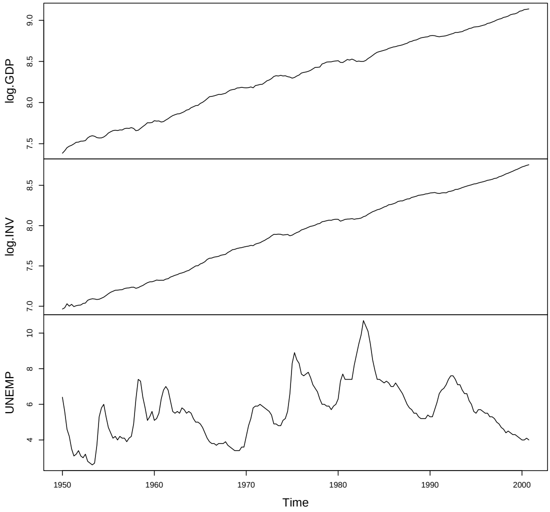
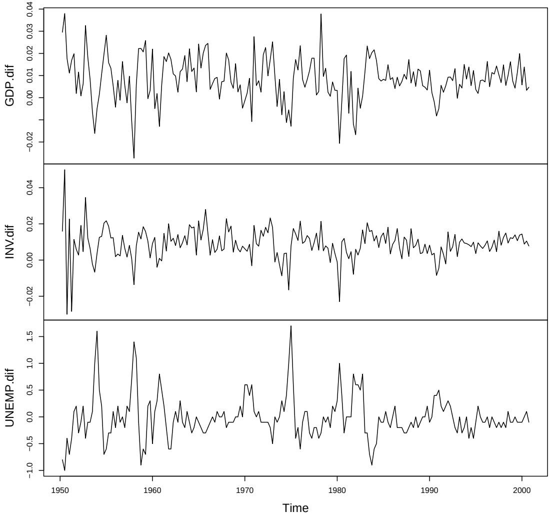
Week 13: Multivariate time series models

Multivariate time series



Multivariate time series

We observe $\boldsymbol{Y}_t = (Y_{1t}, \ldots, Y_{mt})^\top$ for $m > 1$ and $t = 1, \ldots, n$



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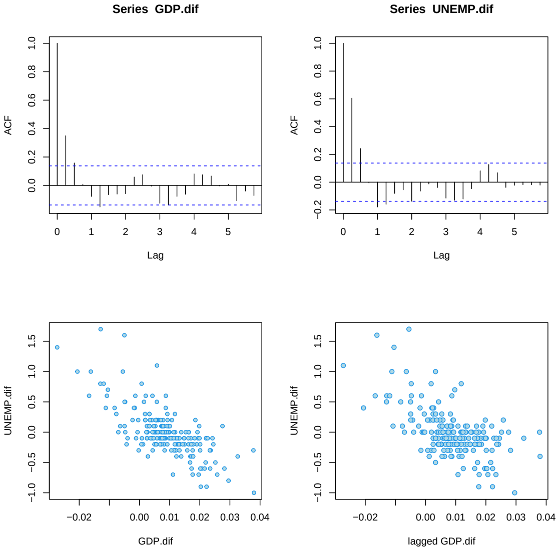

Multivariate time series

Observed characteristics

- $\hookrightarrow$ $\{Y_{it}\}$ is a time series of possibly correlated variables
- $\hookrightarrow$ components Y_{it} and Y_{jt} can be correlated
- $\hookrightarrow$ variables Y_{it} and $Y_{j,t+k}$ can be correlated

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Objectives of multivariate time series analysis

- ↪ to study dynamic relationships between the components of $\boldsymbol{Y}_t$
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- to study dynamic relationships between the components of $\boldsymbol{Y}_t$
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Aim:

- build a model

$$\boldsymbol{Y}_{t+1} = g_t(\boldsymbol{Y}_t, \cdots \boldsymbol{Y}_1)$$

for a suitable function g_t

Random vectors

$\boldsymbol{Y} = (Y_1, \dots, Y_m)^\top$ a random vector with a continuous distribution

- mean vector

$$\mathsf{E}\boldsymbol{Y} = \begin{pmatrix} \mathsf{E}Y_1, \\ \dots \\ \mathsf{E}Y_m \end{pmatrix}$$

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- variance matrix

$$\begin{aligned} \operatorname{Var} \boldsymbol{Y} &= \mathrm{E}(\boldsymbol{Y} - \mathrm{E}\,\boldsymbol{Y})(\boldsymbol{Y} - \mathrm{E}\,\boldsymbol{Y})^\top \\ &= \begin{pmatrix} \operatorname{Var} Y_1 & \operatorname{Cov}(Y_1, Y_2) & \ldots & \operatorname{Cov}(Y_1, Y_m) \\ \vdots & & & \\ \operatorname{Cov}(Y_m, Y_1) & \operatorname{Cov}(Y_m, Y_2) & \ldots & \operatorname{Var}(Y_m) \end{pmatrix} \end{aligned}$$

