square$

Here, we used the classification of subvarieties of $p\mathbb{AL}^+$ to obtain the classification of all subvarieties of MV-algebras. We should note that we could proceed in the opposite way: using Komori's classification, Corollary 2.19, and Theorem 2.23, we can derive the semantic part of Theorem 2.44. For further details, see the discussion in [98].

2.6.1 Gispert's classification of universal classes of totally ordered MV-algebras

In the rest of this section we focus on quasivarieties and universal classes. Let $\mathbf{S}$ denote any finitely generated dense ℓ-subgroup of $\mathbf{R}$ such that $\mathbf{S} \cap \mathbf{Q} = \mathbf{Z}$. Recall the result from [46]:

Theorem 2.48. *Let $\mathbb{M}$ be a class of totally ordered MV-algebras. Then the following properties are equivalent:*

1. *$\mathbb{M}$ is universal.*

2. *There exist $I, J, K \subseteq \mathbb{N}$ and for every $j \in J$, a nonempty subset $D_j \subseteq \operatorname{div}(j)$ such that $\mathbb{M}$ is equal to*

$$\mathbf{ISP}_{\mathrm{U}}(\{\Gamma(\boldsymbol{Z}_i) \mid i \in I\} \cup \{\Gamma(\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j}) \mid j \in J, d_j \in D_j\} \cup \{\Gamma(\mathbf{S}_k) \mid k \in K\}).$$

We need to show how Γ behaves with respect to ultraproducts.

Lemma 2.49. *Let $\mathbb{K} \cup \{\boldsymbol{A}_a\}$ be a class of strongly positively pointed Abelian ℓ-groups. Then we have $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$ iff $\Gamma(\boldsymbol{A}_a) \in \mathbf{ISP}_{\mathrm{U}}(\Gamma[\mathbb{K}])$.*

Proof. Throughout this proof, let I be an arbitrary fixed set, $\mathfrak{U}$ be an ultrafilter over I and $\{\boldsymbol{A}^i_{a_i} \mid i \in I\} \subseteq \mathbb{K}$. First, let us note that $\Gamma(\prod_{i\in I} \boldsymbol{A}^i_{a_i}/\mathfrak{U}) = \Gamma(\prod_{i\in I} \boldsymbol{A}^i_{a_i})/\mathfrak{U} = (\prod_{i\in I} \Gamma(\boldsymbol{A}^i_{a_i})/\mathfrak{U})$. Consequently, $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$ implies $\Gamma(\boldsymbol{A}_a) \in \mathbf{ISP}_{\mathrm{U}}(\Gamma[\mathbb{K}])$.

For the other inclusion first assume $\Gamma(\boldsymbol{A}_a) \in \mathbf{IS}(\prod_{i\in I} \Gamma(\boldsymbol{A}^i_{a_i})/\mathfrak{U})$. Let $\boldsymbol{B}_b$ denote the strongly pointed convex ℓ-subgroup of $\prod_{i\in I} \boldsymbol{A}^i_{a_i}/\mathfrak{U}$. We know that we have $\Gamma^{-1}(\Gamma(\prod_{i\in I} \boldsymbol{A}^i_{a_i}/\mathfrak{U})) = \boldsymbol{B}_b$. Since Γ^{-1} preserves embeddings, we have $\Gamma^{-1}(\Gamma(\boldsymbol{A}_a)) \in \mathbf{IS}(\Gamma^{-1}(\prod_{i\in I} \Gamma(\boldsymbol{A}^i_{a_i})/\mathfrak{U}))$. Since $\boldsymbol{A}_a \cong \Gamma^{-1}(\Gamma(\boldsymbol{A}_a))$ and

$$\Gamma^{-1}(\prod_{i\in I} \Gamma(\boldsymbol{A}^i_{a_i})/\mathfrak{U}) = \Gamma^{-1}(\Gamma(\prod_{i\in I} \boldsymbol{A}^i_{a_i}/\mathfrak{U})) = \boldsymbol{B}_b,$$

we get $\boldsymbol{A}_a \in \mathbf{IS}(\boldsymbol{B}_b)$. Since $\boldsymbol{B}_b \in \mathbf{IS}(\prod_{i\in I} \boldsymbol{A}^i_{a_i}/\mathfrak{U})$, we have $\boldsymbol{A}_a \in \mathbf{IS}(\prod_{i\in I} \boldsymbol{A}^i_{a_i}/\mathfrak{U})$ and thus $\boldsymbol{A}_a \in \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$. This completes the proof. $\square$

Theorem 2.50. *Let $\mathbb{M}$ be a class of totally ordered positively pointed Abelian ℓ-groups. Then the following properties are equivalent:*

1. *$\mathbb{M}$ is universal.*
2. *There exist $I, J, K \subseteq \mathbb{N}$ and for every $j \in J$, a nonempty subset $D_j \subseteq \operatorname{div}(j)$ such that*

$$\mathbb{M} = \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{Z}_i \mid i \in I\} \cup \{\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j} \mid j \in J, d_j \in D_j\} \cup \{\mathbf{S}_k \mid k \in K\}).$$

Proof. The implication 2 to 1 is trivial. To prove the other implication assume $\mathbb{M}$ is universal. Therefore, by Theorem 2.3 we have $\mathbb{M} = \mathbf{ISP}_{\mathrm{U}}(\mathbb{M})$. Let us denote by $\mathbb{K}$ the subclass of $\mathbb{M}$ consisting of all strongly pointed Abelian ℓ-groups.

Using Lemma 2.49 it follows $\Gamma[\mathbb{K}] = \mathbf{ISP}_{\mathrm{U}}(\Gamma[\mathbb{K}])$. Thus $\Gamma[\mathbb{K}]$ is universal and hence by Theorem 2.48 we have that there exist $I, J, K \subseteq \mathbb{N}$ and for every $j \in J$, a nonempty subset $D_j \subseteq \operatorname{div}(j)$ such that

$$\Gamma[\mathbb{K}] = \mathbf{ISP}_{\mathrm{U}}(\{\Gamma(\boldsymbol{Z}_i) \mid i \in I\} \cup \{\Gamma(\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j}) \mid j \in J, d_j \in D_j\} \cup \{\Gamma(\mathbf{S}_k) \mid k \in K\}).$$

Now, again using Lemma 2.49 we obtain

$$\mathbb{K} = \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{Z}_i \mid i \in I\} \cup \{\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j} \mid j \in J, d_j \in D_j\} \cup \{\mathbf{S}_k \mid k \in K\}) \cap sp\mathbb{AL}^+.$$

We discuss two options:

1. If $\boldsymbol{Z}_0 \in \mathbb{M}$ then $p\mathbb{AL}^0_{\mathrm{FSI}} \subseteq \mathbb{M}$ by Lemma 2.13 and thus by Theorem 2.23 we have

$$\begin{aligned}\mathbb{M} &= \mathbf{ISP}_{\mathrm{U}}(\mathbb{K}) \cup p\mathbb{AL}^0_{\mathrm{FSI}} = \mathbf{ISP}_{\mathrm{U}}(\mathbb{K}) \cup \mathbf{ISP}_{\mathrm{U}}(\boldsymbol{Z}_0)\\ &= \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{Z}_i \mid i \in I\} \cup \{\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j} \mid j \in J, d_j \in D_j\} \cup \{\mathbf{S}_k \mid k \in K\}).\end{aligned}$$

2. If $\boldsymbol{Z}_0 \notin \mathbb{M}$, we obtain by Lemma 2.13 that $p\mathbb{AL}^0_{\mathrm{FSI}} \cap \mathbb{M} = \emptyset$ and thus

$$\begin{aligned}\mathbb{M} &= \mathbf{ISP}_{\mathrm{U}}(\mathbb{K}) \\ &= \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{Z}_i \mid i \in I\} \cup \{\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j} \mid j \in J, d_j \in D_j\} \cup \{\mathbf{S}_k \mid k \in K\}).\end{aligned}$$

This completes the proof. □

Corollary 2.51. *Let $\mathbb{K}$ be the quasivariety generated by a class of totally ordered positively pointed Abelian ℓ-groups. Then there exists $I, J, K \subseteq \mathbb{N}$ and for every $j \in J$, a nonempty subset $D_j \subseteq \mathrm{div}(j)$ such that*

$$\mathbb{K} = \mathbf{ISPP}_{\mathrm{U}}(\{\boldsymbol{Z}_i \mid i \in I\} \cup \{\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j} \mid j \in J, d_j \in D_j\} \cup \{\mathbf{S}_k \mid k \in K\}).$$

Let us note that analogical statements to Theorem 2.50 and Corollary 2.51 can be proved also for negatively pointed Abelian ℓ-groups. Therefore, we can generalize the Theorem 2.50 and Corollary 2.51 as follows.

Corollary 2.52. *Let $\mathbb{M}$ be a class of totally ordered pointed Abelian ℓ-groups. Then the following properties are equivalent:*

1. *$\mathbb{M}$ is universal.*
2. *There exist $I, J, K \subseteq \mathbb{Z}$ and for every $j \in J$, a nonempty subset $D_j \subseteq \mathrm{div}(j)$ such that*

$$\mathbb{M} = \mathbf{ISP}_{\mathrm{U}}(\{\boldsymbol{Z}_i \mid i \in I\} \cup \{\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j} \mid j \in J, d_j \in D_j\} \cup \{\mathbf{S}_k \mid k \in K\}).$$

Corollary 2.53. *Let $\mathbb{K}$ be the quasivariety generated by a class of totally ordered pointed Abelian ℓ-groups. Then there exists $I, J, K \subseteq \mathbb{Z}$ and for every $j \in J \setminus \{0\}$, a nonempty subset $D_j \subseteq \mathrm{div}(j)$ such that*

$$\mathbb{K} = \mathbf{ISPP}_{\mathrm{U}}(\{\boldsymbol{Z}_i \mid i \in I\} \cup \{\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j} \mid j \in J, d_j \in D_j\} \cup \{\mathbf{S}_k \mid k \in K\}).$$

2.7 Future Work

In this section, we outline possible directions for future research on the topics covered in this paper.

- In this paper, we discuss the characterization of all subquasivarieties generated by totally ordered ℓ-groups. However, we provide only the structural characterization of these subquasivarieties without any quasiequational basis. The obvious future step is to provide these axiomatizations, which we can obtain by analyzing the algebras $\boldsymbol{Z}_i$, $\boldsymbol{Z}_j \ltimes \boldsymbol{Z}_{d_j}$ and $\mathbf{S}_k$ from Corollary 2.53.
- One of the main motivations for this paper was to show that when we discuss the subvarieties of $p\mathbb{AL}$, we can do so purely in the theory of pointed Abelian ℓ-groups in a self-contained way without any circular reasoning. However, the same still needs to be done regarding the characterization of subquasivarieties of $p\mathbb{AL}$ that are generated by their totally ordered members.

- One of the most significant drawbacks of Corollary 2.25 is that we require the strongly pointed ℓ-group to be *totally ordered*. It is known that there are subquasivarieties of $p\mathbb{AL}$ that are not generated by their totally ordered members and they have no immediate counterpart in MV-algebras. For example, consider the quasivariety given by the quasiequation $\mathtt{f} \geq 0 \implies x \geq 0$ (see [27]), which is known to not be generated by its totally ordered members. We do not aim to provide an explicit description of all subquasivarieties of $p\mathbb{AL}$, as such a class is likely Q-universal, similarly to the class of all subquasivarieties of MV-algebras [72]. Instead, we aim to explicitly describe the additional subquasivarieties that do not correspond to any subquasivariety of MV-algebras.

- An additional question is connected to expanding the language of pointed Abelian ℓ-groups with a modal operator $\langle \boldsymbol{A}_a, \Box \rangle$ [62]. By doing so, we obtain a variety of modal pointed Abelian ℓ-groups that is not generated by totally ordered algebras. As such, we cannot utilize Theorem 2.23 here, meaning we need to reformulate more general criteria for when a pointed modal Abelian ℓ-group is in a variety generated by its strongly pointed modal convex ℓ-subgroup.

3 Satisfiability in Łukasiewicz logic and its unbounded relative

3.1 Introduction

In the realm of nonclassical logics, and particularly in its subarea known as many-valued logics, the infinite-valued Łukasiewicz logic Ł has long occupied a prominent position. Its evolution (discussed in detail in, e.g., [21, 82, 53]) spanned the introduction of its three-valued antecedent by Łukasiewicz in his famous analysis of future contingents [70], as well as the infinite-valued calculus also introduced by Łukasiewicz and elaborated by himself and Tarski in the paper [71], which first discussed, but did not prove, completeness w.r.t. the intended semantics. A completeness theorem was given much later by Rose and Rosser [92]. MV-algebras, the equivalent algebraic semantics, were introduced and analyzed by Chang, and completeness was proved [20, 19]. Numerous monographs deal exclusively or substantially in the topic, e.g., [21, 53, 79, 82, 26]. Moreover, Łukasiewicz logic and its extensions can fruitfully be investigated in the framework of substructural logics [90, 45].

Increasingly then, the well-established area of Łukasiewicz logic and MV-algebras has been taken as a vantage point for various theoretical or applied problems, be it expansions of its language with constants in the pioneering works of Pavelka [88], modal expansions [39, 38, 54, 96], or reasoning about probabilities [40], to name a few.

Mundici initiated the study of computational complexity problems raised by Łukasiewicz logic in his famous work [83] which showed satisfiability and positive satisfiability of terms in the MV-algebra on the real unit interval with the natural order, $[0,1]_{Ł}$, to be NP-complete. An earlier important work, relevant to many subsequent contributions to the area of algorithmic problems in Łukasiewicz logic, came in McNaughton's work [76] that characterized term-definable functions in $[0,1]_{Ł}$ as continuous, piecewise linear functions with integer coefficients. Algebraic methods are the mainstay of studying computational problems in the area, see especially [2, 3, 4, 1]. Komori's classification of subvarieties of MV [65, 66] yielded the coNP-completeness result for axiomatic extensions of Ł [25], and Mundici's NP-completeness result for term satisfiability generalizes to the existential theory of $[0,1]_{Ł}$ [55]. While Mundici's results place the satisfiability and the tautology problems in Ł on a par with their counterparts in classical propositional logic (either being NP-complete and coNP-complete respectively), this is not a given in the infinite-valued setting. More generally, Łukasiewicz logic is dissimilar from classical logic in many respects and a list to showcase this phenomenon ought not to omit Ł not being structurally complete [36], the admissible rules being PSPACE complete [63], the first-order standard tautologies not being recursively enumerable [94] and in fact Π_2-complete in the arithmetical hierarchy [89], or the complex structure of the lattice of extensions of Ł, which is dually isomorphic to the lattice of subquasivarieties of the variety of MV-algebras. This lattice is Q-universal [72]. Consequently, the extensions cannot all be decidable for cardinality reasons; this is also the case already for term satisfiability problems for various MV-algebras on the rational domain [57].

Unbounded Łukasiewicz logic Łu was considered recently in the papers [24, 27]. Formally, it is obtained by expanding the language of lattice-ordered abelian groups with a propositional constant -1 with suitable axioms ensuring that in any algebra within its equivalent algebraic semantics, the upset of the interpretation of 0 is the set of designated elements, representing truth, whereas the downset of -1 represents falsity. By design, then, the logic bears resemblance to both Łukasiewicz logic and to abelian logic, as considered in Meyer and Slaney's paper [81]. Unlike Ł, and like abelian logic, Łu is inconsistent with classical logic.

Weispfenning [97] proved that the existential theory of the variety of ℓ-groups is NP-complete. So the universal theory of ℓ-groups is coNP-complete and *a fortiori*, the quasiequational theory and the equational theory are in coNP. It is also known that ℓ-groups are generated as a quasivariety by the totally ordered members [30], in fact, by any nontrivial totally ordered member, since the universal theories of each two nontrivial totally ordered abelian groups coincide [50]. One easily shows that the equational theory of the totally ordered additive group on the integers is coNP-hard, thus any of the above universal fragments is coNP-complete.

The purpose of this paper is to provide a complexity estimate for Łu by exhibiting a polynomial-time reduction of the existential theory of the intended semantics of Łu—the algebra $\boldsymbol{R}_{-1}$, the pointed abelian totally ordered additive group on the reals where -1 is interpreted by itself—to the existential theory of the intended semantics of Ł, namely $[0,1]_{Ł}$, the standard MV-algebra on the reals. In both cases, the logics are finitely strongly complete w.r.t. the intended semantics. In our opinion the method surpasses the result in terms of interest, since the result might be anticipated. The method consists in taking a linear map of a sufficiently large neighbourhood of the element 0 to the interval $[0,1]$, with the coefficient of the linear map determined by the instance (i.e., a given existential sentence).

In combination with the extant method of [27] that reduces in polynomial time the set of valid universal formulas (valid identities) of $[0,1]_{Ł}$ to the set of valid universal formulas (valid identities respectively) of the algebra $\boldsymbol{R}_{-1}$ one obtains firstly, a coNP-completeness of each of the universal fragments (i.a., the logic Łu is coNP-complete) and secondly, a non-trivial interpretation of the existential theory of the standard MV-algebra in itself.

The result invites a comparison with the work of Marchioni and Montagna [75], which concerns the logic ŁΠ½ and its relation to the universal theory of real closed fields (RCF). The authors show that the universal fragment of RCF has a polynomial time reduction to the (theorems of) the logic ŁΠ½. The relevance consists in, on the one hand, the logic ŁΠ½ being a conservative expansion of the logic Ł (the theorems of Ł are exactly those theorems of ŁΠ½ that only use the language $\mathcal{L}_{\mathrm{MV}}$) and on the other hand, the universal theory of pointed abelian totally ordered group being term-equivalent to the multiplication- and division-free fragment of the universal theory of RCF. Their reduction starts with fixing a bijection between $\boldsymbol{R}$ and the interval $(0,1)$ and then defining the operations of the real closed field on $\boldsymbol{R}$ in the ŁΠ½-algebra on the reals. This approach cannot be used in the present case because it relies on the multiplicative function symbols of the logic Π even when defining the real addition; so it does not scale well to language fragments.

This paper is structured as follows. Key notions are presented in section 3.2. Section 3.3 establishes that assignments that make an open formula of the language of pointed ℓ-groups true, if any, can be found in a suitably bounded interval. Section 3.4 presents the main result, the polynomial-time reduction. Section 3.5 then composes two reductions in order to obtain a nontrivial reduction of the existential theory of $[0,1]_{Ł}$ into itself.

3.2 Preliminaries

Let $\mathrm{Var} = \{x_1, x_2, \dots\}$ be a denumerable set of variables. A first-order language $\mathcal{L}$ uniquely determines the set $Tm_{\mathcal{L}}$ of terms of $\mathcal{L}$. The logical symbol $=$ is the only predicate symbol of any algebraic language; however, we may additionally use binary predicate symbols $\leq, <$ (in Section 3.3). If φ and ψ are terms of $\mathcal{L}$, then $\varphi \diamond \psi$ is an atomic formula (or "atom") of $\mathcal{L}$ for $\diamond \in \{=, \leq, <\}$. If there are no predicate symbols in $\mathcal{L}$ then $\mathcal{L}$-atoms are just $\mathcal{L}$-identities $\varphi = \psi$. We use lowercase Greek for terms and uppercase Greek for first-order formulas. For boolean symbols occurring in first-order formulas (boolean combinations of atoms of $\mathcal{L}$) we use $\neg$ for boolean negation, $\wedge$ and $\vee$ for boolean conjunction and disjunction respectively, $\bigwedge$ and $\bigvee$ for finite boolean conjunctions and disjunctions respectively. An open formula of $\mathcal{L}$ is an arbitrary finite boolean combination (i.e., a $\{\neg, \wedge, \vee\}$-combination) of $\mathcal{L}$-atoms; we do not in general assume a normal form for first-order boolean formulas. Moreover, if $\Rightarrow$ denotes boolean implication, quasiidentities of $\mathcal{L}$ are formulas of the form $\bigwedge_{i=1}^{n}(\varphi_i = \psi_i) \Rightarrow (\varphi_0 = \psi_0)$ where $\{\varphi_i, \psi_i\}_{0 \leq i \leq n}$ is a set of $\mathcal{L}$-terms. Since quasiequational theories are not of particular interest in this work, $\Rightarrow$ is not taken as primitive and a formula $\Phi \Rightarrow \Psi$ is understood as $\neg\Phi \vee \Psi$ with only an insubstantial increase in length. An existential (universal) sentence of a language $\mathcal{L}$ is an existential (universal respectively) closure of an open formula of $\mathcal{L}$.

If φ is an $\mathcal{L}$-term, $\mathrm{Var}(\varphi)$ denotes the set of variables that occur in φ, and analogously for $\mathrm{Var}(\Phi)$ if Φ is an $\mathcal{L}$-formula. We use $\bar{x}$ for $\langle x_1, \dots, x_n \rangle$ if n is of no consequence.

The following languages will be used:[1]

- $\mathcal{L}_{\mathrm{Ab}} = \langle +, -, 0, \wedge, \vee \rangle$ is the language of abelian lattice-ordered groups (ℓ-groups) and Tm_{Ab} is the corresponding set of well-formed terms;
- $\mathcal{L}_{\mathrm{pAb}} = \langle +, -, 0, -1, \wedge, \vee \rangle$ is the language of pointed ℓ-groups and Tm_{pAb} the set of its well-formed terms;
- $\mathcal{L}_{\mathrm{MV}} = \langle \oplus, \otimes, \neg, \rightarrow, 0, 1, \wedge, \vee \rangle$ is the language of MV-algebras and Tm_{MV} the set of its well-formed terms;
- $\mathcal{L}_{\mathrm{MV}[\frac{1}{2}]} = \langle \oplus, \otimes, \neg, \rightarrow, 0, 1, {}^1\!/_2, \wedge, \vee \rangle$ expands the language of MV-algebras with a new constant $1/2$ and $Tm_{\mathrm{MV}[\frac{1}{2}]}$ is the set of its well-formed terms.

$\boldsymbol{R} = \langle \mathbb{R}, +, -, \wedge, \vee, 0 \rangle$ is an abelian lattice-ordered group (ℓ-group) on the real numbers with the usual order, (the usual) addition, subtraction and 0. The algebra $\boldsymbol{R}$ provides a semantics to the language $\mathcal{L}_{\mathrm{Ab}}$.

[1]We write Tm_{Ab} instead of $Tm_{\mathcal{L}_{\mathrm{Ab}}}$. We use the same notation for a function symbol of a language and for its interpretation in a particular algebra where possible, to avoid cluttering.

$\boldsymbol{R}_{-1} = \langle \mathbb{R}, +, -, \wedge, \vee, 0, -1 \rangle$ is a *pointed* abelian ℓ-group whose $\mathcal{L}_{\mathrm{Ab}}$-reduct is $\boldsymbol{R}$ and the element -1 interprets the constant -1.[2]

In the *standard MV-algebra* $[0,1]_{Ł}$ on the reals, which provides an interpretation to the language $\mathcal{L}_{\mathrm{MV}}$, the domain is $[0,1]$, as for the basic operations, $\neg a$ is $1-a$ and $a \otimes b$ is $\max(0, a+b-1)$ for $a, b \in [0,1]$. The remaining function symbols are definable.

- $a \oplus b$ is $\neg(\neg a \otimes \neg b)$
- $a \to b$ is $\neg a \oplus b$
- $a \wedge b$ is $a \otimes (a \to b)$
- $a \vee b$ is $(a \to b) \to b$
- 0 is $a \otimes \neg a$ for some a
- 1 is $\neg 0$

In any MV-algebra, $\neg\neg a = a$ holds (involutive law), analogously in any abelian ℓ-group, we have $- - a = a$. In view of this, we assume that terms are given in a reduced form within their respective languages, containing no subterms of the form $\neg\neg\varphi$ or $- - \varphi$, respectively.

The two element boolean algebra is a subalgebra of $[0,1]_{Ł}$. The algebra $[0,1]_{Ł}^{1/2}$ interprets the language $\mathcal{L}_{\mathrm{MV}[\frac{1}{2}]}$. Its $\mathcal{L}_{\mathrm{MV}}$-reduct is the algebra $[0,1]_{Ł}$ and the element $1/2$ interprets the constant $1/2$.

It is assumed throughout that no well-formed term of the languages under consideration contains two consecutive occurrences of the symbol $-$ or the symbol $\neg$. This can be assumed without a loss of generality, relying on the valid involutive laws.

One can introduce the (infinite-valued) Łukasiewicz logic Ł and the unbounded Łukasiewicz logic Łu axiomatically and then obtain that Ł is finitely strongly complete w.r.t. $[0,1]_{Ł}$ and Łu is finitely strongly complete w.r.t. $\boldsymbol{R}_{-1}$. We take these results for granted, referring the reader to the works [21, 53, 27]. Moreover, without further ado we use Mundici's result on the NP-completeness of the term satisfiability problem in $[0,1]_{Ł}$ and the coNP-completeness of term validity therein. Recall that the designated element of $[0,1]_{Ł}$ is 1; accordingly, a term $\varphi(\bar{x})$ holds under an assignment $\bar{a}$ provided that $\varphi(\bar{a}) = 1$, and $\varphi(\bar{x})$ is *satisfiable* in $[0,1]_{Ł}$ provided that an assignment exists that makes it true there. We retain this terminology also for first-order open formulas: such a formula $\Phi(\bar{x})$ is satisfiable in $[0,1]_{Ł}$ provided an assignment to its variables exists that makes it true therein. A term φ of $\mathcal{L}_{\mathrm{MV}}$ is *valid* in $[0,1]_{Ł}$ provided it is true under any assignment therein. The set of designated elements of $\boldsymbol{R}_{-1}$ is the set of its nonnegative elements (the upset of 0), and accordingly a term is satisfiable in $\boldsymbol{R}_{-1}$ provided some assignment sends it to a nonnegative element; the rest is analogous.

If $\mathcal{L}$ is a language and $\boldsymbol{A}$ is an $\mathcal{L}$-algebra, the existential theory ($\exists$-theory) of $\boldsymbol{A}$ is the set of all existential $\mathcal{L}$-sentences valid in $\boldsymbol{A}$; dually for the universal theory ($\forall$-theory) and universal $\mathcal{L}$-sentences. The (quasi)equational theory of $\boldsymbol{A}$ is the set

[2] While the algebra $\boldsymbol{R}_{-1}$ seems very similar to $\boldsymbol{R}$, the additional constant in the language grants much more expressivity and brings the logic Łu closer to the logic Ł.

of universal closures of $\mathcal{L}$-(quasi)identities valid in $\boldsymbol{A}$. Both the equational theory and the quasiequational theory of an algebra $\boldsymbol{A}$ are fragments of the universal theory of $\boldsymbol{A}$.

Each term φ of $\mathcal{L}_{\mathrm{Ab}}$ in n variables $x_1, \ldots, x_n$ in Var uniquely determines a function $f_\varphi \colon \mathbb{R}^n \to \mathbb{R}$ (on the domain of the algebra $\boldsymbol{R}$) using the following inductive steps:

- $f_x = x$ for each variable x in Var;
- $f_{\varphi+\psi} = f_\varphi + f_\psi$;
- $f_{-\varphi} = -f_\varphi$;
- $f_{\varphi\wedge\psi} = \min(f_\varphi, f_\psi)$;
- $f_{\varphi\vee\psi} = \max(f_\varphi, f_\psi)$;
- $f_0 = 0$.

Analogously for a term φ of $\mathcal{L}_{\mathrm{pAb}}$. Furthermore, a term φ of $\mathcal{L}_{\mathrm{MV}}$ uniquely determines a function $f_\varphi \colon [0,1]^n \to [0,1]$ (on the domain of the standard MV-algebra), i.e.,

- $f_{\varphi\otimes\psi} = \max(0, f_\varphi + f_\psi - 1)$,
- $f_{\neg\varphi} = 1 - f_\varphi$,

etc., and analogously for a term of the language $\mathcal{L}_{\mathrm{MV}[\frac{1}{2}]}$.

For a term φ of $\mathcal{L}_{\mathrm{Ab}}$, the *depth* of φ is defined by induction: $\mathrm{depth}(0) = 0$; $\mathrm{depth}(x) = 0$ if $x \in \mathrm{Var}$; $\mathrm{depth}(-\varphi) = \mathrm{depth}(\varphi) + 1$; and $\mathrm{depth}(\varphi \circ \psi) = \max(\mathrm{depth}(\varphi), \mathrm{depth}(\psi)) + 1$ if $\circ$ is one of $\{+, \wedge, \vee\}$. Analogously for the other languages, with depth of constants set to 0.

For a term φ and a first-order formula Φ of $\mathcal{L}_{\mathrm{Ab}}$, $\sharp(\varphi)$ and $\sharp(\Phi)$ will denote the *length* (or *size*) of φ and Φ respectively. We define $\sharp(\varphi)$ as the total number of occurrences of variables and constants in φ; should one view φ as a tree, $\sharp(\varphi)$ is the number of leaves. Recalling that in φ no two consecutive negations '$-$' can occur, the number of occurrences of function symbols in φ (i.e., of the inner nodes of the tree) is bounded by $3\sharp(\varphi)$. The sizes of representations of each variable and each constant are neglected and representation in unit length is considered.

For an open Φ, we use $\mathrm{At}^\sharp(\Phi)$ for the number of occurrences of atomic formulas. We define $\sharp(\Phi)$ to be the sum of the sizes of all the occurrences of terms in the atomic identities; we have $\sharp(\Phi) \leq \mathrm{At}^\sharp(\Phi) \cdot 2\max\{\sharp(\varphi) \mid \varphi \text{ atom in } \Phi\}$. Again any two consecutive boolean negations are omitted. Analogous considerations apply to the remaining languages.

Let $\mathcal{L}$ be a language and Φ an open $\mathcal{L}$-formula. A *Tseitin variant* of Φ is an open formula Φ' with some desirable properties: $\sharp(\Phi')$ is of polynomial size in $\sharp(\Phi)$; every atom of Φ' contains at most one function symbol (in particular, terms of Φ' have depth at most 1); and Φ' is equisatisfiable with Φ in any $\mathcal{L}$-structure $\boldsymbol{M}$ (i.e., the existential closure of Φ holds in $\boldsymbol{M}$ iff so does the existential closure of Φ'). We proceed to define Tseitin variants formally.

Let T be the set of all subterms of $\mathcal{L}$-terms that occur in $\Phi(x_1, \ldots, x_n)$. Let $|T|$ be the cardinality of T. To each term φ in T that is not a variable among

$x_1, \ldots, x_n$, assign a new variable x_φ from Var, not occurring in Φ, with distinct variables assigned to distinct terms. We have $|T| \leq 4\sharp(\Phi)$. Define a new open formula Ψ from Φ as follows. For every atom $\varphi \diamond \psi$ occurring in Φ (here $\diamond$ is among $\{=, <\}$), if either of φ or ψ is not a variable among $x_1, \ldots, x_n$, replace this occurrence of φ or ψ with the new variable x_φ or x_ψ respectively. For convenience, if the term φ is a variable, x_φ denotes the original variable φ. Then the Tseitin variant of Φ is

$$\Phi' \coloneqq \Psi \wedge \bigwedge_{\substack{\varphi \text{ occurs in } \Phi \\ \varphi \text{ is } f(\varphi_1,\ldots,\varphi_n) \text{ for some } n\text{-ary function symbol } f \in \mathcal{L}}} x_\varphi = f(x_{\varphi_1}, \ldots, x_{\varphi_n}),$$

where $x_\varphi, x_{\varphi_1}, \ldots, x_{\varphi_n}$ are metavariables for elements of Var. Observe Φ' has no compound terms. Furthermore, $\mathrm{At}^\sharp(\Phi') \leq \mathrm{At}^\sharp(\Phi) + |T|$. Lastly, Φ' is indeed equisatisfiable with Φ in any $\mathcal{L}$-structure $\boldsymbol{M}$: for any assignment $v_{\boldsymbol{M}}$ in $\boldsymbol{M}$ one has that $v_{\boldsymbol{M}}$ satisfies Φ in $\boldsymbol{M}$ if and only if Φ' is satisfied by the assignment $v'_{\boldsymbol{M}}$, where $v'_{\boldsymbol{M}}$ extends $v_{\boldsymbol{M}}$ by sending each added variable x_φ to $v_{\boldsymbol{M}}(\varphi)$.

3.3 Bounded models

This section establishes a technical result to be used below: if an existential sentence $\exists x_1 \exists x_2 \ldots \exists x_n \Phi$ in the language $\mathcal{L}_{\mathrm{pAb}}$ holds in $\boldsymbol{R}_{-1}$, then there is a satisfying (rational) assignment whose binary representation has size polynomial in $\sharp(\Phi)$. Our exposition follows (in some detail) the one in the paper [37] as to the general method.

Lemma 3.1. *Let k, m, n be positive natural numbers. Let $Ax \leq b$ be a system of linear inequalities with A an $n \times m$ integer matrix, b a rational n-tuple, and $a_{i,j}, b_i \in [-k, k]$ for $i \leq n, j \leq m$. Assume the system has a solution in $\mathbb{R}$. Then there is a (rational) solution in the interval $[-(mk)^m, (mk)^m]$.*

Proof. The polytope $P = \{x \mid Ax \leq b\}$ is nonempty. Fix a non-empty face of P minimal under inclusion. Such a minimal face is given by a system of equations $\hat{A}x = \hat{b}$, a subsystem of $Ax = b$. To solve this system, it is enough to solve a system $A^0 y = b^0$, where A^0 and b^0 are obtained from $\hat{A}$ and $\hat{b}$ by fixing d columns that contain pivots after Gauss elimination (permuting columns if necessary, assume the first d columns of A) and fixing the corresponding d linearly independent rows. The matrix A^0 is regular of rank d and b^0 is a d-tuple of rationals. A solution of $A^0 y = b^0$ also yields a solution of $\hat{A}x = \hat{b}$ by assigning the value 0 to all variables $x_{d+1}, \ldots, x_m$. Cramer's rule yields the solution

$$\langle y_i \rangle_{i \leq d} = \langle \frac{\det(A_i^0)}{\det(A^0)} \rangle_{i \leq d}.$$

Since A has integer entries, $|\det(A^0)| \geq 1$ and thus $|y_i| \leq |\det(A_i^0)|$ for each $i \leq d$. We have $|\det(A_i^0)| \leq d! \cdot k^d \leq d^d \cdot k^d = (kd)^d$. So $y_i \in [-(dk)^d, (dk)^d]$ for each $i \leq d$, hence for each $i \leq m$. Since $d \leq \min(m, n)$ we have $x_i \in [-(mk)^m, (mk)^m]$ for $i \leq m$. $\square$

Lemma 3.2 (Bounded assignment)**.** *There exists an integer C such that for every open formula $\Phi(x_1, \ldots, x_n)$ of the language $\mathcal{L}_{\mathrm{pAb}}$ with $N = \sharp(\Phi)$, the following holds:*

If $\boldsymbol{R}_{-1} \models \exists x_1 \ldots \exists x_n \Phi$, then there are real numbers $\langle a_1, \ldots, a_n \rangle$ and an integer M such that $\Phi(a_1, \ldots, a_n)$ holds in $\boldsymbol{R}_{-1}$, $-2^M \leq a_i \leq 2^M$ for each $1 \leq i \leq n$, and $M \leq CN \log_2 N$.