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- covariance matrix of $\boldsymbol{Y}_{m\times 1}$ and $\boldsymbol{Z}_{p\times 1}$

$$\begin{aligned} \mathrm{Cov}\,(\boldsymbol{Y}, \boldsymbol{Z}) &= \mathrm{E}(\boldsymbol{Y} - \mathrm{E}\,\boldsymbol{Y})(\boldsymbol{Z} - \mathrm{E}\boldsymbol{Z})^\top = \mathrm{Cov}\,(\boldsymbol{Z}, \boldsymbol{Y})^\top \\ &= \big(\mathrm{Cov}\,(Y_i, Z_j)\big)_{i=1,j=1}^{m,p} \end{aligned}$$

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- correlation matrix $\boldsymbol{Y}_{m\times 1}$ and $\boldsymbol{Z}_{p\times 1}$

$$\operatorname{cor}(\boldsymbol{Y}, \boldsymbol{Z}) = \left(\operatorname{cor}(Y_i, Z_j)\right)_{i,j=1}^{m,p}$$

Multivariate normal distribution

$\boldsymbol{Y} \sim \mathrm{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ can be constructed as

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{Z} + \boldsymbol{\mu},$$

where $\boldsymbol{Z} = (Z_1, \ldots, Z_m)^\top$ such that

$$Z_i \sim \mathrm{N}(0, 1) \quad \text{are independent and } \boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{\Sigma}$$

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- $\hookrightarrow$ $\mathrm{E}\, Y = \boldsymbol{\mu}$ and Var $\boldsymbol{Y} = \boldsymbol{\Sigma}$
- $\hookrightarrow$ If $\boldsymbol{\Sigma}$ regular $\leadsto$ continuous distribution with density

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{m/2}\sqrt{\det \boldsymbol{\Sigma}}} e^{\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$$

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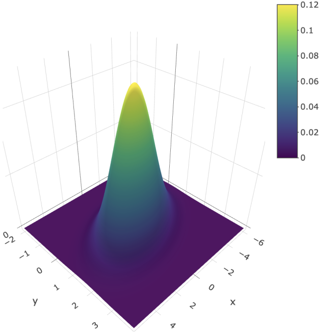
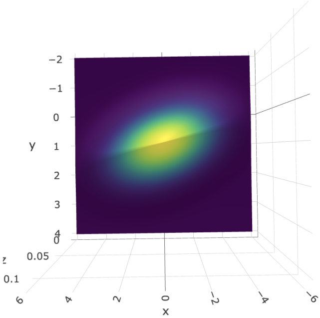
↪ each Y_i normal distribution,
conditional distributions are normal

Multivariate normal distribution





Multivariate white noise

$\{\varepsilon_t\}$ such that

$\hookrightarrow$

$$\mathrm{E}\varepsilon_t = \mathbf{0} \quad \text{and} \quad \mathrm{Var}\,\varepsilon_t = \mathbf{\Sigma}$$

for all t

$\hookrightarrow$ ε_t and ε_s uncorrelated for $t \neq s$, i.e.

$$\mathrm{Cov}\,(\varepsilon_t, \varepsilon_s) = \mathrm{E}\varepsilon_t \varepsilon_s^\top = \mathbf{0} \quad \text{for } s \neq t$$

Example: ε_t iid from $\mathrm{N}_m(\mathbf{0}, \mathbf{\Sigma})$

Basic TS concepts

Time series $\{\boldsymbol{Y}_t\}_{t=-\infty}^{\infty}$ of m-dimensional vectors

$\hookrightarrow$ Weakly stationary if for all t, s and k:

$$\mathrm{E}\,\boldsymbol{Y}_t = \boldsymbol{\mu}$$

$$\mathrm{Cov}\,(\boldsymbol{Y}_s, \boldsymbol{Y}_t) = \mathrm{E}(\boldsymbol{Y}_s - \boldsymbol{\mu})(\boldsymbol{Y}_t - \boldsymbol{\mu})^\top = \mathrm{Cov}\,(\boldsymbol{Y}_{s+k}, \boldsymbol{Y}_{t+k})$$

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$\hookrightarrow$ Matrix autocovariance function $\{\boldsymbol{\Gamma}_k\}_{k=-\infty}^{\infty}$

$$\boldsymbol{\Gamma}_k = \mathrm{Cov}\,(\boldsymbol{Y}_t, \boldsymbol{Y}_{t-k}) = \mathrm{Cov}\,(\boldsymbol{Y}_{t+k}, \boldsymbol{Y}_t) = \left(\gamma_{ij}(k)\right)_{i,j=1}^{m}$$

with $\gamma_{ij}(k) = \mathrm{Cov}\,(Y_{i,t+k}, Y_{j,t})$, and $\boldsymbol{\Gamma}_k = \boldsymbol{\Gamma}_{-k}^\top$