Proof. For convenience, the symbol and $<$ is added to the language $\mathcal{L}_{\mathrm{pAb}}$. Let $\Phi(x_1, \ldots, x_n)$ be given. Applying de Morgan rules and removing any double boolean negations that occur, Φ can be rewritten so that any boolean negation is applied to an atom. Using trichotomy, any atom $\neg(\varphi = \psi)$ can be replaced with $(\varphi < \psi) \vee (\psi < \varphi)$. This yields an equivalent negation-free formula $\Phi'(x_1, \ldots, x_n)$ whose atoms are identities and strict inequalities. Moreover $\mathrm{At}^\sharp(\Phi') \leq 2\mathrm{At}^\sharp(\Phi)$.

Φ' can be brought to a DNF without negations, obtaining a

$$\Phi'' = \Psi_1 \vee \Psi_2 \vee \ldots \vee \Psi_m$$

for some m, where each Ψ_j for $1 \leq j \leq m$ is a conjunction of atoms of the form $(\varphi = \psi)$ or $(\varphi < \psi)$. The value of m need not be polynomial in $\mathrm{At}^\sharp(\Phi')$. The remaining part of the proof establishes that a bounded satisfying assignment to Φ exists on the basis of the existence of the formula $\Phi'' = \mathrm{DNF}(\Phi')$ and the fact, which we proceed to show, of an existence of a bounded satisfying assignment to at least one among the Ψ_j. Both Φ' and Φ'' are equivalent to Φ; namely, a tuple $\langle a_1, \ldots, a_n \rangle$ makes Φ true in $\boldsymbol{R}_{-1}$ if and only if it makes Φ' true, and the latter is the case if and only if it makes some Ψ_j true for some $j \leq m$. Trivially $\mathrm{At}^\sharp(\Psi_j) \leq \mathrm{At}^\sharp(\Phi')$ for $1 \leq j \leq m$.

Next, we remove any complex terms from Φ'' using a Tseitin transformation. Let $\Psi'_1, \ldots \Psi'_m$ be Tseitin variants of $\Psi_1, \ldots \Psi_m$. Without loss of generality, we assume that the Tseitin transformation assigns the same variable x_t to identical terms t, regardless of the disjunct in which they appear. Consequently, $|\bigcup_{j=1}^m \mathrm{V}ar(\Psi'_j)| = |T|$, where $|T|$ is the cardinality of the set T of all $\mathcal{L}_{\mathrm{pAb}}$-(sub)terms that occur in Φ'.

Let us set

$$(\Phi''') = (\Psi'_1 \vee \ldots \vee \Psi'_m).$$

Since Ψ_j is equisatisfiable with Ψ'_j for each $j \leq m$, we obtain that Φ'' is equisatisfiable with Φ'''. Let k denote the total number of variables in Φ'''. Clearly, we have $k = |T| \leq 4\sharp(\Phi')$.

The formula Φ''' is still in DNF and we have $\mathrm{At}^\sharp(\Psi'_j) \leq |T| + \sharp(\Psi_j) \leq 5\sharp(\Phi')$ for each $j \leq m$.

By the assumption, there is an assignment v in $\boldsymbol{R}_{-1}$ with $v(x_i) = a_i$ for $i \leq k$ such that the vector $\langle a_1, \ldots, a_k \rangle$ makes both Φ and Φ''' true in $\boldsymbol{R}_{-1}$. Fix such an assignment v. Then there is a total ordering given by a permutation σ of $\{1, \ldots, k\}$ such that, in $\boldsymbol{R}$, $a_{\sigma(i)} \leq a_{\sigma(i+1)}$ for each $i < k$. Let

$$\rho(x_1, \ldots, x_k) \coloneqq \bigwedge_{\{i<k\};\, a_{\sigma(i)} < a_{\sigma(i+1)}} (x_{\sigma(i)} < x_{\sigma(i+1)}) \wedge \bigwedge_{\{i<k\}; a_{\sigma(i)} = a_{\sigma(i+1)}} (x_{\sigma(i)} = x_{\sigma(i+1)}).$$

We have $\mathrm{At}^\sharp(\rho) \leq k$. For some $j \leq m$, $(\rho \wedge \Psi'_j)(a_1, \ldots, a_k)$ is true in $\boldsymbol{R}_{-1}$.

Fix such $j \leq m$ and let us show that there exists an assignment that makes the formula $(\rho \wedge \Psi'_j)$ true $\boldsymbol{R}_{-1}$ and whose values are bounded as required. We refer to the formula Ψ'_j simply as Ψ.

Using ρ, one can omit atoms with the function symbols $\wedge$ and $\vee$. Indeed, since $\rho \wedge \Psi$ is true under v, it must also be the case that if, e.g., $x_\varphi \wedge x_\psi = x_\chi$ occurs as an atom in Ψ (which means either that the term χ is the term $\varphi \wedge \psi$ in Φ, or that $\chi = \varphi \wedge \psi$ is an identity in Φ), then the identity $\varphi(a) \wedge \psi(a) = \chi(a)$ holds and hence, this is captured by ρ, namely $x_\chi =_\rho \min_\rho(x_\varphi, x_\psi)$. The other cases featuring $\vee$ are analogous. Thus ρ already captures all the information that is provided by those atoms in $\rho \wedge \Psi$ in which the lattice function symbols $\wedge$ and $\vee$ occur, whereby these atomic formulas are redundant and can be omitted from the conjunction.

Such an omission yields a new conjunction of identities and strict inequalities $\rho \wedge \Psi^\star$. This formula has k variables, and

$$l := \mathrm{At}^\sharp(\rho \wedge \Psi^\star) = \mathrm{At}^\sharp(\rho) + \mathrm{At}^\sharp(\Psi'_j) \leq \sharp(\Phi') + 5\sharp(\Phi') = 6\sharp(\Phi') \leq 12\sharp(\Phi).$$

Now $\rho \wedge \Psi^\star$ defines a system S of linear equations and strict inequalities in $\boldsymbol{R}$ as follows. Any atomic formula in $\rho \wedge \Psi^\star$ has at most three distinct variables and at most one function symbol. Subtracting the expression on right-hand side of each atomic formula from both sides, we get a system consisting of a subsystem $Ax = c$ with l_1 rows, and a subsystem $Bx < d$ with l_2 rows, both in the variables $x_1, \ldots, x_k$ and with $l_1 + l_2 = l$, thus $l_1, l_2 \leq 12\sharp(\Phi)$. Since $\langle a_1, \ldots, a_k \rangle$ is a solution to the system, there has to exist a rational vector $d' = \langle d'_1, \ldots, d'_{l'_2} \rangle$ where $d_i - 1 \leq d'_i < d_i$, such that a modified system S' consisting of $Ax = c$ and $Bx \leq d'$ still has a solution. Clearly, any solution to S' is also a solution to S.

Rewriting each of the equations in $Ax = c$ as two inequalities and adding the rows of $Bx \leq d'$ yields a system $A'x \leq b'$, s.t. there is an integer $c \leq 36$ with the number of both rows and columns in $A'x \leq b'$ bounded by $C \cdot \sharp(\Phi)$. A' is an integer matrix, b' is a rational vector, and the absolute values of entries in either are bounded by 3. Applying Lemma 3.1 yields a solution whose absolute value is bounded by $(3C\sharp(\Phi))^{C\sharp(\Phi)}$. It is then enough to pick $M = \lceil C\sharp(\Phi) \log_2(3C\sharp(\Phi)) \rceil$. □

Using the tools of this section, one might proceed to show that the existential theory of $\boldsymbol{R}_{-1}$ is in NP. Indeed, one gets almost for free from the above, for any open $\mathcal{L}_{\mathrm{pAb}}$-formula Φ with a real assignment that satisfies it, also a satisfying assignment that is rational with both the numerators and the denominators of polynomial size in $\sharp(\Phi)$. This assignment is then a witness of the fact that the existential closure of Φ is true, whence the existential theory of $\boldsymbol{R}_{-1}$ is in NP. However, our intent is to use the results of this section to obtain suitable maps on the reals in the next section, and hence a polynomial-time reduction of the existential theory of $\boldsymbol{R}_{-1}$ to the existential theory of $[0,1]_{Ł}$. In other words, we answer the satisfiability question, the instances of which are open formulas, indirectly. It is worth remarking that, if one established NP-completeness of the existential theory of $\boldsymbol{R}_{-1}$ anyhow, then one would still get a poly-time reduction between this problem and the existential theory of $[0,1]_{Ł}$, because the latter problem is known to be NP-complete (using, e.g., [52, 55]). Our reduction has the added value of being explicit.

3.4 Reducing the $\exists$-theory of $\boldsymbol{R}_{-1}$ to the $\exists$-theory of $[0,1]_{Ł}$

Consider the mapping $\tau\colon Tm_{\mathrm{Ab}} \to Tm_{\mathrm{MV}[\frac{1}{2}]}$, assigning to each term φ of $\mathcal{L}_{\mathrm{Ab}}$ a term of $\mathcal{L}_{\mathrm{MV}[\frac{1}{2}]}$ as follows:

1. $\tau(x) = x$ for $x \in \mathrm{V}ar(\varphi)$;
2. $\tau(0) = \frac{1}{2}$;
3. $\tau(-\varphi) = \neg(\tau(\varphi))$;
4. $\tau(\varphi + \psi) = (\tau(\varphi) \oplus \tau(\psi)) \otimes (\frac{1}{2} \oplus (\tau(\varphi) \otimes \tau(\psi)))$;
5. $\tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi)$;
6. $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$.

For each pair of nonnegative integers M and k, let $r_{M,k} : \mathbb{R} \to \mathbb{R}$ be defined as

$$r_{M,k}(a) = \frac{a}{2^{M+k+1}} + \frac{1}{2}. \tag{3.1}$$

Lemma 3.3. *Let M and k be fixed nonnegative integers. Let $\varphi(x_1, \ldots, x_n)$ be a term in Tm_{Ab} with n variables, let* $\mathrm{depth}(\varphi) = l$ *with $l \leq k$. Assume $a_1, \ldots, a_n \in [-2^M, 2^M]$. Then*

$$f_{\tau(\varphi)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = r_{M,k}(f_\varphi(a_1, \ldots, a_n)).$$

Moreover, $r_{M,k}(f_\varphi(a_1, \ldots, a_n)) \in [\frac{1}{2} - \frac{1}{2^{k-l+1}}, \frac{1}{2} + \frac{1}{2^{k-l+1}}]$.

Proof. First let us note that for each $a \in [-2^M, 2^M]$ it holds that $r_{M,k}(a) \in [\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$. We will proceed by induction on term depth.

- Let φ be a variable x. By definition we have

$$f_{\tau(\varphi)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = r_{M,k}(f_\varphi(a_1, \ldots, a_n)) \in [\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}].$$

- Let $\varphi = 0$. We have

$$f_{\tau(\varphi)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = \frac{1}{2} = r_{M,k}(f_\varphi(a_1, \ldots, a_n)).$$

Now assume $\mathrm{depth}(\varphi) = l$ and the statement holds for any $l' < l$.

- Let φ be $-\psi$. We assume $f_{\tau(\psi)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = r_{M,k}(f_\psi(a_1, \ldots, a_n))$. We have

$$\begin{aligned} f_{\tau(-\psi)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) &= f_{\neg\tau(\psi)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) \\ = 1 - f_{\tau(\psi)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) &= 1 - r_{M,k}(f_\psi(a_1, \ldots, a_n)) \\ = 1 - (\frac{f_\psi(a_1, \ldots, a_n)}{2^{M+k+1}} + \frac{1}{2}) &= \frac{1}{2} - \frac{f_\psi(a_1, \ldots, a_n)}{2^{M+k+1}} = \frac{1}{2} + \frac{f_{-\psi}(a_1, \ldots, a_n)}{2^{M+k+1}} \\ &= r_{M,k}(f_{-\psi}(a_1, \ldots, a_n)). \end{aligned}$$

Moreover, since by (the induction) hypothesis

$$f_{\tau(\psi)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) \in [\frac{1}{2}-\frac{1}{2^{k-(l-1)+1}},\frac{1}{2}+\frac{1}{2^{k-(l-1)+1}}],$$

we obtain

$$\begin{aligned}f_{\tau(-\psi)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) &= 1-f_{\tau(\psi)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n))\\ &\in [\frac{1}{2}-\frac{1}{2^{k-l+2}},\frac{1}{2}+\frac{1}{2^{k-l+2}}]\subseteq[\frac{1}{2}-\frac{1}{2^{k-l+1}},\frac{1}{2}+\frac{1}{2^{k-l+1}}].\end{aligned}$$

- Let $\varphi=\psi\wedge\gamma$. We assume

$$\begin{aligned}f_{\tau(\psi)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) &= r_{M,k}(f_\psi(a_1,\ldots,a_n)),\\ f_{\tau(\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) &= r_{M,k}(f_\gamma(a_1,\ldots,a_n)).\end{aligned}$$

We obtain

$$\begin{aligned}f_{\tau(\psi\wedge\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) &= f_{\tau(\psi)\wedge\tau(\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n))\\ =(f_{\tau(\psi)}\wedge f_{\tau(\gamma)})(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) &= ((r_{M,k}\circ f_\psi)\wedge(r_{M,k}\circ f_\gamma))(a_1,\ldots,a_n).\end{aligned}$$

Since $r_{M,k}$ is order preserving we obtain

$$\begin{aligned}((r_{M,k}\circ f_\psi)\wedge(r_{M,k}\circ f_\gamma))(a_1,\ldots,a_n) &= r_{M,k}((f_\psi\wedge f_\gamma)(a_1,\ldots,a_n))\\ &= r_{M,k}((f_{\psi\wedge\gamma})(a_1,\ldots,a_n)).\end{aligned}$$

Since

$$f_{\tau(\psi\wedge\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) = f_{\tau(\psi)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)),$$

or

$$f_{\tau(\psi\wedge\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) = f_{\tau(\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)),$$

then necessarily

$$f_{\tau(\psi\wedge\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) \in [\frac{1}{2}-\frac{1}{2^{k-(l-1)+1}},\frac{1}{2}+\frac{1}{2^{k-(l-1)+1}}]$$

and therefore

$$f_{\tau(\psi\wedge\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) \in [\frac{1}{2}-\frac{1}{2^{k-l+1}},\frac{1}{2}+\frac{1}{2^{k-l+1}}].$$

- The case $\varphi=\psi\vee\gamma$ is analogous.
- Now let φ be $\psi+\gamma$. We assume

$$\begin{aligned}f_{\tau(\psi)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) &= r_{M,k}(f_\psi(a_1,\ldots,a_n)),\\ f_{\tau(\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) &= r_{M,k}(f_\gamma(a_1,\ldots,a_n))\end{aligned}$$

and since $l=\mathrm{depth}(\varphi)=\max(\mathrm{depth}(\psi),\mathrm{depth}(\gamma))+1$ we obtain

$$f_{\tau(\psi)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) \in [\frac{1}{2}-\frac{1}{2^{k-l+2}},\frac{1}{2}+\frac{1}{2^{k-l+2}}].$$

and

$$f_{\tau(\gamma)}(r_{M,k}(a_1),\ldots,r_{M,k}(a_n)) \in [\frac{1}{2}-\frac{1}{2^{k-l+2}},\frac{1}{2}+\frac{1}{2^{k-l+2}}].$$

We distinguish two cases:

1. In case
$$(f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) \geq 1$$
we have
$$f_{\tau(\psi)\oplus\tau(\gamma)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = 1$$
and
$$f_{\tau(\psi)\otimes\tau(\gamma)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = (f_{\tau(\psi)}+f_{\tau(\gamma)})(r_{M,k}(a_1), \ldots, r_{M,k}(a_n))-1.$$
Thus, we have
$$\begin{aligned}
&f_{\tau(\psi+\gamma)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) \\
=&f_{(\tau(\psi)\oplus\tau(\gamma))\otimes(\frac{1}{2}\oplus(\tau(\psi)\otimes\tau(\gamma)))}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) \\
=&f_{1\otimes(\frac{1}{2}\oplus(\tau(\psi)\otimes\tau(\gamma)))}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) \\
=&f_{\frac{1}{2}\oplus(\tau(\psi)\otimes\tau(\gamma))}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n) \\
=&\min(1, f_{\tau(\psi)\otimes\tau(\gamma)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) + \frac{1}{2}) \\
=&\min(1, (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) - 1 + \frac{1}{2}) \\
=&\min(1, (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) - \frac{1}{2}) \\
=&\min(1, (r_{M,k} \circ f_\psi + r_{M,k} \circ f_\gamma)(a_1, \ldots, a_n) - \frac{1}{2}).
\end{aligned}$$
Moreover, by the induction hypothesis
$$(r_{M,k} \circ f_\psi + r_{M,k} \circ f_\gamma)(a_1, \ldots, a_n) - \frac{1}{2} \in [\frac{1}{2} - \frac{1}{2^{k-l+1}}, \frac{1}{2} + \frac{1}{2^{k-l+1}}]$$
and thus, we can conclude
$$\min(1, (r_{M,k}(f_\psi+f_\gamma)(a_1, \ldots, a_n))-\frac{1}{2}) = (r_{M,k}\circ f_\psi+r_{M,k}\circ f_\gamma)(a_1, \ldots, a_n)-\frac{1}{2}.$$
Therefore,
$$\begin{aligned}
&f_{\tau(\psi+\gamma)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = (r_{M,k} \circ f_\psi + r_{M,k} \circ f_\gamma)(a_1, \ldots, a_n) - \frac{1}{2} \\
=&\frac{(f_\psi + f_\gamma)(a_1, \ldots, a_n)}{2^{M+k+1}} + \frac{1}{2} = r_{M,k}(f_{\psi+\gamma}(a_1, \ldots, a_n)).
\end{aligned}$$
This completes the proof of the first case.

2. In case
$$(f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) \leq 1,$$
we have
$$f_{\tau(\psi)\otimes\tau(\gamma)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = 0$$
and
$$f_{\tau(\psi)\oplus\tau(\gamma)}(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) = (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)).$$

Thus, we have

$$\begin{aligned}
&f_{\tau(\psi+\gamma)}(r_{M,k}(a_1),\dots,r_{M,k}(a_n))\\
=&f_{(\tau(\psi)\oplus\tau(\gamma))\otimes(\frac{1}{2}\oplus(\tau(\psi)\otimes\tau(\gamma)))}(r_{M,k}(a_1),\dots,r_{M,k}(a_n))\\
=&f_{(\tau(\psi)\oplus\tau(\gamma))\otimes(\frac{1}{2}\oplus 0)}(r_{M,k}(a_1),\dots,r_{M,k}(a_n))\\
=&f_{(\tau(\psi)\oplus\tau(\gamma))\otimes\frac{1}{2}}(r_{M,k}(a_1),\dots,r_{M,k}(a_n))\\
=&\max(0,(f_{\tau(\psi)}+f_{\tau(\gamma)})(r_{M,k}(a_1),\dots,r_{M,k}(a_n))+\frac{1}{2}-1)\\
=&\max(0,(f_{\tau(\psi)}+f_{\tau(\gamma)})(r_{M,k}(a_1),\dots,r_{M,k}(a_n))-\frac{1}{2})\\
=&\max(0,(r_{M,k}\circ f_\psi+r_{M,k}\circ f_\gamma)(a_1,\dots,a_n)-\frac{1}{2}).
\end{aligned}$$

Since by the induction hypothesis

$$(r_{M,k}\circ f_\psi+r_{M,k}\circ f_\gamma)(a_1,\dots,a_n)-\frac{1}{2}\in[\frac{1}{2}-\frac{1}{2^{k-l+1}},\frac{1}{2}+\frac{1}{2^{k-l+1}}]$$

we can conclude that

$$\begin{aligned}
&f_{\tau(\psi+\gamma)}(r_{M,k}(a_1),\dots,r_{M,k}(a_n))=(r_{M,k}\circ f_\psi+r_{M,k}\circ f_\gamma)(a_1,\dots,a_n))-\frac{1}{2}\\
&=\frac{(f_\psi+f_\gamma)(a_1,\dots,a_n)}{2^{M+k+1}}+\frac{1}{2}=r_{M,k}(f_{\psi+\gamma}(a_1,\dots,a_n)).
\end{aligned}$$

This completes the second case and thus concludes the proof. □

We will extend τ to open $\mathcal{L}_{\text{Ab}}$-formulas. For Φ of $\mathcal{L}_{\text{Ab}}$ define $\tau'(\Phi)$ by replacing each occurrence of an atom $\varphi=\psi$ with the atom $\tau(\varphi)=\tau(\psi)$. By definition then, τ' does not interfere with the boolean structure of Φ.

Lemma 3.4. *Let M and k be fixed nonnegative integers, let Φ be an open formula in $\mathcal{L}_{\text{Ab}}$ with n variables such that k is an upper bound on the depth of terms in Φ. Then for any $a_1,\dots,a_n\in[-2^M,2^M]$*

$$\boldsymbol{R}\models\Phi(a_1,\dots,a_n)\ \text{if and only if}\ [0,1]_{\text{Ł}}^{1/2}\models\tau'(\Phi)(r_{M,k}(a_1),\dots,r_{M,k}(a_n)).\quad(3.2)$$

Equivalently, for any $b_1,\dots,b_n\in[\frac{1}{2}-\frac{1}{2^{k+1}},\frac{1}{2}+\frac{1}{2^{k+1}}]$

$$[0,1]_{\text{Ł}}^{1/2}\models\tau'(\Phi)(b_1,\dots,b_n)\ \text{if and only if}\ \boldsymbol{R}\models\Phi(r_{M,k}{}^{-1}(b_1),\dots,r_{M,k}{}^{-1}(b_n)).\quad(3.3)$$

Proof. We only prove the first equivalence, the second following from the first one using the fact that $r_{M,k}$ is a bijection between $[-2^M,2^M]$ and $[\frac{1}{2}-\frac{1}{2^{k+1}},\frac{1}{2}+\frac{1}{2^{k+1}}]$. The proof is by induction on boolean structure of Φ. For an atom $\varphi=\psi$ (where φ, ψ are terms), if $f_\varphi(a_1,\dots,a_n)=f_\psi(a_1,\dots,a_n)$ then $r_{M,k}(f_\varphi(a_1,\dots,a_n))=r_{M,k}(f_\psi(a_1,\dots,a_n))$ and using Lemma 3.3,

$$f_{\tau(\varphi)}(r_{M,k}(a_1),\dots,r_{M,k}(a_n))=f_{\tau(\psi)}(r_{M,k}(a_1),\dots,r_{M,k}(a_n)),$$

which implies $[0,1]_{\text{Ł}}^{1/2}\models\tau'(\varphi=\psi)(r_{M,k}(a_1),\dots,r_{M,k}(a_n))$. The other implication is obtained analogously using the fact that $r_{M,k}$ is one-to-one.

Now assume the induction hypothesis for open formulas Ψ and Δ, that is

$$\boldsymbol{R} \models \Psi(a_1, \ldots, a_n) \text{ iff } [0,1]_{\text{Ł}}^{1/2} \models \tau'(\Psi)(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)) \text{ and} \tag{3.4}$$

$$\boldsymbol{R} \models \Delta(a_1, \ldots, a_n) \text{ iff } [0,1]_{\text{Ł}}^{1/2} \models \tau'(\Delta)(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)). \tag{3.5}$$

First, since

$$\boldsymbol{R} \not\models \Psi(a_1, \ldots, a_n) \text{ iff } [0,1]_{\text{Ł}}^{1/2} \not\models \tau'(\Psi)(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)),$$

and $\tau'(\neg\Psi) = \neg\tau'(\Psi)$, it follows that

$$\boldsymbol{R} \models \neg\Psi(a_1, \ldots, a_n) \text{ iff } [0,1]_{\text{Ł}}^{1/2} \models \tau'(\neg\Psi)(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)).$$

Also, $\tau'(\Psi \wedge \Delta) = \tau'(\Psi) \wedge \tau'(\Delta)$ and thus clearly

$$\boldsymbol{R} \models (\Psi \wedge \Delta)(a_1, \ldots, a_n) \text{ iff } [0,1]_{\text{Ł}}^{1/2} \models \tau'(\Psi \wedge \Delta)(r_{M,k}(a_1), \ldots, r_{M,k}(a_n)).$$

The induction step for the connective $\vee$ can be checked analogously. □

Let us recap what has been established. Let M and k be fixed nonnegative integers. Let Φ be an open formula with n variables in $\mathcal{L}_{\text{Ab}}$ and term depth bounded by k be given. Using the bijection $r_{M,k}$ of $[-2^M, 2^M]$ onto $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$ one can map satisfying assignments of (variables in) Φ in $\boldsymbol{R}_{-1}$ to those of $\tau'(\Phi)$ in $[0,1]_{\text{Ł}}$ (which fall into $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$), and conversely, any satisfying assignment to $\tau'(\Phi)$ in the interval $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$ of $[0,1]_{\text{Ł}}$ can be mapped to a satisfying assignment to Φ (in $[-2^M, 2^M]$).

The translation τ', as presented, can lead to an exponential increase in term size. For example $\tau(\ldots(x_1 + x_2) + x_3) + \cdots + x_n)$ has 2^{n-1} occurrences of variable x_1. To attain a polynomial-size translation of open $\mathcal{L}_{\text{pAb}}$-formulas to open $\mathcal{L}_{\text{MV}}$-formulas, complex terms need to be removed from Φ while preserving satisfying assignments, before applying τ'. This can be done relying (again) on Tseitin transformations.

Lemma 3.5. *Let Φ be a $\mathcal{L}_{\text{Ab}}$ formula. Let Φ' be the result of a Tseitin transformation of Φ. Then for any assignment v in $[0,1]_{\text{Ł}}^{1/2}$ one has that v satisfies $\tau'(\Phi)$ if and only if $\tau'(\Phi')$ is satisfied by the assignment v', where v' uniquely extends v by sending each added variable x_φ to $v(\tau(\varphi))$. Moreover, $\sharp(\tau'(\Phi'))$ is polynomial in $\sharp(\Phi)$.*

Lemma 3.6. *Let M, k be fixed nonnegative integers. Consider the following set of equations in $\mathcal{L}_{\text{MV}}$:*

1. $z_1 = \neg z_1$.
2. $(z_{i+1} \oplus z_{i+1} = z_i)$ *for each* $i \leq M + k$.
3. $q = \neg(z_1 \oplus z_{M+k+1})$ *and* $r = \neg(z_1 \oplus z_{k+1})$.

Any assignment v in $[0,1]_{\text{Ł}}$ that satisfies all these equations has $v(z_i) = \frac{1}{2^i}$, $v(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}$ and $v(r) = \frac{1}{2} - \frac{1}{2^{k+1}}$.

Notice that in the following lemma, the translation S operates on $\mathcal{L}_{\text{pAb}}$-formulas and brings them to $\mathcal{L}_{\text{MV}}$-formulas.

Lemma 3.7. *Let $\Phi(x_1,\dots,x_n)$ be an open formula in the language $\mathcal{L}_{\mathrm{pAb}}$. Let Φ' be a Tseitin variant of Φ. Assume that the variables $\{z_1,\dots,z_{M+k+1},r,q\}$ do not occur in* $\mathrm{Var}(\Phi')$. *Let $\zeta(\Phi')$ be obtained from Φ' by replacing each occurrence of the constant -1 in Φ' with the variable q, applying τ' to the resulting formula and then replacing all occurrences of the constant $1/2$ with the variable z_1. Let k be the maximal depth of terms in Φ and, denoting $N := \sharp(\Phi)$, set $M := \lceil cN\log_2 N\rceil$ for some large integer c. Let $S(\Phi,M,k)$ be the following open formula in $\mathcal{L}_{\mathrm{MV}}$:*

$$\zeta(\Phi') \wedge (z_1 = \neg z_1) \wedge (r = \neg(z_1 \oplus z_{k+1})) \wedge$$
$$(q = \neg(z_1 \oplus z_{M+k+1})) \wedge \bigwedge_{i\le M+k} (z_{i+1}\oplus z_{i+1} = z_i) \wedge \bigwedge_{i\le n}(r \le x_i) \wedge \bigwedge_{i\le n}(x_i \le \neg r).$$

Then Φ has a satisfying assignment in $\boldsymbol{R}_{-1}$ if and only if $S(\Phi,M,k)$ has a satisfying assignment in $[0,1]_{\text{Ł}}$. Moreover, $\sharp(S(\Phi,M,k))$ is polynomial in $\sharp(\Phi)$.

Proof. Assume first that $\Phi(x_1,\dots,x_n)$ is satisfied in $\boldsymbol{R}_{-1}$. By Lemma 3.2, there is a satisfying assignment $v(x_i) = a_i$ with $a_i \in [-2^M, 2^M]$. Then Φ' in variables $x_1,\dots,x_m$ (with $m \ge n$) is satisfied by any assignment that extends v on $x_1,\dots,x_n$ by sending the extra variables $x_{n+1},\dots,x_m$ to the values $a_{n+1},\dots,a_m$ of subterms in Φ in the interval $[-2^{M+k+1}, 2^{M+k+1}]$. W.l.o.g. we assume that v itself is this extended, satisfying assignment. Apply Lemma 3.4, sending -1 to $\frac12 - \frac{1}{2^{M+k+1}}$ via the mapping $r_{M,k}$. Then $\zeta(\Phi')$ has a satisfying assignment $r_{M,k}(a_1),\dots,r_{M,k}(a_m)$ in $[0,1]_{\text{Ł}}$ and moreover, by construction, one that sends q to $\frac12 - \frac{1}{2^{M+k+1}}$ and z_1 to $\frac12$. By Lemma 3.6, the formula

$$(z_1 = \neg z_1) \wedge (q = \neg(z_1 \oplus z_{M+k+1})) \wedge (r = \neg(z_1 \oplus z_{k+1})) \wedge \bigwedge_{i\le M+k} (z_{i+1}\oplus z_{i+1} = z_i) \tag{3.6}$$

has the unique satisfying assignment in $[0,1]_{\text{Ł}}$, which sends z_i to $1/2^i$ for each $i \le M+k$, q to $\frac12 - \frac{1}{2^{M+k+1}}$ and r to $\frac12 - \frac{1}{2^{k+1}}$. By construction, we have that $r_{M,k}(a_i)$ for $i \le n$ belongs to $[\frac12 - \frac{1}{2^{k+1}}, \frac12 + \frac{1}{2^{k+1}}]$, so $\bigwedge_{i\le n}(r \le x_i) \wedge \bigwedge_{i\le n}(x_i \le \neg r)$ is satisfied too.

For the converse implication assume $S(\Phi,M,k)$ is satisfied in $[0,1]_{\text{Ł}}$ by an assignment v. Let $v(x_i) = b_i$ for $i \le n$. In particular, then, Equation 3.6 is satisfied and hence by Lemma 3.6, $v(z_1) = \frac12$, $v(q) = \frac12 - \frac{1}{2^{M+k+1}}$ and $v(r) = \frac12 - \frac{1}{2^{k+1}}$. This implies that $v(x_i)$ for $i \le n$ are in $[\frac12 - \frac{1}{2^{k+1}}, \frac12 + \frac{1}{2^{k+1}}]$. Using Lemma 3.5 one can easily show that any evaluation satisfying $\zeta(\Phi')$ will also satisfy $\zeta(\Phi)$. Thus v satisfies $\zeta(\Phi)$ with $v(q) = \frac12 - \frac{1}{2^{M+k+1}}$ and $v(z_1) = \frac12$. By Lemma 3.4 the assignment $v'(x_i) = r^{-1}_{M,k}(b_i)$ for $i \le n$ satisfies Φ, which completes the proof. □

Theorem 3.8. *The existential theory of $\boldsymbol{R}_{-1}$ reduces in polynomial time to the existential theory of $[0,1]_{\text{Ł}}$. Therefore, the former theory is in NP.*

It follows that the universal theory of $\boldsymbol{R}_{-1}$ is in coNP. It was already established in the paper [27] that the equational theory of $\boldsymbol{R}_{-1}$ is coNP-hard. Hence the universal theory and the quasiequational theory of $\boldsymbol{R}_{-1}$ are coNP-complete and the existential theory is NP-complete.

3.5 Reducing the $\exists$-theory of $[0,1]_{\text{Ł}}$ to itself

This section investigates the composition of the reduction from $[0,1]_{\text{Ł}}$ to $\boldsymbol{R}_{-1}$ and that from $\boldsymbol{R}_{-1}$ to $[0,1]_{\text{Ł}}^{1/2}$. First let us introduce a variant of the translation used in [27] which provided a faithful interpretation of the universal theory of $[0,1]_{Ł}$ (in the language $\mathcal{L}_{\mathrm{MV}}$) in the universal theory of $\boldsymbol{R}_{-1}$ (in the language $\mathcal{L}_{\mathrm{pAb}}$), mentioned in Section 3.1. Such a mapping δ from Tm_{MV} to Tm_{pAb} is defined recursively as follows:

1. $\delta(x) = (x \vee -1) \wedge 0$ for $x \in \mathrm{Var}$;
2. $\delta(\varphi \to \xi) = (\delta(\xi) - \delta(\varphi)) \wedge 0$;
3. $\delta(\varphi \otimes \psi) = (\delta(\varphi) + \delta(\psi)) \vee -1$;
4. $\delta(0) = -1$;
5. $\delta(1) = 0$;
6. $\delta(\varphi \circ \xi) = \delta(\varphi) \circ \delta(\xi)$ for $\circ \in \{\wedge, \vee\}$.

Lemma 3.9. *Let $\varphi(x_1, \dots, x_n)$ be a term in $\mathcal{L}_{\mathrm{MV}}$ with n variables. Assume $a_1, \dots, a_n$ are elements of the domain of $[0,1]_{Ł}$. Then $f_{\delta(\varphi)}((a_1 - 1), \dots, (a_n - 1)) + 1 = f_\varphi(a_1, \dots, a_n)$.*

Proof. As in the proof of [27, Theorem 5.8]. □

As in the case of τ defined in Section 3.4, the mapping δ extends to a mapping on open formulas of $\mathcal{L}_{\mathrm{MV}}$: for Φ of $\mathcal{L}_{\mathrm{MV}}$ define $\delta'(\Phi)$ by replacing each occurrence of an atom $\varphi = \psi$ with the atom $\delta(\varphi) = \delta(\psi)$.

Lemma 3.10. *Let Φ be an open formula in $\mathcal{L}_{\mathrm{MV}}$ with n variables and let $a_1, \dots, a_n \in [0,1]$. Then*

$$[0,1]_{Ł} \models \Phi(a_1, \dots, a_n) \textit{ if and only if } \boldsymbol{R}_{-1} \models \delta'(\Phi)(a_1 - 1, \dots, a_n - 1).$$

Proof. As in the proof of Lemma 3.4. □

Our aim is to compose δ with τ. This is not immediately possible, since τ is defined only on Tm_{Ab} (lacking the constant -1). First, therefore, the definition of τ will be extended to its variant τ_q. Let $\varphi \in Tm_{\mathrm{pAb}}$ and let q be a variable not used in φ. The function $\tau_q(\varphi)$ is defined recursively as follows:

1. $\tau_q(x) = (x \vee q) \wedge \frac{1}{2}$ for $x \in \mathrm{V}ar(\varphi)$;
2. $\tau_q(0) = \frac{1}{2}$;
3. $\tau_q(-1) = q$;
4. $\tau_q(-\varphi) = \neg(\tau_q(\varphi))$;
5. $\tau_q(\varphi \circ \psi) = \tau_q(\varphi) \circ \tau_q(\psi)$ for $\circ \in \{\vee, \wedge\}$;
6. $\tau_q(\varphi + \psi) = (\tau_q(\varphi) \oplus \tau_q(\psi)) \otimes (\frac{1}{2} \oplus (\tau_q(\varphi) \otimes \tau_q(\psi)))$.