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$\hookrightarrow$ Matrix autocorrelation function $\{\boldsymbol{\rho}_k\}_{k=-\infty}^{\infty}$

$$\boldsymbol{\rho}_k = \boldsymbol{D}^{-1/2}\boldsymbol{\Gamma}_k\boldsymbol{D}^{-1/2} = (\rho_{ij}(k))_{i,j=1}^{m} \quad \text{where } \boldsymbol{D} = \mathrm{diag}(\mathrm{Var}\,\boldsymbol{Y}_1)$$

with

$$\rho_{ij}(k) = \frac{\mathrm{Cov}\,(Y_{i,t+k}, Y_{j,t})}{\sqrt{\mathrm{Var}\,Y_{i,t} \cdot \mathrm{Var}\,Y_{j,t}}} = \mathrm{cor}(Y_{i,t+k}, Y_{j,t})$$

and $\boldsymbol{\rho}_k = \boldsymbol{\rho}_{-k}^\top$

Autocorrelation function

$$\boldsymbol{\rho}_k = (\rho_{ij}(k))_{i,j=1}^m$$

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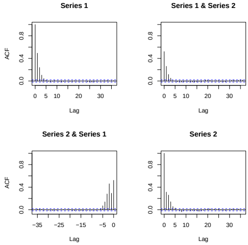
- $\hookrightarrow$ $\{\rho_{ii}(k)\}$ ACF of series $\{Y_{it}\}$
- $\hookrightarrow$ $\boldsymbol{\rho}_0$ correlation matrix of $\boldsymbol{Y}_t$ and $\boldsymbol{Y}_t$ $\rightsquigarrow$ 1s on the diagonal
- $\hookrightarrow$ $\boldsymbol{\rho}_k$ for $k \neq 0$ correlation matrix of $\boldsymbol{Y}_{t+k}$ and $\boldsymbol{Y}_t$
- $\hookrightarrow$ $\rho_{ij}(k)$ for $i \neq j$ are called cross-correlations
- $\hookrightarrow$ visualization: plot $\{\rho_{ij}(k)\}$ as a function of k for each $i, j = 1, \ldots, m$



Cross-correlations

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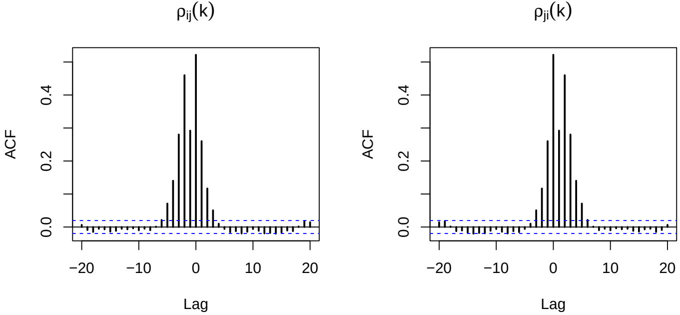
$$\rho_{ij}(k) = \text{cor}(Y_{i,t+k}, Y_{j,t}) = \rho_{ji}(-k)$$



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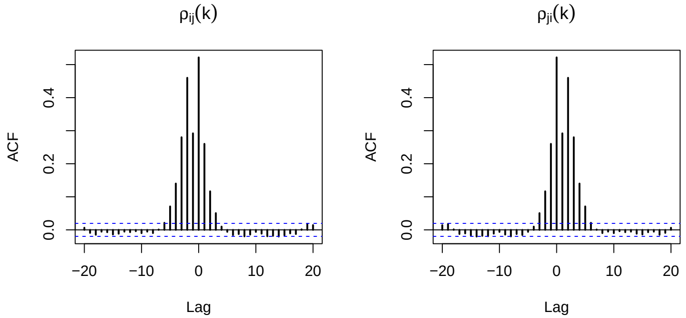

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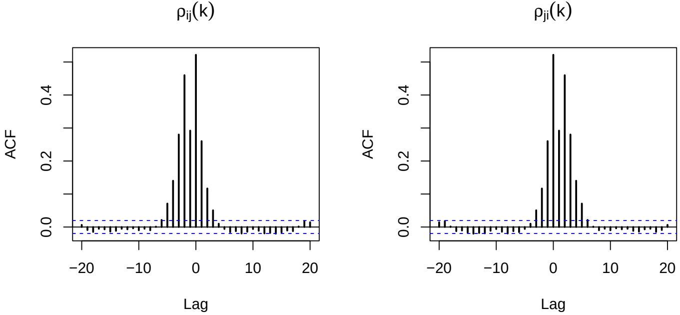
↪ if

$$\rho_{ij}(k) = 0 = \rho_{ji}(k) \quad \text{for all } k$$

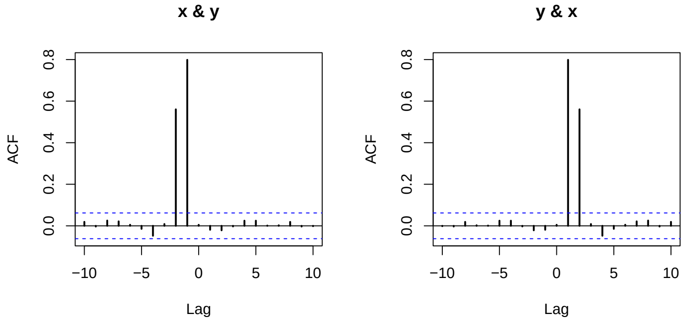
$\rightsquigarrow$ series mutually uncorrelated