We now look into the composition $\tau_q \circ \delta$ which maps Tm_{MV} into $Tm_{\mathrm{MV}[\frac{1}{2}]}$. The following lemma shows that, as far as the semantics provided by $[0,1]_{Ł}$ is concerned, the composition of these two translations can be expressed by a function that increases the length of a term polynomially, rather than exponentially as one might expect, given the behaviour of τ discussed in Section 3.4.

Lemma 3.11. *Let $\varphi \in Tm_{\mathrm{MV}}$ and let q be a variable not among* $\mathrm{Var}(\varphi)$*. The functions $f_{\tau_q(\delta(\varphi))}$ and $f_{\sigma_q(\varphi)}$ coincide on the domain of $[0,1]_{Ł}$, where the mapping σ_q on Tm_{MV} is defined recursively as follows:*

1. $\sigma_q(x) = (x \vee q) \wedge \frac{1}{2}$ *for* $x \in \mathrm{V}ar(\varphi)$*;*
2. $\sigma_q(1) = \frac{1}{2}$*;*
3. $\sigma_q(0) = q$*;*
4. $\sigma_q(a \to b) = ((\sigma_q(a) \to \sigma_q(b)) \otimes \frac{1}{2}) \wedge \frac{1}{2}$*;*
5. $\sigma_q(x \otimes y) = ((\sigma_q(x) \oplus \sigma_q(y)) \otimes \frac{1}{2}) \vee q$*;*
6. $\sigma_q(\varphi \circ \xi) = \sigma_q(\varphi) \circ \sigma_q(\xi)$ *for* $\circ \in \{\wedge, \vee\}$*.*

Moreover, for any term ψ in $\mathcal{L}_{\mathrm{MV}}$, $\sharp(\sigma_q(\psi))$ is polynomial in $\sharp(\psi)$.

Proof. By induction on term depth.

- Base case (depth 0): By definition $\sigma_q(x) = (x \vee q) \wedge \frac{1}{2} = \tau_q(\delta(x))$, $\sigma_q(1) = \frac{1}{2} = \tau_q(\delta(1))$ and $\sigma_q(0) = q = \tau_q(\delta(0))$.

Now assume that $f_{\sigma_q(\psi)} = f_{\tau_q(\delta(\psi))}$ and $f_{\sigma_q(\gamma)} = f_{\tau_q(\delta(\gamma))}$.

- Clearly, for $\varphi = \psi \circ \gamma$ we have $f_{\tau_q(\delta(\psi\circ\gamma))} = f_{\sigma_q(\psi\circ\gamma)}$ for $\circ \in \{\vee, \wedge\}$.
- For $\varphi = \psi \to \gamma$ we have

  $$f_{\tau_q(\delta(\psi\to\gamma))} = f_{\tau_q((\delta(\gamma)-\delta(\psi))\wedge 0)} = f_{((\neg\tau_q(\delta(\psi))\oplus\tau_q(\delta(\gamma)))\otimes(\frac{1}{2}\oplus(\neg\tau_q(\delta(\psi))\otimes\tau_q(\delta(\gamma)))))\wedge\frac{1}{2}}.$$

  By the induction hypothesis this is equivalent to

  $$\begin{aligned}&f_{((\neg\sigma_q(\psi)\oplus\sigma_q(\gamma))\otimes(\frac{1}{2}\oplus(\neg\sigma_q(\psi)\otimes\sigma_q(\gamma))))\wedge\frac{1}{2}}\\ =&\max(0, f_{\neg\sigma_q(\psi)\oplus\sigma_q(\gamma)} + f_{\frac{1}{2}\oplus(\neg\sigma_q(\psi)\otimes\sigma_q(\gamma))} - 1) \wedge \frac{1}{2}.\end{aligned}$$

  Let us distinguish two cases.

  1. First assume $(f_{\neg\sigma_q(\psi)} + f_{\sigma_q(\gamma)})(a_1, \ldots, a_n) \geq 1$. Thus,

     $$f_{\neg\sigma_q(\psi)\oplus\sigma_q(\gamma)}(a_1, \ldots, a_n) = 1$$

     and consequently

     $$\begin{aligned}&\max(0, (f_{\neg\sigma_q(\psi)\oplus\sigma_q(\gamma)} + f_{\frac{1}{2}\oplus(\neg\sigma_q(\psi)\otimes\sigma_q(\gamma))})(a_1, \ldots, a_n) - 1) \wedge \frac{1}{2}\\ =&\max(0, f_{\frac{1}{2}\oplus(\neg\sigma_q(\psi)\otimes\sigma_q(\gamma))}(a_1, \ldots, a_n)) \wedge \frac{1}{2} = \frac{1}{2}\\ =&f_{((\sigma_q(\psi)\to\sigma_q(\gamma))\otimes\frac{1}{2})\wedge\frac{1}{2}}(a_1, \ldots, a_n) = f_{\sigma_q(\psi\to\gamma)}(a_1, \ldots, a_n).\end{aligned}$$

  2. For the other case assume $(f_{\neg\sigma_q(\psi)} + f_{\sigma_q(\gamma)})(a_1, \ldots, a_n) \leq 1$. Then we have

     $$f_{((\neg\sigma_q(\psi)\oplus\sigma_q(\gamma))\otimes(\frac{1}{2}\oplus(\neg\sigma_q(\psi)\otimes\sigma_q(\gamma))))\wedge\frac{1}{2}} = f_{((\neg\sigma_q(\psi)\oplus\sigma_q(\gamma))\otimes\frac{1}{2})\wedge\frac{1}{2}} = f_{\sigma_q(\psi\to\gamma)}.$$

     This completes the second case.

- For $\varphi = \psi \otimes \gamma$ we have
$$f_{\tau_q(\delta(\psi\otimes\gamma))} = f_{\tau_q((\delta(\psi)+\delta(\gamma))\vee -1)} = f_{((\tau_q(\delta(\psi))\oplus\tau_q(\delta(\gamma)))\otimes(\frac{1}{2}\oplus(\tau_q(\delta(\psi))\otimes\tau_q(\delta(\gamma)))))\vee q}.$$
By induction hypothesis this is equal to
$$f_{((\sigma_q(\psi)\oplus\sigma_q(\gamma))\otimes(\frac{1}{2}\oplus(\sigma_q(\psi)\otimes\sigma_q(\gamma))))\vee q}.$$
Since $f_{\sigma_q(\gamma)}(a_1,\dots,a_n) \leq \frac{1}{2}$ for each $a_1,\dots,a_n \in [0,1]_Ł$ and for each $\gamma \in Tm_{\mathrm{MV}}$, we have for each $a_1,\dots,a_n \in [0,1]_Ł$ that $f_{\sigma_q(\psi)\otimes\sigma_q(\gamma)}(a_1,\dots,a_n) = 0$. Thus,
$$f_{\tau_q(\delta(\psi\otimes\gamma))} = f_{((\sigma_q(\psi)\oplus\sigma_q(\gamma))\otimes\frac{1}{2})\vee q}.$$
This completes the second case and also the proof. □

We extend definitions of τ_q and σ_q to open formulas. For Φ of $\mathcal{L}_{\mathrm{pAb}}$ define $\tau'_q(\Phi)$ by replacing each occurrence of an atom $\varphi = \psi$ with the atom $\tau_q(\varphi) = \tau_q(\psi)$. Similarly for Φ of $\mathcal{L}_{\mathrm{MV}}$ define $\sigma'_q(\Phi)$ by replacing each occurrence of an atom $\varphi = \psi$ with the atom $\sigma_q(\varphi) = \sigma_q(\psi)$.

Corollary 3.12. *Let Ψ be open formula of $\mathcal{L}_{\mathrm{MV}}$ with n variables and let $q \notin \mathrm{Var}(\Psi)$. Let $a_1,\dots a_n \in [0,1]$. Then*
$$[0,1]_Ł^{1/2} \vDash \tau'_q(\delta'(\Psi))(a_1,\dots,a_n) \text{ if and only if } [0,1]_Ł^{1/2} \vDash \sigma'_q(\Psi)(a_1,\dots,a_n).$$

Notice that if φ is a $\mathcal{L}_{\mathrm{MV}}$-term and v an assignment in $\boldsymbol{R}_{-1}$ such that $v(x_i) \in [-1,0]$ for each $x_i \in \mathrm{Var}(\varphi)$, then for each subterm ψ of φ, $v(\delta(\psi)) \in [-1,0]$. Using this observation, we obtain the following variant of Lemma 3.4, now featuring the translation τ'_q in the role of τ', and moreover avoiding the assumption regarding the depth of terms in the open first-order formula. The proof of this lemma is a variant of those of Lemma 3.3 and Lemma 3.4 with only minor modifications.

Lemma 3.13. *Let M and k be fixed nonnegative integers, $\Phi(x_1,\dots,x_n)$ an open formula in the language $\mathcal{L}_{\mathrm{MV}}$ and $q \notin \mathrm{Var}(\Phi)$. For any assignment v in $\boldsymbol{R}_{-1}$ where $v(x_i) = a_i \in [-1,0]$, let v' be an assignment in $[0,1]_Ł^{1/2}$ with*
$$v'(x_i) = r_{M,k}(a_i) \text{ for each } i \leq n \text{ and } v'(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}.$$
Then v satisfies the formula $\delta'(\Phi)$ in the algebra $\boldsymbol{R}_{-1}$ if and only if v' satisfies the formula $\tau'_q(\delta'(\Phi))$ in $[0,1]_Ł^{1/2}$.

This is equivalent to the following. For any assignment v in $[0,1]_Ł^{1/2}$ where $v(x_i) = b_i \in [\frac{1}{2} - \frac{1}{2^{M+k+1}}, \frac{1}{2}]$ let v' be an assignment in $\boldsymbol{R}_{-1}$ with $v'(x_i) = r^{-1}_{M,k}(b_i)$. Assume that $v(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}$. Then v satisfies the formula $\tau'_q(\delta'(\Phi))$ in the algebra $[0,1]_Ł^{1/2}$ if and only if v' satisfies the formula $\delta'(\Phi)$ in $\boldsymbol{R}_{-1}$.

Theorem 3.14 (A translation of the existential theory of $[0,1]_Ł$ to itself)**.** *Let $\Phi(x_1,\dots,x_n)$ be an open formula of $\mathcal{L}_{\mathrm{MV}}$. Let us assume that the variables $\{z_1,\dots,z_{M+k+1},q\}$ do not occur in Φ. Let $M,k \in \mathbb{N}$. Let $\zeta(\Phi)$ be obtained from Φ by applying σ'_q on Φ and then replacing all occurrences of the constant $\frac{1}{2}$ with the variable z_1. Let $S(\Phi,M,k)$ be the following open formula in $\mathcal{L}_{\mathrm{MV}}$:*
$$\zeta(\Phi) \wedge (z_1 = \neg z_1) \wedge (q = \neg(z_1 \oplus z_{M+k+1})) \wedge \bigwedge_{i\leq M+k} (z_{i+1} \oplus z_{i+1} = z_i).$$
Then Φ has a satisfying assignment in $[0,1]_Ł$ if and only if $S(\Phi,M,k)$ has a satisfying assignment in $[0,1]_Ł$. Moreover, $\sharp(S(\Phi,M,k))$ is polynomial in $\sharp(\Phi)$.

Proof. Assume first that $\Phi(x_1,\dots,x_n)$ has a satisfying assignment $v(x_i)=a_i$ with $a_i\in[0,1]_{Ł}$. By Lemma 3.10 $\delta'(\Phi)$ has a satisfying assignment $x_i\mapsto a_i-1$ with $(a_i-1)\in[-1,0]$.

By Lemma 3.13 $\tau'_q(\delta'(\Phi))$ has a satisfying assignment $x_i\mapsto r_{M,k}(a_i-1)$ and $q\mapsto \frac{1}{2}-\frac{1}{2^{M+k+1}}$ in $[0,1]_{Ł}^{1/2}$. By Lemma 3.12 $\sigma'_q(\Phi)$ is satisfied by the very same assignment. Consequently, $\zeta(\Phi)$ has satisfying assignment $x_i\mapsto r_{M,k}(a_i-1), q\mapsto \frac{1}{2}-\frac{1}{2^{M+k+1}}$ and $z_1\mapsto\frac{1}{2}$ in $[0,1]_{Ł}$.

By Lemma 3.6, the formula $(z_1=\neg z_1)\wedge(q=\neg(z_1\oplus z_{M+k+1}))\wedge\bigwedge_{i\le M+k}(z_{i+1}\oplus z_{i+1}=z_i)$ has the unique satisfying assignment in $[0,1]_{Ł}$, which sends z_i to ${}^1\!/_{2^i}$ for each $i\le M+k$ and q to $\frac{1}{2}-\frac{1}{2^{M+k+1}}$.

Conversely, assume $S(\Phi,M,k)$ has a satisfying assignment v in $[0,1]_{Ł}$. The conjunct

$$(z_1=\neg z_1)\wedge(q=\neg(z_1\oplus z_{M+k+1}))\wedge\bigwedge_{i\le M+k}(z_{i+1}\oplus z_{i+1}=z_i)$$

forces by Lemma 3.6 that $v(z_1)=\frac{1}{2}$ and $v(q)=\frac{1}{2}-\frac{1}{2^{M+k+1}}$. Observe that by the definition of ζ, every occurrence of a variable x_i in $\zeta(\Phi)$ appears strictly within the term $(x_i\vee q)\wedge z_1$ for each $i\le n$. Therefore, we can assume without loss of generality that $\frac{1}{2}-\frac{1}{2^{M+k+1}}=v(q)\le v(x_i)\le v(z_1)=\frac{1}{2}$ for all $i\le n$. Since v satisfies $S(\Phi,M,k)$, it must satisfy $\zeta(\Phi)$. By definition of ζ, this means the formula $\sigma'_q(\Phi)$ is satisfied by the assignment v in $[0,1]_{Ł}^{1/2}$. By Lemma 3.12 the assignment v also satisfies $\tau'_q(\delta'(\Phi))$ in $[0,1]_{Ł}^{1/2}$. We can now apply Lemma 3.13.

Since $\frac{1}{2}-\frac{1}{2^{M+k+1}}\le v(x_i)\le\frac{1}{2}$ and $v(q)=\frac{1}{2}-\frac{1}{2^{M+k+1}}$, by Lemma 3.13 the assignment $v'(x_i)=r^{-1}_{M,k}(v(x_i))$ satisfies $\delta'(\Phi)$ in $\boldsymbol{R}_{-1}$. Finally, applying Lemma 3.10, the assignment $v''(x_i)=v'(x_i)+1$ satisfies the original formula Φ in $[0,1]_{Ł}$. This completes the proof. □

The last theorem can be viewed as providing a way of testing the validity of existential sentences in $[0,1]_{Ł}$ on a subinterval of its domain (rather than the full domain), provided the value $1/2$ belongs to this interval. The statement relies on the well-known fact that rational values in $[0,1]$ are implicitly definable by $\mathcal{L}_{\mathrm{MV}}$-terms, and moreover, the size of the defining term is polynomial in the length of the positional representation of the rational number in question.

4 Pointed Modal Abelian Logic, Algebraically

4.1 Introduction

Abelian Logic, independently introduced by Meyer & Slaney [81] and Casari [18], is a well-known contraclassical, paraconsistent logic. It is also referred to as the *logic of Abelian ℓ-groups* [17] or *Abelian Group Logic* [87] since the matrix models of Abelian logic are based on Abelian ℓ-groups, their positive cones being the filters of designated elements. A primary motivation for studying Abelian logic is its connection to *Łukasiewicz logic*, since there exists a well-known categorical equivalence between the categories of strongly pointed Abelian ℓ-groups and MV-algebras, the algebras of Łukasiewicz logic [21].

Recently, *modal Abelian logic* has also gained some attention, particularly from the perspective of its real-valued Kripke semantics [34, 33, 80]. Following this, we investigate *pointed real-valued modal Abelian logic* from an algebraic perspective. Our motivation for this focus is twofold. First, it structurally parallels real-valued modal Łukasiewicz logic. Exploiting this connection, *pointed* modal Abelian logic (albeit not explicitly named therein) is already used in [33, Sect. 2.3] to directly analyze real-valued modal Łukasiewicz logic [58]. Specifically, their analysis requires the inclusion of a designated constant c to act as a fixed point under the $\Box$-modality (i.e., $\Box c = c$), which serves as the falsum for Łukasiewicz logic. Furthermore, by substituting bounded MV-algebras for Abelian ℓ-groups, our framework lifts the traditional restriction to the bounded interval $[0, 1]$, allowing formulas to instead be evaluated over $\mathbb{R}$. Second, this framework allows us to introduce a modal extension of *Łukasiewicz unbound logic*. Originally defined in [27], this logic is strongly complete with respect to the Abelian ℓ-group of real numbers equipped with the designated element -1. Consequently, by equipping our real-valued Kripke models with the same constant -1, we provide a natural modal semantics for Łukasiewicz unbound logic. Despite these connections, the *algebraic* aspects of pointed modal Abelian logic have hitherto remained unexplored.

In this article we close this gap, introducing the variety of (negatively) *pointed modal Abelian ℓ-groups* $\mathsf{mAb}_{\mathsf{p}}^{-}$ (Definition 4.11) and explore its relationship to real-valued Kripke semantics of pointed modal Abelian logic (with constant -1). We do this by constructing complex algebras in $\mathsf{mAb}_{\mathsf{p}}^{-}$ from Kripke frames (entailing an algebraic soundness result, Proposition 4.15) and vice versa. Our first main result is a Truth Lemma for *strongly* negatively pointed modal Abelian ℓ-groups (Corollary 4.22). Using this, we characterize the validities of pointed modal Abelian logic via an equational condition ($\mathsf{A}\infty$) on the variety $\mathsf{mAb}_{\mathsf{p}}^{-}$, which may be seen as a form of (infinitary) *algebraic completeness* (Theorem 4.25).