(but be careful with the bounds in graphs)



Cross-correlations

$\hookrightarrow$ if $\rho_{ij}(k) = 0$ for all $k > 0$ and $\rho_{ji}(l) \neq 0$ for some $l > 0 \rightsquigarrow Y_{it}$ is uncorrelated with past values Y_{jt}, but Y_{jt} is correlated with some past values of Y_{it}



Left: Efect of Y_t on X_{t+k}

Right: Efect of X_t on Y_{t+k}

Linear process

$\{\boldsymbol{Y}_t\}$ is a **linear process** if it has a representation

$$\boldsymbol{Y}_t = \boldsymbol{\mu} + \sum_{j=0}^{\infty} \boldsymbol{\Psi}_j \boldsymbol{\varepsilon}_{t-j},$$

where $\{\boldsymbol{\varepsilon}_t\}$ is $\mathrm{WN}(\mathbf{0}, \boldsymbol{\Sigma})$ and $\{\boldsymbol{\Psi}_j\}$ is a sequence of $m \times m$ matrices such that

$$\boldsymbol{\Psi}_0 = \boldsymbol{I}, \quad \sum_{j=1}^{\infty} \|\boldsymbol{\Psi}_j\|^2 < \infty$$

for Frobenius matrix norm

$$\|\boldsymbol{A}\| = \sqrt{\mathrm{trace}(\boldsymbol{A}^\top \boldsymbol{A})} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|^2}.$$

Properties of a linear process:

- ↪ stationary
- ↪ $\mathrm{E}\,\boldsymbol{Y}_t = \boldsymbol{\mu}$ and

$$\boldsymbol{\Gamma}_k = \sum_{j=0}^{\infty} \boldsymbol{\Psi}_{j+k} \boldsymbol{\Sigma} \boldsymbol{\Psi}_j^\top$$

Estimates

Data $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ from a series $\{\boldsymbol{Y}_t\}_{t=-\infty}^{\infty}$

$\hookrightarrow$ estimated mean

$$\overline{\boldsymbol{Y}}_n = \frac{1}{n} \sum_{t=1}^{n} \boldsymbol{Y}_t$$

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$\hookrightarrow$ estimated (matrix) autocovariance function

$$\boldsymbol{C}_k = \frac{1}{n} \sum_{t=1}^{n-k} (\boldsymbol{Y}_{t+k} - \overline{\boldsymbol{Y}})(\boldsymbol{Y}_t - \overline{\boldsymbol{Y}})^\top \quad \text{for } k > 0,$$

and $\boldsymbol{C}_k = \boldsymbol{C}_{-k}^\top$ for $k < 0$

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↪ estimated (matrix) autocorrelation function

$$\boldsymbol{R}_k = \widehat{\boldsymbol{D}}^{-1/2} \boldsymbol{C}_k \widehat{\boldsymbol{D}}^{-1/2},$$

where

$$\widehat{\boldsymbol{D}} = \operatorname{diag}(\boldsymbol{C}_0)$$

Special case $m = 2$

We observe $\boldsymbol{Y}_t = (X_t, Z_t)^\top$ for $t = 1, \ldots, n$

- estimated (matrix) autocovariances

$$\boldsymbol{C}_0 = \begin{pmatrix} S_X^2 & S_{XZ} \\ S_{XZ} & S_Z^2 \end{pmatrix} \quad \boldsymbol{C}_k = \begin{pmatrix} c_{11}(k) & c_{12}(k) \\ c_{21}(k) & c_{22}(k) \end{pmatrix}$$

where

↪ $\{c_{11}(k)\}$ is the sample autocovariance function of $\{X_t\}$

↪

$$c_{12}(k) = \frac{1}{n}\sum_{t=1}^{n-k}(X_{t+k}-\overline{X})(Z_t-\overline{Z}), \quad c_{21}(k) = \frac{1}{n}\sum_{t=1}^{n-k}(Z_{t+k}-\overline{Z})(X_t-\overline{X})$$

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$$\boldsymbol{R}_k = \widehat{\boldsymbol{D}}^{-1/2} \boldsymbol{C}_k \widehat{\boldsymbol{D}}^{-1/2} = \begin{pmatrix} r_{11}(k) & r_{12}(k) \\ r_{21}(k) & r_{22}(k) \end{pmatrix}$$

  so

  ↪ $r_{11}(k)$ is sample ACF of $\{X_t\}$

  ↪

$$r_{12}(k) = \frac{c_{12}(k)}{\sqrt{S_X^2 S_Z^2}} \text{ estimates } \mathrm{cor}(X_{t+k}, Z_t)$$