The primary technical challenges to overcome here stem from the algebraic behavior of strong units and the non-trivial intersection of maximal ℓ-ideals. In general, a non-zero element of a negatively pointed modal Abelian ℓ-group may belong to every maximal ℓ-ideal. This effectively prevents us from using such an element as a falsifier for a formula, since no world in the corresponding canonical frame would evaluate it as non-zero. The rule ($\mathsf{A}\infty$) ensure this does not happen.

Furthermore, this rule allows us to argue via *strongly* pointed algebras in our proof of Theorem 4.25. While it is a well-known fact that the property of possessing a strong unit is not first-order definable, enforcing a strong unit condition is justified within our framework since it corresponds directly to the restriction to *bounded* valuations in real-valued Kripke models (required to guarantee the semantics of the $\Box$-modality is well-defined). We leave it up for future research to determine whether these issues could be circumvented in other ways.

• **Related Work.** Prior research on *real-valued modal logic* [33, 34] and *modal Łukasiewicz logic* [58] has been a main inspiration for the current paper. More specifically, we build on the definition of (pointed) real-valued modal logic of the former and introduce an infinitary rule similar to the latter (here, let us also point out [7] about *continuous modal logic* and [69] about *modal Riesz spaces*). More generally, our work stands in the research tradition on *many-valued modal logic* (e.g. [39, 14, 58, 16, 91]), note that we are using crisp accessibility relations (like [91] for Gödel and [58] for Łukasiewicz modal logic). Naturally the current paper is also related to the *algebraic* theory of *lattice-ordered Abelian groups* and MV-*algebras* [21]. In particular, for more context about *pointed* Abelian ℓ-groups and their logic see [27, 98, 61].

• **Structure of the article.** In Section 4.2, we provide some preliminaries about modal Abelian ℓ-groups and proving some helpful results about ℓ-ideals. In Section 4.3, we introduce the real-valued relational semantics of pointed modal Abelian logic (Subsection 4.3.1) as well as the variety of pointed modal Abelian ℓ-groups (Subsection 4.3.2). We also explain how to relate these two via complex algebras and canonical frames (Subsection 4.3.3). Section 4.4 contains the main results of the article, namely a Truth Lemma for strongly pointed algebras (Subsection 4.4.1) and the infinitary algebraic completeness derived thereof (Subsection 4.4.2). We discuss potential directions for future research in the concluding Section 4.5.

4.2 Pointed Abelian ℓ-groups

In this (mostly preliminary) section, we recall some important facts about pointed Abelian ℓ-groups, focusing in particular on structures equipped with a negative strong unit (the reader should feel free to skip some of the more technical proofs here until we refer back to the corresponding results later on). We give a brief exposition of ℓ-ideal theory, and analyze the structure of ℓ-ideals in the presence of such a designated unit. Adapting techniques from the theory of MV-algebras, we characterize the radical, the intersection of all maximal ℓ-ideals, via infinitesimal elements. Lastly, we establish a pivotal correspondence between the maximal ℓ-ideals of a strongly negatively pointed Abelian ℓ-group and its homomorphisms into the pointed Abelian ℓ-group of reals $\mathbf{R}_{-1}$, which will be an important 'bridge' between the abstract algebraic syntax and our concrete real-valued semantics.

Abelian logic (e.g. see [81, 18, 27]) can be elegantly axiomatized as the negation-free fragment of FL_e (i.e. Full Lambek calculus with exchange) extended by the axiom of relativity $((\varphi \to \psi) \to \psi) \to \varphi$. Due to our algebraic focus, here we only treat Abelian logic via its algebraic models, i.e. Abelian ℓ-groups. Nevertheless, let us mention that the logics corresponding to the varieties defined in this paper inherit similar axiomatizations, since they are superabelian (see [27]).

Definition 4.1 (Abelian ℓ-group). *An algebra* $\mathbf{A} = \langle A, +, -, \vee, \wedge, 0\rangle$ *is an* Abelian ℓ-group *if* $\langle A, +, -, 0\rangle$ *is an Abelian group,* $\langle A, \vee, \wedge\rangle$ *is a lattice and* $\mathbf{A}$ *satisfies the monotonicity condition* $x \leq y \Rightarrow x + z \leq y + z$.

When brackets are omitted the lattice operations take precedence over the group operations, e.g. $x + y \wedge z$ is understood as $x + (y \wedge z)$. For $a \in A$ we define the *absolute value* $|a| := a \vee -a$ and we let $n \cdot a := a + \cdots + a$ denote the n-fold sum for every $n \in \mathbb{N}$. For $z \in \mathbb{Z}\backslash\mathbb{N}$ we set $z \cdot a := (-z) \cdot (-a)$.

Definition 4.2 (Strong Unit). *Let* $\mathbf{A}$ *be an Abelian* ℓ*-group. We say that* $p \in A$ *is a* strong unit *of* $\mathbf{A}$ *if*

$$\forall a \in A\colon \exists z \in \mathbb{Z}\colon z \cdot p \geq a. \tag{SU}$$

Note that for a strictly negative strong unit $p < 0$ *the condition* (SU) *simplifies to*

$$\forall a \in A\colon \exists n \in \mathbb{N}\colon n \cdot p \leq a. \tag{SU-}$$

An algebra $\mathbf{A}_p = \langle A, +, -, \vee, \wedge, 0, p\rangle$ *is a* pointed Abelian ℓ-group *if* $\langle A, +, -, \vee, \wedge, 0\rangle$ *is an Abelian* ℓ*-group and* $p \in A$. *It is* strongly pointed *if* p *is a strong unit and* negatively pointed *if* $p \leq 0$.

Remark 4.3. Traditionally, a strong unit is required to be strictly positive (e.g. see [10, p. 300]). Note that we use $\mathbb{Z}$ in (SU), which allows negative strong units. This distinction purely serves for notational convenience: since we primarily work in the negative cone, it avoids a proliferation of minus signs in our proofs. While our definition differs from the standard one, the structural role is identical: $p < 0$ is a strong unit in our sense if and only if its additive inverse $-p > 0$ is a standard strong unit.

As a standard example, let $\mathbf{R}_{-1} = \langle \mathbb{R}, +, -, \vee, \wedge, 0, -1\rangle$ be the Abelian group of real numbers with addition together with its linear order and the designated point -1. This clearly is a strongly negatively pointed Abelian ℓ-group (in fact, it is our *algebra of truth-degrees* later on).

The following properties of Abelian ℓ-groups are immediate consequences of the well-known fact that the variety of Abelian ℓ-groups is generated by the ℓ-group of integers $\mathbf{Z}$ (see [6, Thm. 6.1]).

Lemma 4.4. *Let* $\mathbf{A} = \langle A, +, -, \vee, \wedge, 0\rangle$ *be an Abelian* ℓ*-group and* $a, b \in A$.

(i) $a = a \wedge 0 + a \vee 0$,

(ii) $\Big((a - b) \vee 0\Big) \wedge \Big((b - a) \vee 0\Big) = 0$,

(iii) $(a \vee b) + c = (a + c) \vee (b + c)$,

(iv) $(a - b) \vee 0 = a - a \wedge b$,

(v) $|a + b| \leq |a| + |b|$.

Definition 4.5 (ℓ-ideal). *An* ℓ-ideal *of an Abelian* ℓ*-group* $\mathbf{A}$ *is an* ℓ*-subgroup* $I \subseteq \mathbf{A}$ *that is* convex *(equivalently* $0 \leq |x| \leq |i|$ *for some* $i \in I$ *implies* $x \in I$*). An* ℓ*-ideal* I *is* prime *if it satisfies*

$$a \wedge b \in I \quad \Rightarrow \quad a \in I \text{ or } b \in I. \tag{PI}$$

First, let us note that for an ℓ-ideal I and $a \in A$ by definition it holds that $a \in I$ iff $|a| \in I$ iff $-|a| \in I$. One can easily prove (using Zorn's Lemma) that for each element $a \in A\backslash\{0\}$ there exists an ℓ-ideal which is maximal among all the ℓ-ideals which do not contain a. Every such ℓ-ideal is prime and we refer to them as *values* of a [95, Thm. 2.4.2].

It is well known that *maximal* ℓ-ideals are prime (again see [95, Thm. 2.4.2]), although in general an Abelian ℓ-group $\mathbf{A}$ need not have any maximal ideals. However, if $p \in A$ is a strong unit, the maximal ℓ-ideals correspond to values of p (since any ℓ-ideal containing p is improper). In fact, in the presence of a strong unit every proper ℓ-ideal $I \subseteq \mathbf{A}$ can be extended to a maximal ℓ-ideal $M \supseteq I$ (which is again easily shown via Zorn's Lemma).

Let $\mathbf{A}$ be an Abelian ℓ-group and $S \subseteq A$. By $C(S)$ we denote the ℓ-ideal generated by S. It is well-known [95, Thm. 2.2.4 (b)] that

$$C(S) := \{x \in A \mid |x| \leq |b_1| + \cdots + |b_n| \text{ for some } b_1, \ldots, b_n \in S\}. \qquad (\mathsf{IG})$$

Let us also recall from [95, Thm. 2.2.4 (d)], that for two sets $S, T \subseteq A$ we have

$$\begin{aligned} C(S) \vee C(T) &= C(S \cup T) \\ &= \{x \in A \mid |x| \leq |b_1| + \cdots + |b_n| \text{ for some } b_1, \ldots, b_n \in S \cup T\}. \end{aligned}$$

Therefore, for a principal ℓ-ideal $J = \langle a \rangle = \{x \mid \exists n \in \mathbb{N} \colon |x| \leq n \cdot |a|\}$, we obtain

$$I \vee J = I \vee \{x \mid \exists n \in \mathbb{N} \colon |x| \leq n \cdot |a|\} = \{x \mid \exists n \in \mathbb{N}\ \exists i \in I \colon |x| \leq i + n \cdot |a|\}.$$

In particular, this implies that for every maximal ℓ-ideal I of an Abelian ℓ-group $\mathbf{A}$ and $a \notin I$:

$$A = \{x \mid \exists n \in \mathbb{N} \colon \exists i \in I \colon |x| \leq i + n \cdot |a|\}. \qquad (\mathsf{IG}*)$$

For a strongly pointed Abelian ℓ-group $\mathbf{A}_p$ we define the *radical* of $\mathbf{A}_p$ as the intersection of all maximal ideals of $\mathbf{A}_p$ and denote it by $\mathsf{rad}(\mathbf{A}_p)$. Equivalently, $\mathsf{rad}(\mathbf{A}_p)$ is the intersection of all values of p. We call $a \in A$ is *infinitely small* (or *infinitesimal*) if

$$\forall n \in \mathbb{N} \colon n \cdot |a| \leq |p|. \qquad (\mathsf{INF})$$

In the case where $a, p \leq 0$, Equation (INF) simplifies to

$$\forall n \in \mathbb{N} \colon n \cdot a - p \geq 0. \qquad (\mathsf{INF\text{-}})$$

One can easily check that this is also equivalent to $\forall k \in \mathbb{N} \colon 2^k \cdot a - p \geq 0$.

We now show that the radical of a strongly pointed Abelian ℓ-group consists exactly of its infinitesimal elements. The proof of the following is similar to the one of [21, Prop. 3.6.4].

Lemma 4.6. *Let $\mathbf{A}_p$ be a strongly pointed Abelian ℓ-group. Then $a \in \mathsf{rad}(\mathbf{A}_p)$ iff a is infinitely small.*

Proof. If $a \notin \mathsf{rad}(\mathbf{A}_p)$ there is a maximal ℓ-ideal I such that $a \notin I$. Consequently, p is in the ℓ-ideal generated by $\{a\} \cup I$ and thus by ($\mathsf{IG}*$) there are $n \in \mathbb{N}$ and $i \in I$ such that $0 < |p| \leq n \cdot |a| + i$, which is equivalent to $|p| - n \cdot |a| \leq i$. Assume towards contradiction that a is infinitely small. Then $|p| - (n+1) \cdot |a| \geq 0$ for each $n \in \mathbb{N}$, equivalently $|p| - n \cdot |a| \geq |a|$ for each $n \in \mathbb{N}$. Thus altogether we have $0 \leq |a| \leq |p| - n \cdot |a| \leq i$, which by convexity of the ℓ- ideal I implies $a \in I$, a contradiction.

Conversely, assume $a \in A$ is not infinitely small. Then there is $n \in \mathbb{N}$ such that $n \cdot |a| - |p| \not\leq 0$, which implies $(n \cdot |a| - |p|) \vee 0 > 0$. Let P be a value of $(n \cdot |a| - |p|) \vee 0$. By Lemma 4.4 (ii) we have

$$\Big((n \cdot |a| - |p|) \vee 0\Big) \wedge \Big((|p| - n \cdot |a|) \vee 0\Big) = 0.$$

Since P is prime, it follows that $(|p| - n \cdot |a|) \vee 0 \in P$. Let M be a maximal ℓ-ideal containing P and towards contradiction assume that $a \in M$. Then $(n \cdot |a|) \vee 0 \in M$ and

$$(n \cdot |a|) \vee 0 \geq (n \cdot |a| - |p|) \vee 0 > 0.$$

Therefore, $(n \cdot |a| - |p|) \vee 0 \in M$. Consequently, by Lemma 4.4(i) we find

$$(|p| - n \cdot |a|) \vee 0 - (n \cdot |a| - |p|) \vee 0 = (|p| - n \cdot |a|) \vee 0 + (|p| - n \cdot |a|) \wedge 0 = (|p| - n \cdot |a|) \in M.$$

Since $p \notin M$ we have $|p| \notin M$ and thus also $n \cdot |a| \notin M$. However, this implies $|a| \notin M$, and thus lastly $a \notin M$, a contradiction our assumption $a \in M$ which shows $a \notin \mathsf{rad}(\mathbf{A}_p)$. ☐

Lemma 4.7. *Let $\mathbf{A}_p$ be a strongly negatively pointed Abelian ℓ-group and I an ℓ-ideal of $\mathbf{A}$. Then I is maximal if and only if $I = v^{-1}(0)$ for some homomorphism $v \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$.*

Proof. ($\Rightarrow$): Let I be maximal. By Hölder's Theorem [95, Thm. 2.3.10] $\mathbf{A}_p/I$ embeds into $\mathbf{R}_r$ for some $r \in \mathbb{R}$. This gives us $v \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_r)$. Since p is a strong unit of $\mathbf{A}$, $p \notin I$ and $v(p) \neq 0$. Since p is negative, also $v(p) \leq 0$. Therefore $v(p) = r < 0$. Lastly, since $\mathbf{R}_{r_1} \cong \mathbf{R}_{r_2}$ for any two negative $r_1, r_2 \in \mathbb{R}$, we can assume without loss of generality that $v \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$.

($\Leftarrow$): For every $v \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$, the subset $v^{-1}(0)$ is an ℓ-ideal of $\mathbf{A}_p$ and any ℓ-subgroup of $\mathbf{R}_{-1}$ is simple, which ensures that the ideal $v^{-1}(0)$ has to be maximal. ☐

This correspondence between infinitely small elements, maximal ℓ-ideals and homomorphisms into $\mathbf{R}_{-1}$ is important later on, since $\mathbf{R}_{-1}$ is the algebra of truth-degrees for pointed modal Abelian logic introduced in the next section (in particular, *canonical frames* are defined on the collections $\mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$).

4.3 Pointed Modal Abelian Logic: Frames and Algebras

We first introduce relational (Kripke) semantics for pointed modal Abelian logic similar to the real-valued logic of [33] and other many-valued modal logics on crisp frames (Subsection 4.3.1). Next, we introduce the variety of pointed modal Abelian ℓ-groups and establish some basic identities therein (Subsection 4.3.2). Lastly, we explain how to 'shift' between these two frameworks constructing complex algebras (in particular showing algebraic soundness) and canonical frames (Subsection 4.3.3). This sets up the scene for our main results which we prove in the subsequent Section 4.4.

4.3.1 Relational Semantics

The *language* $\mathfrak{L}_p$ of pointed modal Abelian logic is defined inductively from a countable set *Prop* $= \{x_1, x_2, x_3, \dots\}$ of *propositional variables*, the signature $\langle +, -, \vee, \wedge, 0, p\rangle$ of pointed Abelian ℓ-groups and the unary modality $\Box$. Let $Fm_{\mathfrak{L}_p}$ be the set of *formulas* over the language $\mathfrak{L}_p$.

Definition 4.8 (Model). *A* model $\mathfrak{M} = \langle W, R, \mathcal{V}\rangle$ *consists of a* Kripke frame $\langle W, R\rangle$ *(i.e. a non-empty set W with a binary relation $R \subseteq W^2$), together with a* bounded propositional valuation, *by which we mean a map $\mathcal{V}\colon W \times Prop \to \mathbb{R}$ such that for every propositional variable x_i there exists a positive real number r_i with $\{\mathcal{V}(u, x_i) \mid u \in W\} \subseteq [-r_i, r_i]$.*

Note that this slightly differs from [33], where a propositional valuation is itself bounded. However, it is clear that our condition suffices to ensure that the following interpretation of $\Box$ is (still) well-defined.

Definition 4.9 (Satisfaction & Validity). *Let $\mathfrak{M} = \langle W, R, \mathcal{V}\rangle$ be a model and $u \in W$. We inductively extend the valuation to $\mathcal{V}\colon W \times Fm_{\mathfrak{L}_p} \to \mathbb{R}$ by interpreting the non-modal connectives in the algebra $\mathbf{R}_{-1}$ in the standard way. Modal formulas are interpreted as follows:*

$$\mathcal{V}(u, \Box\varphi) = \bigwedge_{uRv} \mathcal{V}(v, \varphi).$$

(In particular, here we set $\bigwedge \varnothing := 0$.) A formula φ is satisfied *at $u \in W$ if and only if*

$$\mathcal{V}(u, \varphi) \geq 0$$

and the formula is valid, *denoted $\models \varphi$, if this holds at every state of every model.*

The following properties correspond to our axiomatization of pointed modal Abelian ℓ-groups later on (Definition 4.11) and essentially yield (algebraic) *soundness* later on.