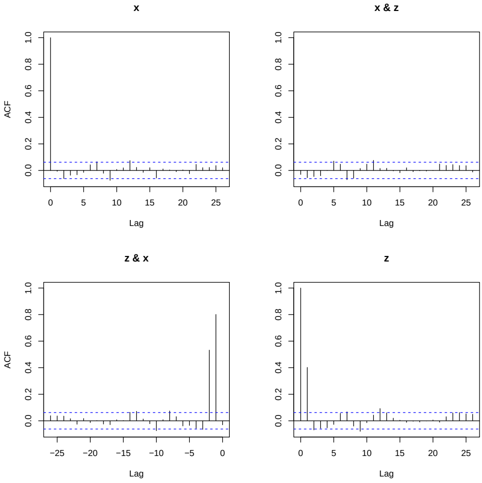
  ↪ $r_{12}(-k) = r_{21}(k)$

Example

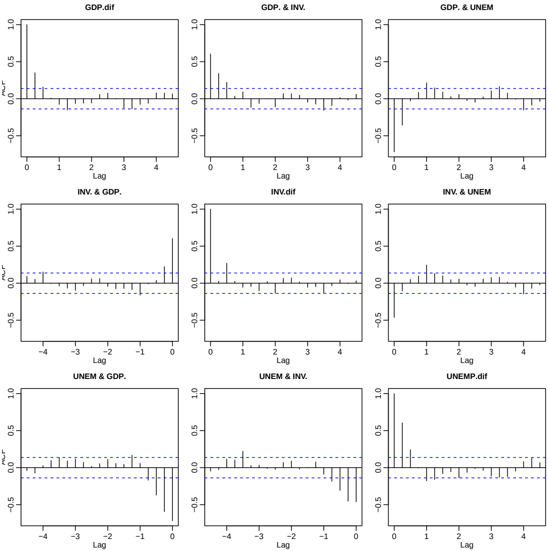
Data $\boldsymbol{Y}_t = (X_t, Z_t)^\top$, $t = 1, \dots, n$

$\hookrightarrow$ $\{X_t\}$ are iid N(0, 1)

$\hookrightarrow$ $Z_t = 3X_{t-1} + 2X_{t-2} + \varepsilon_t$, ε_t iid N(0, 1)



US data example



Generalization of Bartlett's formula

$m = 2 \rightsquigarrow \boldsymbol{Y}_t = (X_t, Z_t)^\top$, $t = 1, \ldots, n$

$\hookrightarrow$ recall: if $\{X_t\}$ white noise $\rightsquigarrow$

$$\sqrt{n} \cdot r_{11}(k) \xrightarrow{D} \mathrm{N}(0, 1) \quad \text{for } k \neq 0$$

$\rightsquigarrow$ bounds in the ACF plot

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$\hookrightarrow$ if $\{\boldsymbol{Y}_t\}$ is a white noise

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$\rightsquigarrow$ the same bounds

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↪ if one of $\{X_t\}$, $\{Z_t\}$ white noise and $\{X_t\}$, $\{Z_t\}$ independent $\rightsquigarrow$ the same bounds in the CCF plot

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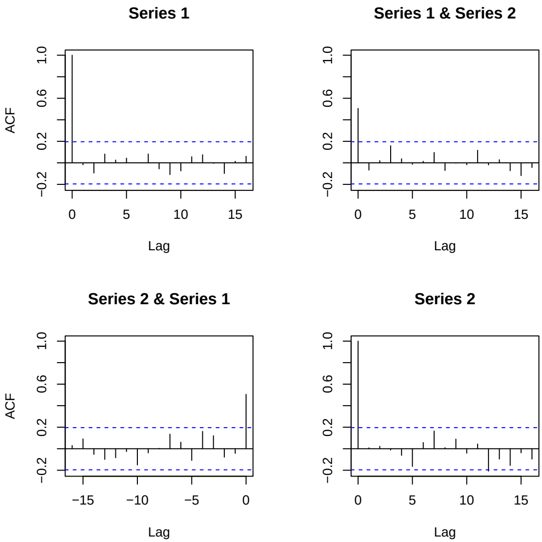
$\hookrightarrow$ if neither of $\{X_t\}$ and $\{Z_t\}$ white noise

$$\sqrt{n} \cdot r_{12}(k) \xrightarrow{D} \mathrm{N}(0, w_k),$$

where w_k depends on ACFs of $\{X_t\}$ and $\{Z_t\}$
$\rightsquigarrow$ the bounds from the CCF plot do not apply for test $\rho_{12}(k) = 0$