Lemma 4.10. *For $\mathfrak{M} = \langle W, R, \mathcal{V}\rangle$ a model, $u \in W$, and $\varphi, \psi \in Fm_{\mathfrak{L}_p}$, the following hold:*

(i) $\mathcal{V}\big(u, \Box(\psi - \varphi)\big) \leq \mathcal{V}(u, \Box\psi) - \mathcal{V}(u, \Box\varphi)$,

(ii) $\mathcal{V}(u, \Box\varphi) + \mathcal{V}(u, \Box\varphi) = \mathcal{V}\big(u, \Box(\varphi + \varphi)\big)$,

(iii) $\mathcal{V}(u, \Box\varphi) \vee 0 = \mathcal{V}\big(u, \Box(\varphi \vee 0)\big)$,

(iv) $\mathcal{V}(u, \Box\varphi) \wedge \mathcal{V}(u, \Box\psi) = \mathcal{V}\big(u, \Box(\varphi \wedge \psi)\big)$,

(v) $\mathcal{V}(u, |\Box p - p| \wedge |\Box\varphi|) = 0$,

(vi) $\mathcal{V}(u, -\Box p) = \mathcal{V}(u, \Box{-p})$.

Proof. (i): By substituting $\chi = \psi - \varphi$ and adding $\mathcal{V}(u, \Box\varphi)$ to both sides of the inequation, it suffices to prove $\mathcal{V}(u, \Box\chi) + \mathcal{V}(u, \Box\varphi) \leq \mathcal{V}(u, \Box(\chi + \varphi))$, or equivalently

$$\bigwedge_{uRv} \mathcal{V}(v, \chi) + \bigwedge_{uRv} \mathcal{V}(v, \varphi) \leq \bigwedge_{uRv} \mathcal{V}(v, \chi + \varphi).$$

This follows, since we have $\bigwedge_{uRv} \mathcal{V}(v, \chi) + \bigwedge_{uRv} \mathcal{V}(v, \varphi) \leq \mathcal{V}(v, \chi + \varphi)$ for each uRv.
(ii): Directly follows from the identity $\bigwedge_{i \in I} r_i + \bigwedge_{i \in I} r_i = \bigwedge_{i \in I} r_i + r_i$ of $\mathbf{R}_{-1}$.
(iii): Directly follows from the infinite distributivity law of ℓ-groups, see [6, Eq. 1.1.11].
(iv): Directly follows from the law $\bigwedge_{i \in I} r_i \wedge \bigwedge_{i \in I} s_i = \bigwedge_{i \in I} r_i \wedge s_i$ of $\mathbf{R}_{-1}$.
(v) & (vi): We distinguish cases (a) $\exists v \colon uRv$ and (b) $\nexists v \colon uRv$. In case (a) note $\mathcal{V}(u, \Box p) = -1 = \mathcal{V}(u, p)$ and thus $\mathcal{V}(u, |\Box p - p| \wedge |\Box\varphi|) = 0$ and $\mathcal{V}(u, -\Box p) = 1 = \mathcal{V}(u, \Box -p)$. In case (b) we have $\mathcal{V}(u, \Box\varphi) = \bigwedge \varnothing = 0$, thus $\mathcal{V}(u, |\Box p - p| \wedge |\Box\varphi|) = 0$. Furthermore, $\mathcal{V}(u, -\Box p) = 0 = \mathcal{V}(u, \Box -p)$. $\Box$

4.3.2 Modal Abelian ℓ-groups

We aim to relate the relational semantics introduced in the previous subsection to the variety of pointed Abelian ℓ-groups with unary operators defined as follows.

Definition 4.11 (Modal Abelian ℓ-group)**.** *An algebra $\langle \mathbf{A}, \Box \rangle$ is a* modal Abelian ℓ-group *if $\mathbf{A}$ is an Abelian ℓ-group and $\Box \colon A \to A$ is a unary operation which satisfies for each $x, y \in A$ the following:*

(A1) $\Box(y - x) \leq \Box y - \Box x$,

(A2) $\Box(x + x) = \Box x + \Box x$,

(A3) $\Box(x \vee 0) = \Box x \vee 0$,

(A4) $\Box(x \wedge 0) = \Box x \wedge 0$.

If $\mathbf{A}_p$ is a pointed Abelian ℓ-group, we say that $\langle \mathbf{A}_p, \Box \rangle$ is a pointed modal Abelian ℓ-group *if it furthermore satisfies the following:*

(A5) $|\Box x| \wedge |\Box p - p| = 0$,

(A6) $\Box -p = -\Box p$.

We denote the varieties of modal Abelian ℓ-groups and of negatively pointed modal Abelian ℓ-groups by mAb *and* $\mathsf{mAb}_\mathsf{p}^-$*, respectively. A pointed modal Abelian ℓ-group $\langle \mathbf{A}_p, \Box \rangle$ is* strongly negatively pointed *if $\mathbf{A}_p$ is strongly negatively pointed.*

Example. Let $\mathbf{A}_p$ be a pointed Abelian ℓ-group. There are always at least two 'trivial' ways to expand this structure to a modal pointed Abelian ℓ-group, namely the identity $\Box_1 x := x$ and the constant zero $\Box_2 x := 0$. It is straightforward to verify that both $\langle \mathbf{A}_p, \Box_1 \rangle$ and $\langle \mathbf{A}_p, \Box_2 \rangle$ satisfy axioms (A1)–(A6).

Clearly $\mathsf{mAb}_\mathsf{p}^-$ is a conservative expansion of mAb, as in every $\mathbf{A} \in \mathsf{mAb}$ one can interpret the additional constant symbol as $p^{\mathbf{A}} := 0^{\mathbf{A}}$. Then Axioms (A5)–(A6) are trivially satisfied by (i) in the following.

Lemma 4.12. *The following hold in all modal Abelian ℓ-groups.*

(i) $\Box 0 = 0$,

(ii) $\Box(2^k \cdot x) = 2^k \cdot \Box x$ *for each* $k \in \mathbb{N}$,

(iii) $x \geq 0 \Rightarrow \Box x \geq 0$,

(iv) $x \leq y \Rightarrow \Box x \leq \Box y$.

Moreover, the following hold in all pointed modal Abelian ℓ-groups.

(v) $\Box(x + p) = \Box x + \Box p$ *and*
$\Box(x - p) = \Box x - \Box p$,

(vi) $\Box(x + z \cdot p) = \Box x + z \cdot \Box p$ *for each* $z \in \mathbb{Z}$,

(vii) $\Box(x \vee p) = \Box x \vee \Box p$,

(viii) $p \leq \Box p$,

(ix) $\Box(-k \cdot p) = -\Box(k \cdot p)$ *for each* $k \in \mathbb{N}$.

Proof. *(i)*: Follows from (A2) with $x = 0$.
(ii): Follows from iterating (A2).
(iii): Follows from (A3).
(iv): If $x \leq y$, we have $0 \leq y - x$ and thus $0 \leq \Box(y - x)$ by the previous item. Therefore $0 \leq \Box(y - x) \leq \Box y - \Box x$ by (A1), which implies $\Box x \leq \Box y$.
(v): Using (A1) and (A6) we compute

$$\begin{aligned}\Box(x + p) &\leq \Box x - \Box(-p) = \Box x + \Box p \\ &= \Box(x + p - p) + \Box p \leq \Box(x + p) - \Box p + \Box p = \Box(x + p).\end{aligned}$$

Therefore, $\Box(x + p) = \Box x + \Box p$ for each $x \in A$. Consequently by substitution, $\Box x = \Box(x - p + p) = \Box(x - p) + \Box p$ for each $x \in A$.
(vi): Follows from iterating the previous item.
(vii): Using (A3), (vi) and Lemma 4.4 (iii) we compute

$$\begin{aligned}\Box(x \vee p) &= \Box\Big(((x - p) \vee 0) + p\Big) && \text{(Lemma 4.4(iii))} \\ &= \Box\Big((x - p) \vee 0\Big) + \Box p && \text{(Item (v))} \\ &= \Big(\Box(x - p) \vee 0\Big) + \Box p && \text{((A3) \& Item (v))} \\ &= \Big(\Box(x - p) + \Box p\Big) \vee \Box p = \Box x \vee \Box p. && \text{(Lemma 4.4(iii) \& Item (v))}\end{aligned}$$

(viii): Due to Lemma 4.4 (iv) & (v) we compute that $(|\Box p| - |\Box p - p|) \leq |p|$ and furthermore that $|\Box p| - |\Box p| \wedge |\Box p - p| = (|\Box p| - |\Box p - p|) \vee 0$. Therefore by (A5) we obtain

$$|\Box p| = |\Box p| - |\Box p| \wedge |\Box p - p| = (|\Box p| - |\Box p - p|) \vee 0 \leq |p|.$$

Consequently, by applying (A6),(vii) and $p \leq 0$ compute

$$|\Box p| = -\Box p \vee \Box p = \Box(-p) \vee \Box p = \Box(-p \vee p) = \Box|p| = \Box(-p) = -\Box p$$

which shows $p \leq \Box p$ as desired.
(ix): Is a straightforward consequence of (i) and (vi) with $x = 0$. $\square$

Remark 4.13. As already discussed in Lemma 4.10, by applying subtraction and substitution, it is easy to see that axiom (A1) can be equivalently replaced with

$$\Box x + \Box y \leq \Box(x + y). \tag{A1$*$}$$

Let us also note that we could replace (A5) and (A6) with a single axiom

$$|\Box x| \wedge |\Box(x + x - p) - \Box x + \Box x - p| = 0 \tag{A5$*$}$$

Although this axiom is harder to verify, together with (A2) it provides an equational counterpart to the modal MV-algebra axioms $\Box(x \oplus x) = \Box x \oplus \Box x$ and $\Box(x \odot x) = \Box x \odot \Box x$ used in [58].

4.3.3 Complex Algebras and Canonical Frames

As usual, to 'shift' between the relational and algebraic frameworks introduced in the previous two subsections, we define complex algebras and canonical frames, starting with the former.

Definition 4.14 (Complex Algebra). *Let $\langle W, R\rangle$ be a Kripke frame. The corresponding* complex algebra *is given by $\mathfrak{C}_{\langle W,R\rangle} = \langle b\mathbf{R}_{-1}^{W}, \Box_R\rangle$ where $b\mathbf{R}_{-1}^{W} \leq \mathbf{R}_{-1}^{W}$ denotes the subalgebra of the product $\mathbf{R}_{-1}^{W}$ defined by the subuniverse of* bounded maps *$\{f \colon W \to \mathbb{R}$ bounded$\}$ and the operator $\Box_R$ is defined by*

$$(\Box_R f)(u) = \bigwedge_{uRv} f(v).$$

(Note that $\Box f$ is a well-defined bounded map if f is bounded.)

It is clear from Lemma 4.10 that all complex algebras are pointed modal Abelian ℓ-groups. Furthermore, the map of constant value -1 is a strong unit. From this we get the following.

Proposition 4.15 (Algebraic Soundness). *If $\mathsf{mAb}_{\mathsf{p}}^{-} \models \varphi \geq 0$ then $\models \varphi$.*

Proof. If $\not\models \varphi$ then there is a model $\langle W, R, \mathcal{V}\rangle$ such that $\mathcal{V}(u, \varphi) < 0$. Then $e_{\mathcal{V}} \colon Prop \to \mathfrak{C}_{\langle W,R\rangle}$ defined by $e_{\mathcal{V}}(x_i) = \mathcal{V}(-, x_i)$ is an assignment under which $\varphi \geq 0$ fails in the complex algebra $\mathfrak{C}_{\langle W,R\rangle}$. □

Similarly, we move from $\mathsf{mAb}_{\mathsf{p}}^{-}$-algebras to Kripke frames via the following construction based on the set of all homomorphisms into our algebra of truth-values $\mathbf{R}_{-1}$.

Definition 4.16 (Canonical Frame). *Let $\langle \mathbf{A}_p, \Box\rangle$ be a strongly negatively pointed modal Abelian ℓ-group and let $u, v \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$ be homomorphisms. We define*

$$uR_{\Box}v \quad \textit{if and only if} \quad u(\Box a) \geq 0 \Rightarrow v(a) \geq 0 \textit{ for all } a \in \mathbf{A}_p.$$

We call $\mathfrak{F}_{\langle \mathbf{A}_p,\Box\rangle} := \langle \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1}), R_{\Box}\rangle$ the canonical frame *of $\langle \mathbf{A}_p, \Box\rangle$.*

Note that from any *algebraic evaluation* $e\colon Prop \to \mathbf{A}_p$ we can define a propositional valuation $\mathcal{V}_e\colon \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1}) \times Prop \to \mathbb{R}$ on the canonical frame $\mathfrak{F}_{(\mathbf{A}_p,\Box)}$ via $\mathcal{V}_e(u, x_i) = u(e(x_i))$. However, these are *not* necessarily bounded valuations, that is, they do not necessarily define models according to Definition 4.8. However, note that these valuations are bounded if p is a *negative strong unit*, since then $e(x_i) \in \mathbf{A}_p$ is bounded by $n \cdot p \leq e(x_i) \leq -n \cdot p$ for some $n \in \mathbb{N}$. Therefore, every homomorphism $u\colon \mathbf{A}_p \to \mathbf{R}_{-1}$ satisfies $u(e(x_i)) \in [-n, n]$. In the next section, we prove that the corresponding Truth Lemma also holds for strongly negatively pointed modal Abelian ℓ-groups.

4.4 Pointed Modal Abelian Logic, Algebraically

In Subsection 4.4.1, we establish the Truth Lemma with respect to the constructions of the previous subsection for *strongly* negatively pointed modal Abelian ℓ-groups (Corollary 4.22). For this, we introduce some terms t_d which can 'separate' elements of $\mathbf{R}_{-1}$.

In Subsection 4.4.2, we introduce an infinitary derivability relation with respect to $\mathsf{mAb}_\mathsf{p}^-$ and prove completeness of real-valued modal logic with respect to this relation (Theorem 4.25).

4.4.1 The Truth Lemma

First let us note that the following is easily derived from ($\mathsf{A5}$) and the fact that $\mathbf{R}_{-1}$ is totally ordered.

Lemma 4.17. *Let $\langle \mathbf{A}_p, \Box\rangle$ be a pointed modal Abelian ℓ-group and let $u \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$ be a homomorphism. Then $u(\Box a) = 0$ for each $a \in \mathbf{A}_p$ or $u(\Box p) = u(p) = -1$.*

The set $D := \{\frac{z}{2^k} \mid z \in \mathbb{Z}, k \in \mathbb{N}\}$ is a dense subset of $\mathbb{R}$. For each $d \in D$ we fix some representation $d = \frac{z}{2^k}$ and define t_d to be a unary term

$$t_d(x) = (2^k \cdot x + z \cdot p) \wedge 0.$$

The following summarizes the main properties of these terms for our purposes.

Lemma 4.18. *The terms t_d defined above satisfy the following.*

(i) $t_d^{\mathbf{A}_p}(x) \leq 0$ *for every* $x \in \mathbf{A}_p$.

(ii) $t_d^{\mathbf{R}_{-1}}(r) = 0$ *iff* $r \geq d$.

(iii) Assume $u(\Box y) \neq 0$ for some $y \in \mathbf{A}_p$. Then $u\Big(t_d(\Box x)\Big) = u\Big(\Box t_d(x)\Big)$.

Proof. Let $d = \frac{z}{2^k}$ be the representation chosen in the definition of t_d. By definition, *(i)* is obvious. *(ii)*: Observe that for each $r \in \mathbb{R}$ we have $t_d(r) \geq 0$ iff $2^k \cdot r + z \cdot (-1) \geq 0$, which is equivalent to $r - d \geq 0$.

(iii): First observe that by Lemma 4.17 and the assumption we obtain $u(\Box p) = u(p)$. Compute

$$
\begin{aligned}
u\Big(t_d(\Box x)\Big) &= u\Big((2^k \cdot \Box x + z \cdot p) \wedge 0\Big) && \text{(Def. } t_d\text{)}\\
&= \Big(u(2^k \cdot \Box x) + z \cdot u(p)\Big) \wedge 0 && (u \text{ homomorphism})\\
&= \Big(u(2^k \cdot \Box x) + z \cdot u(\Box p)\Big) \wedge 0 && (u(\Box p) = p)\\
&= u\Big((2^k \cdot \Box x + z \cdot \Box p) \wedge 0\Big) && (u \text{ homomorphism})\\
&= u\Big((\Box(2^k \cdot x) + z \cdot \Box p) \wedge 0\Big) && \text{(Lemma 4.12(ii))}\\
&= u\Big(\Box(2^k \cdot x + z \cdot p) \wedge 0\Big) && \text{(Lemma 4.12(vi))}\\
&= u\Big(\Box((2^k \cdot x + z \cdot p) \wedge 0)\Big) = u\Big(\Box t_d(x)\Big) && \text{(Axioms (A1),(A4)\&Def. } t_d\text{)}
\end{aligned}
$$

as desired, finishing the proof. $\square$

Among others, we use these terms to prove the following (recall the Definition 4.16 of the canonical frame $\langle \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1}), R_\Box \rangle$).