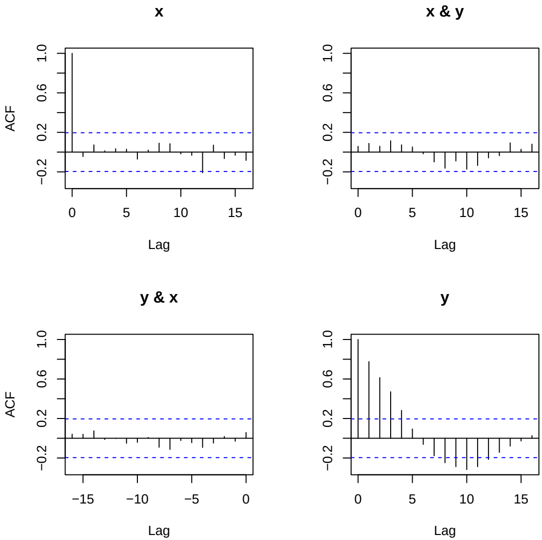
Example 1

Simulated white noise



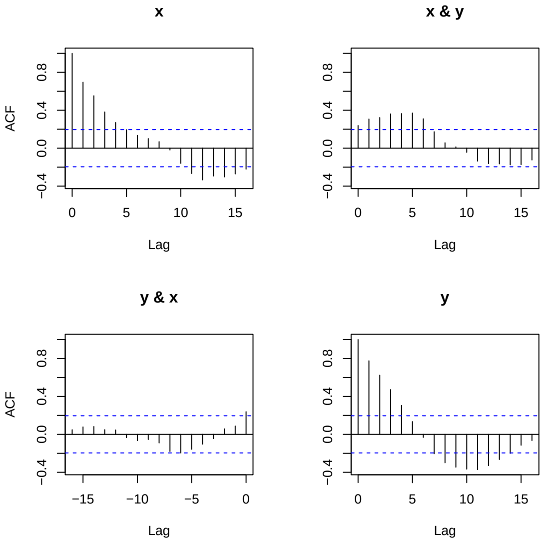
Example 2

Two simulated independent series: white noise and AR(1) series



Example 3

Two simulated independent AR(1) series



Vector Autoregression VAR

$=$ generalization of AR model for *m*-variate series

VAR(1) model

$$\boldsymbol{Y}_t = \boldsymbol{\alpha} + \boldsymbol{\Phi} \boldsymbol{Y}_{t-1} + \boldsymbol{\varepsilon}_t$$

where $\boldsymbol{\varepsilon}_t$ is *m*-variate white noise $\mathrm{WN}(\mathbf{0}, \boldsymbol{\Sigma})$

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Example: for $m = 2$

$$Y_{1,t} = \alpha_1 + \varphi_{11} Y_{1,t-1} + \varphi_{12} Y_{2,t-1} + \varepsilon_{1,t},$$
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- common model for time dynamics of $(Y_{1,t}, Y_{2,t})^\top \rightsquigarrow$ no need to distinguish the output and input

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VAR(p):

$$\boldsymbol{Y}_t = \boldsymbol{\alpha} + \boldsymbol{\Phi}_1 \boldsymbol{Y}_{t-1} + \cdots + \boldsymbol{\Phi}_p \boldsymbol{Y}_{t-p} + \boldsymbol{\varepsilon}_t$$

Properties of VAR(1)

$$\boldsymbol{Y}_t = \boldsymbol{\alpha} + \boldsymbol{\Phi} \boldsymbol{Y}_{t-1} + \varepsilon_t$$

↪ stationary if all roots of

$$\det(\boldsymbol{I} - \boldsymbol{\Phi} z) = 0$$

lie outside the unit circle
(i.e. all *m* eigenvalues of $\boldsymbol{\Phi}$ lie inside the unit circle)

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↪ under stationarity: $\mathsf{E}\boldsymbol{Y}_t = \boldsymbol{\mu}$ and

$$\mathsf{E}\boldsymbol{Y}_t = \boldsymbol{\alpha} + \boldsymbol{\Phi}\mathsf{E}\boldsymbol{Y}_{t-1} + \mathsf{E}\boldsymbol{\varepsilon}_t$$

so

$$\boldsymbol{\mu} = (\boldsymbol{I} - \boldsymbol{\Phi})^{-1} \boldsymbol{\alpha}$$

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$\hookrightarrow$ autocovariance function

$$\boldsymbol{\Gamma}_k = \mathrm{E}(\boldsymbol{Y}_{t+k} - \boldsymbol{\mu})(\boldsymbol{Y}_t - \boldsymbol{\mu})^\top = \boldsymbol{\Phi}\boldsymbol{\Gamma}_{k-1} = \boldsymbol{\Phi}^k\boldsymbol{\Gamma}_0$$

and $\boldsymbol{\Gamma}_0 = \boldsymbol{\Phi}\boldsymbol{\Gamma}_0\boldsymbol{\Phi}^\top + \boldsymbol{\Sigma}$

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↪ MA(∞) representation:

$$\boldsymbol{Y}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \boldsymbol{\Phi}\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Phi}^2\boldsymbol{\varepsilon}_{t-2} + \dots$$

Properties of VAR(1) for $m = 2$

$$Y_{1,t} = \alpha_1 + \varphi_{11} Y_{1,t-1} + \varphi_{12} Y_{2,t-1} + \varepsilon_{1,t},$$
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- ↪ so called reduced form (other forms also exist, but we do not discuss them)
- ↪ if $\varphi_{12} = \varphi_{21} = 0 \rightsquigarrow$

$$\mathbf{\Phi} = \begin{pmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{pmatrix}$$

and

$$\mathbf{\Gamma}_0 = \begin{pmatrix} \frac{\sigma_{11}}{1-\varphi_{11}^2} & \frac{\sigma_{12}}{1-\varphi_{11}\varphi_{22}} \\ \frac{\sigma_{12}}{1-\varphi_{11}\varphi_{22}} & \frac{\sigma_{22}}{1-\varphi_{22}^2} \end{pmatrix}, \quad \mathbf{\Gamma}_k = \begin{pmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{pmatrix}^k \mathbf{\Gamma}_0$$

$\rightsquigarrow$ the two series still correlated via ε_t

- ▶ only if $\varphi_{12} = \varphi_{21} = 0$ and $\mathbf{\Sigma}$ diagonal $\rightsquigarrow \gamma_{12}(k) = 0$ for all $k \rightsquigarrow$ *uncoupled* series

Individual series

For simplicity $\alpha_1 = \alpha_2 = 0$

$$
\begin{aligned}
Y_{1,t} - \varphi_{11} Y_{1,t-1} - \varphi_{12} Y_{2,t-1} = \varepsilon_{1,t}, \\
Y_{2,t} - \varphi_{21} Y_{1,t-1} - \varphi_{22} Y_{2,t-1} = \varepsilon_{2,t},
\end{aligned}
$$

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so

$$\begin{pmatrix} 1 - \varphi_{11} B & -\varphi_{12} B \\ -\varphi_{21} B & 1 - \varphi_{22} B \end{pmatrix} \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix}$$

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then

$$\begin{pmatrix} 1 - \varphi_{22} B & \varphi_{12} B \\ \varphi_{21} B & 1 - \varphi_{11} B \end{pmatrix} \cdot \begin{pmatrix} 1 - \varphi_{11} B & -\varphi_{12} B \\ -\varphi_{21} B & 1 - \varphi_{22} B \end{pmatrix} \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 - \varphi_{22} B & \varphi_{12} B \\ \varphi_{21} B & 1 - \varphi_{11} B \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix}$$

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So

$$[(1 - \varphi_{22} B)(1 - \varphi_{11} B) - \varphi_{12}\varphi_{21} B^2] Y_{1,t} = \underbrace{(1 - \varphi_{22} B)\varepsilon_{1,t} + \varphi_{12} B \varepsilon_{2,t}}_{MA(1)}$$

$\rightsquigarrow$ each series follows an ARMA(2,1)

VAR model building

1. Choice of the order p: information criteria or statistical testing

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   - $\hookrightarrow$ Least squares method
   - $\hookrightarrow$ Maximum likelihood under normality

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   - exploration of residuals (m-variate): sample ACF
   - multivariate Ljung Box portmanteau test for testing $H_0 : \boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = \cdots = \boldsymbol{\rho}_K = \mathbf{0}$ of the innovations

$$Q^2 = n^2 \sum_{l=1}^{K} \frac{1}{n-l} \boldsymbol{b}_l^\top \widehat{\mathrm{Var}}(\boldsymbol{b}_l) \boldsymbol{b}_l, \quad \boldsymbol{b}_l = \mathrm{vec}(\boldsymbol{R}_l)$$

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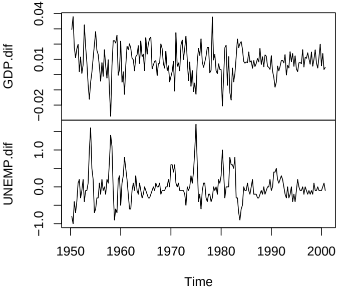
Prediction:

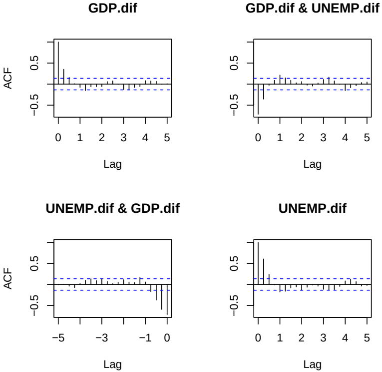
$$\widehat{\boldsymbol{Y}}_{n+1} = \widehat{\alpha} + \widehat{\boldsymbol{\Phi}}_1 \boldsymbol{Y}_n + \widehat{\boldsymbol{\Phi}}_2 \boldsymbol{Y}_{n-1} + \cdots + + \widehat{\boldsymbol{\Phi}}_p \boldsymbol{Y}_{n+1-p},$$