Lemma 4.19. *Let* $\langle \mathbf{A}_p, \Box \rangle \in \mathsf{mAb}_\mathsf{p}^-$ *be strongly negatively pointed and* $u, v \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$*. The following conditions are equivalent:*

(1) $uR_\Box v$*,*

(2) $u(\Box a) = 0 \Rightarrow v(a) = 0$ *for all* $a \leq 0$*,*

(3) $\Box^{-1}u^{-1}(0) \cap \{a \in A \colon a \leq 0\} \subseteq v^{-1}(0)$*,*

(4) $u(\Box a) \leq v(a)$ *for all* $a \in A$*.*

Proof. (1) ⇒ (2): Let $a \leq 0$, and $u(\Box a) = 0$. The former yields $v(a) \leq 0$ and the latter implies together with $uR_\Box v$ that $v(a) \geq 0$.
(2) ⇔ (3) is obvious.
(2) ⇒ (4): We proceed by contraposition. Assume there is a such that $v(a) < u(\Box a)$. Then there is $d \in D$ such that $v(a) < d \leq u(\Box a)$. By Lemma 4.18(ii) we have $t_d^{\mathbf{R}_{-1}}(v(a)) = v(t_d(a)) < 0$ and $t_d^{\mathbf{R}_{-1}}(u(\Box a)) = 0$. By Lemma 4.18(iii) this implies $u(\Box t_d(a)) = 0$. Also, by Lemma 4.18(i) we have $t_d(a) \leq 0$.
(4) ⇒ (1): If $u(\Box a) \leq v(a)$ for each $a \in A$ then clearly whenever $0 \leq u(\Box a)$ then also $0 \leq v(a)$. $\square$

We show one more technical result before proving the main theorem of this subsection.

Lemma 4.20. *Let* $\langle \mathbf{A}_p, \Box \rangle$ *be a strongly negatively pointed modal Abelian* ℓ*-group,* $u \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$*, and let* I *be the* ℓ*-ideal generated by the set* $S := \{x \in A \mid x \leq 0\} \cap \Box^{-1}u^{-1}(0)$*. Let* $a \in I$ *such that* $a \leq 0$*. Then* $a \in \Box^{-1}u^{-1}(0)$*.*

Proof. Since a is in the ℓ-ideal generated by the set S, due to (IG), there exist $b_1, \ldots, b_n \in S$ such that $\sum_{i=1}^n b_i \leq a \leq 0$. Since by Lemma 4.12 (iv) the operator $\Box$ is order preserving, by iterating Equation ($\mathsf{A1}*$) we have the following in $\mathbf{A}_p$:

$$\Box b_1 + \cdots + \Box b_n \leq \Box(b_1 + \cdots + b_n) \leq \Box a \leq \Box 0 = 0.$$

By applying u, we obtain:

$$0 = u(\Box b_1 + \cdots + \Box b_n) \leq u\Big(\Box(b_1 + \cdots + b_n)\Big) \leq u(\Box a) \leq u(\Box 0) = 0.$$

Thus, $u(\Box a) = 0$, which completes the proof. $\Box$

The following is crucial, essentially being the clause of modal formulas in the Truth Lemma.

Theorem 4.21. *Let $\langle \mathbf{A}_p, \Box\rangle$ be a strongly negatively pointed modal Abelian ℓ-group. Then $u(\Box a) = \bigwedge_{uR_\Box v} v(a)$ holds for every homomorphism $u \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$.*

Proof. We already know by Lemma 4.19(4) that $u(\Box a) \leq \bigwedge_{uR_\Box v} v(a)$ (from this we also see that the meet exists). First, we consider the case $u(\Box a) = 0$ for every $a \in \mathbf{A}_p$. Since for any $v \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$ we have $v(p) = -1$, it follows there is no v such that $uR_\Box v$. Consequently, $\bigwedge_{uR_\Box v} v(a) = 0 = u(\Box a)$.

For the remaining case, by Lemma 4.17 we know that $u(\Box p) = -1$. Towards contradiction, assume $u(\Box a) < \bigwedge_{uR_\Box v} v(a)$. Since D is dense there exists $d \in D$ such that $u(\Box a) < d \leq \bigwedge_{uR_\Box v} v(a)$. Therefore, by Lemma 4.18(ii) & (iii) we have $t_d^{\mathbf{R}_{-1}}(u(\Box a)) = u(t_d(\Box a)) = u(\Box t_d(a)) < 0$ and $t_d^{\mathbf{R}_{-1}}(v(a)) = v(t_d(a)) = 0$ for each $uR_\Box v$. By Lemmata 4.19 & 4.7 each maximal ideal of $\mathbf{A}_p$ containing $\{x \leq 0\} \cap \Box^{-1}u^{-1}(0)$ has to contain $t_d(a)$. Let I be the ℓ-ideal generated by the set $\{x \leq 0\} \cap \Box^{-1}u^{-1}(0)$. Now $[t_d(a)]_I \in \mathsf{rad}(\mathbf{A}_p/I)$ and thus by Lemma 4.6 we obtain $2^k \cdot [t_d(a)]_I - [p]_I \geq [0]_I$ for each $k \in \mathbb{N}$. Therefore, there exists $x \in I$ such that $2^k \cdot t_d(a) - p \geq x$ and without loss of generality we can assume $x \leq 0$ (since otherwise we can replace it by $x \wedge 0 \in I$). From Lemma 4.20 we get $x \in \Box^{-1}u^{-1}(0)$ and thus we have by Lemma 4.12(ii),(iv) & (v) that

$$0 = u(\Box x) \leq u\Big(\Box(2^k \cdot t_d(a) - p)\Big) = u\Big(\Box(2^k \cdot t_d(a))\Big) - u(\Box p) = 2^k \cdot u\Big(\Box t_d(a)\Big) + 1$$

for each $k \in \mathbb{N}$. Since -1 is a negative strong unit in $\mathbf{R}_{-1}$ this means that $u(\Box t_d(a))$ is infinitesimal in $\mathbf{R}_{-1}$. However, since the algebra $\mathbf{R}_{-1}$ is simple this implies $u(\Box t_d(a)) = 0$, a contradiction. $\Box$

The following Truth Lemma is a standard consequence. Note that the set of formulas $Fm_{\mathfrak{L}_p}$ can be seen as the absolutely free algebra in the language of pointed modal Abelian ℓ-groups and we identify *algebraic evaluations* $e\colon Prop \to \langle \mathbf{A}_p, \Box\rangle$ with the corresponding homomorphisms $e\colon \mathfrak{L}_p \to \langle \mathbf{A}_p, \Box\rangle$.

Corollary 4.22 (Truth Lemma)**.** *Let $\langle \mathbf{A}_p, \Box\rangle \in \mathsf{mAb}_\mathsf{p}^-$ be strongly negatively pointed, $u \in \mathrm{Hom}(\mathbf{A}_p, \mathbf{R}_{-1})$ and $e\colon Prop \to \mathbf{A}_p$ an algebraic evaluation. Let $\mathcal{V}_e$ be the propositional valuation on $\mathfrak{F}_{\langle \mathbf{A}_p, \Box\rangle}$ defined by $\mathcal{V}_e(u, x) = u(e(x))$. Then the property $\mathcal{V}_e(u, \varphi) = u(e(\varphi))$ extends to every formula φ.*

Proof. Induction on the formula, where the case $\Box\varphi$ follows from Theorem 4.21:

$$\mathcal{V}_e(u, \Box\varphi) = \bigwedge_{uR_\Box v} \mathcal{V}_e(v, \varphi) = \bigwedge_{uR_\Box v} v(e(\varphi)) = u(e(\Box\varphi)).$$

All other cases are clear (since u is a homomorphism with respect to the non-modal language). $\Box$

4.4.2 An Infinitary Algebraic Completeness

Similar to the case of modal MV-algebras [58], the Truth Lemma of the previous section does not immediately yield algebraic completeness. In particular, if the inequation $\varphi \geq 0$ does not hold in some $\langle \mathbf{A}_p, \Box \rangle \in \mathsf{mAb}_\mathsf{p}^-$ we have yet to obtain the *witnessing homomorphism* which shows that φ is not valid in the corresponding canonical frame $\mathfrak{F}_{\langle \mathbf{A}_p, \Box \rangle}$. To construct such a homomorphism, we will use Lemma 4.6 and introduce an infinitary rule $(\mathsf{A}\infty)$ based on $(\mathsf{INF}\text{-})$ to capture that the corresponding term is not evaluated inside the radical.

Moreover, we must ensure that a potential counterexample is witnessed in a well-defined model, that is, with a *bounded valuation* (recall Definition 4.8). We observed in Subsection 4.3.3 that this is ensured for canonical frames of *strongly* pointed modal Abelian ℓ-groups.

To formalize the requirement that it suffices to check φ within these bounds, let us first set the following notation. For a variable x_i and $k \in \mathbb{N}$, we define terms

$$x_i^k := (x_i \vee k \cdot p) \wedge k \cdot (-p).$$

For a formula $\varphi(x_1, \ldots, x_n)$ and $k \in \mathbb{N}$ we let φ^k denote the substitution instance

$$\varphi^k(x_1, \ldots, x_n) = \varphi(x_1^k, \ldots, x_n^k).$$

Now the *bounded valuations* rule is expressed as follows.

$$\mathsf{mAb}_\mathsf{p}^- \models \varphi^k \geq 0 \text{ for all } k \in \mathbb{N} \quad \Longrightarrow \quad \mathsf{mAb}_\mathsf{p}^- \models \varphi \geq 0 \tag{BV}$$

Note that the other implication trivially holds. Let us emphasize that rule (BV) is *not* a single infinitary rule in the standard sense. Because the syntactic substitutions happen *within* the formula itself, this property cannot be captured by one generalized quasi-equation. Instead, (BV) represents an infinite family of generalized quasi-equations, one for every formula $\varphi \in Fm_{\mathfrak{L}_p}$.

As previously mentioned, we need to combine this with $(\mathsf{INF}\text{-})$ to ensure the existence of homomorphisms witnessing non-validity. Thus, we end up with the following.

Definition 4.23 (Infinitary Algebraic Derivability)**.** *We define*

$$\vdash_\infty \varphi \quad :\Longleftrightarrow \quad \mathsf{mAb}_\mathsf{p}^- \models n \cdot (\varphi^k \wedge 0) - p \geq 0 \textit{ for every } n, k \in \mathbb{N}. \tag{A∞}$$

Recall that our Truth lemma was only proved for strongly negatively pointed modal Abelian ℓ-groups. The following lemma will allow us to 'move' to such an algebra in the proof of Theorem 4.25 later on.

Lemma 4.24. *Let $\langle \mathbf{A}_p, \Box \rangle$ be a negatively pointed modal Abelian ℓ-group with $p \neq 0$ and let $\mathbf{B}$ be the ℓ-ideal of $\mathbf{A}$ generated by p. Then $\langle \mathbf{B}_p, \Box \rangle$ is a strongly negatively pointed modal Abelian ℓ-group.*

Proof. We only need to verify that $\mathbf{B}_p$ is closed under $\Box$. Let $b \in \mathbf{B}_p$. We have $|b| \leq n \cdot |p|$ and hence we obtain $n \cdot p \leq b \leq -n \cdot p$. By Lemma 4.12 (iv) we know that the operator $\Box$ is monotone, from which we get (using $-\Box(n \cdot p) = \Box(-n \cdot p)$) that $\Box(n \cdot p) \leq \Box b \leq \Box(-n \cdot p)$ and $\Box(n \cdot p) \leq -\Box b \leq \Box(-n \cdot p)$. Therefore, we showed $|\Box b| \leq |n \cdot p|$ and thus $\Box b \in \mathbf{B}_p$. $\Box$

We are now ready to prove the main result of this section, completeness of pointed modal Abelian logic with respect to the derivability relation of Definition 4.23.

Theorem 4.25 (Infinitary Algebraic Completeness). *Let φ be a formula. Then $\vDash \varphi$ if and only if $\vdash_\infty \varphi$.*

Proof. For the implication ($\Rightarrow$), assume $\nvdash_\infty \varphi$. By definition, there exist $n, k \in \mathbb{N}$ such that $\mathsf{mAb}_\mathsf{p}^- \nvDash n \cdot (\varphi^k \wedge 0) - p \geq 0$. Let $\mathbf{A}_p \in \mathsf{mAb}_\mathsf{p}^-$ and e be an evaluation on $\mathbf{A}_p$ such that $n \cdot (e(\varphi^k) \wedge 0) - p \ngeq 0$. Note that $\mathbf{A}_p$ satisfies $p \neq 0$, since otherwise it would satisfy $e(x_i^k) = (e(x_i) \vee 0) \wedge 0 = 0$ for each x_i and thus $n \cdot (e(\varphi^k) \wedge 0) - p = 0$. Let $\mathbf{B}$ be the ℓ-ideal of $\mathbf{A}$ generated by p. By Lemma 4.24 we know that $\mathbf{B}_p \in \mathsf{mAb}_\mathsf{p}^-$ is strongly negatively pointed. We define an evaluation $\hat{e}$ on $\mathbf{B}_p$ by $\hat{e}(x) = e(x^k)$. By the above, it holds that $n \cdot (\hat{e}(\varphi) \wedge 0) - p \ngeq 0$. From Lemma 4.6 we obtain that $\hat{e}(\varphi \wedge 0) \notin \mathsf{rad}(\mathbf{B}_p)$. Consequently, there exists a homomorphism $u \in \mathrm{Hom}(\mathbf{B}_p, \mathbf{R}_{-1})$ such that $u(\hat{e}(\varphi \wedge 0)) < 0$. Let $\mathcal{V}_{\hat{e}}$ be the valuation on the canonical frame of $\mathbf{B}_p$ defined by $\mathcal{V}_{\hat{e}}(v, x_i) = v(\hat{e}(x_i))$ for all $v \in \mathrm{Hom}(\mathbf{B}_p, \mathbf{R}_{-1})$. The Truth Lemma yields

$$\mathcal{V}_{\hat{e}}(u, \varphi) \wedge 0 = \mathcal{V}_{\hat{e}}(u, \varphi \wedge 0) = u\Big(\hat{e}(\varphi \wedge 0)\Big) < 0$$

and therefore clearly $\mathcal{V}_{\hat{e}}(u, \varphi) < 0$, which witnesses $\nvDash \varphi$.

For the implication ($\Leftarrow$), assume $\nvDash \varphi$ and let $\langle W, R, \mathcal{V} \rangle$ be a model such that $\mathcal{V}(u, \varphi) < 0$ for some $u \in W$. Since $\mathcal{V}$ is a bounded valuation there is some $k \in \mathbb{N}$ such that $\mathcal{V}(u, \varphi^k) < 0$ also holds. Let e be the evaluation on the complex algebra $\mathfrak{C}_{\langle W,R \rangle}$ defined by $e(x_i) = \mathcal{V}(-, x_i)$. We have $e(\varphi^k) = \mathcal{V}(-, \varphi^k)$. Since $e(\varphi^k)(u) = \mathcal{V}(u, \varphi^k) < 0$ it follows that $e(\varphi^k \wedge 0) < 0$. Since $\{f \in \mathfrak{C}_{\langle W,R \rangle} \colon f(u) = 0\}$ is a maximal ℓ-ideal of $\mathfrak{C}_{\langle W,R \rangle}$ (because it is the preimage $\mathsf{ev}_u^{-1}(0)$ of the homomorphism $\mathsf{ev}_u \colon \mathfrak{C}_{\langle W,R \rangle} \to \mathbf{R}_{-1}$ evaluating at u) and $e(\varphi^k \wedge 0) \notin \{f \in \mathfrak{C}_{\langle W,R \rangle} \colon f(u) = 0\}$ by Lemma 4.6 it follows there is $n \in \mathbb{N}$ such that $e(n \cdot (\varphi^k \wedge 0) - p) \ngeq 0$, which by definition shows $\nvdash_\infty \varphi$. This completes the proof. □

4.5 Conclusion

In this paper, we introduced the variety of negatively pointed modal Abelian ℓ-groups and established its connection to real-valued modal logic, culminating in Theorem 4.25 characterizing modal validity via the rule ($\mathsf{A}\infty$). We take this as evidence that the variety $\mathsf{mAb}_\mathsf{p}^-$ deserves further investigation in the future.

Naturally, this also applies to the study of modal Abelian ℓ-groups *without* a designated point. The crucial role played by the non-zero designated element was highlighted throughout the present work. Without the constant symbol p in the language, every term function $t(x_1, \ldots, x_n)$ over any modal Abelian ℓ-group trivially satisfies $t(0, \ldots, 0) = 0$, which effectively prevents the construction of the separating terms t_d of Lemma 4.18. Moreover, the strong unit, the radical, and the boundedness of valuations are all dependent on the assumption of pointedness. Nevertheless, in future work we aim to establish a similar algebraic completeness result for (point-free) real-valued modal logic with respect to the variety mAb.

It is common in many-valued modal logic to not only consider crisp, but *labeled accessibility relations* (e.g. see [14, 16] for the case of finitely-valued modal Łukasiewicz logic). In (pointed) modal Abelian logic, one may consider *bounded* labeled accessibility relations $R\colon W^2 \to [-r, r] \subseteq \mathbb{R}$ with the modality interpreted via $\mathcal{V}(u, \Box\varphi) = \bigwedge_{v \in W} \mathcal{V}(v, \varphi) - R(u, v)$. Studying this logic and the relationship to modal Łukasiewicz logic would clearly be an interesting (albeit challenging) future direction.

There is a well-known categorical equivalence between the class of strongly pointed Abelian ℓ-groups and the class of MV-algebras established by the *Mundici functor* (e.g. see [21]). Similarly, we plan to investigate the category-theoretic relationship between strongly negatively pointed modal Abelian ℓ-groups and the category MMV of modal MV-algebras of [58] (let us note here that the Mundici equivalence was extended to *monadic* MV-algebras in [22]). As a first step we can show that, given a negatively pointed modal Abelian ℓ-group, its interval $[p, 0]$ is a modal MV-algebra. The construction in the reverse direction appears to be less straightforward.

Another open question is whether we can replace rule ($\mathsf{A}\infty$) with a *single* generalized quasi-equation. As discussed previously, rule (BV) corresponds to an infinite family of generalized quasi-equations (one for each formula φ) and consequently defines a generalized quasi-variety, and we want to investigate whether there exists a single generalized quasi-equation defining it.

Finally, we also want to address questions regarding computational *complexity*. To this end, we plan to generalize and apply the approach presented in [56] for Łukasiewicz unbound logic to the setting of strongly pointed modal Abelian ℓ-groups.

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