$$\widehat{\boldsymbol{Y}}_{n+2} = \widehat{\alpha} + \widehat{\boldsymbol{\Phi}}_1 \widehat{\boldsymbol{Y}}_{n+1} + \widehat{\boldsymbol{\Phi}}_2 \boldsymbol{Y}_n + \cdots + + \widehat{\boldsymbol{\Phi}}_p \boldsymbol{Y}_{n+2-p},$$

$$\vdots$$

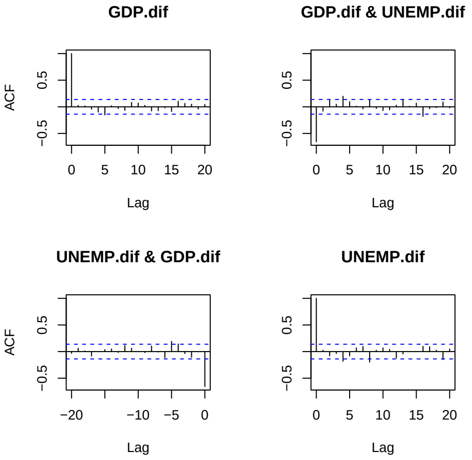
Example: Growths of GDP and UNEMP





Example: Growths of GDP and UNEMP

VAR(1): Residuals $\rightsquigarrow$ unsatisfactory fit

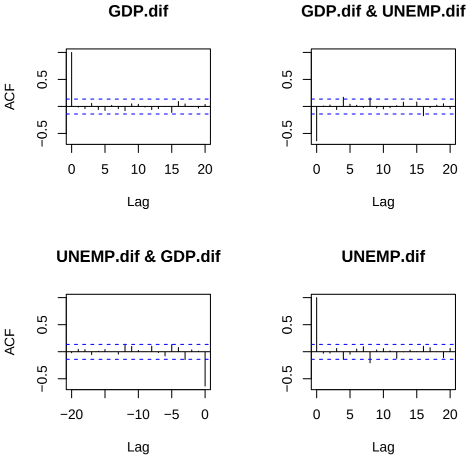


Ljung-Box Statistics:

	m	Q(m)	df	p-value
[1,]	1.00	1.85	0.00	1.00
[2,]	2.00	13.63	4.00	0.01
[3,]	3.00	17.20	8.00	0.03
[4,]	4.00	27.84	12.00	0.01
[5,]	5.00	36.75	16.00	0.00
[6,]	6.00	40.84	20.00	0.00
[7,]	7.00	44.57	24.00	0.01
[8,]	8.00	52.89	28.00	0.00

Example: Growths of GDP and UNEMP

VAR(2): acceptable, but try also a larger *p* (quarterly data)




```
Ljung-Box Statistics:
            m       Q(m)      df      p-value
[1,]    1.000      0.190  -4.000       1.00
[2,]    2.000      0.558   0.000       1.00
[3,]    3.000      6.469   4.000       0.17
[4,]    4.000     14.166   8.000       0.08
[5,]    5.000     18.696  12.000       0.10
[6,]    6.000     21.012  16.000       0.18
[7,]    7.000     24.638  20.000       0.22
[8,]    8.000     33.890  24.000       0.09
```

```
Roots of the characteristic polynomial:
0.5936 0.5936 0.3108 0.3108
```

Fitted model

```
Estimation results for equation GDP.dif:
========================================
GDP.dif = GDP.dif.l1 + UNEMP.dif.l1 + GDP.dif.l2 + UNEMP.dif.l2 + const

              Estimate Std. Error t value Pr(>|t|)
GDP.dif.l1    0.148468   0.090643   1.638 0.103038
UNEMP.dif.l1 -0.008893   0.002787  -3.191 0.001649 **
GDP.dif.l2    0.136568   0.093774   1.456 0.146894
UNEMP.dif.l2  0.008433   0.002392   3.525 0.000528 ***
const         0.005968   0.001299   4.594 7.78e-06 ***
---

Estimation results for equation UNEMP.dif:
==========================================
UNEMP.dif = GDP.dif.l1 + UNEMP.dif.l1 + GDP.dif.l2 + UNEMP.dif.l2 + const

              Estimate Std. Error t value Pr(>|t|)
GDP.dif.l1   -12.31685    2.97799  -4.136 5.24e-05 ***
UNEMP.dif.l1   0.40792    0.09155   4.456 1.40e-05 ***
GDP.dif.l2    -8.34704    3.08086  -2.709 0.007339 **
UNEMP.dif.l2  -0.26620    0.07860  -3.387 0.000855 ***
const          0.17476    0.04268   4.095 6.18e-05 ***
```

Notes to VAR model

- $\hookrightarrow$ a richer structure than univariate processes AR
- $\hookrightarrow$ if m large $\rightsquigarrow$ large number of parameters to be estimated
- $\hookrightarrow$ many further topics (we do not discuss):
  - causality testing
  - cointegration
  - ...