EE 261 The Fourier Transform and its Applications
Fall 2007
Solutions to Problem Set Three

1. (5 points) *Still another reciprocal relationship* The *equivalent width* of a signal $f(t)$, with $f(0) \neq 0$, is the width of a rectangle having height $f(0)$ and area the same as under the graph of $f(t)$. Thus

$$W_f = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t)\,dt\,.$$

This is a measure for how spread out a signal is.

Show that $W_f W_{\mathcal{F}f} = 1$. Thus, the equivalent widths of a signal and its Fourier transform are reciprocal.

Solution: The things we need to note are

$$\mathcal{F}f(0) = \int_{-\infty}^{\infty} e^{-2\pi i 0 \cdot t} f(t)\,dt = \int_{-\infty}^{\infty} f(t)\,dt\,,$$

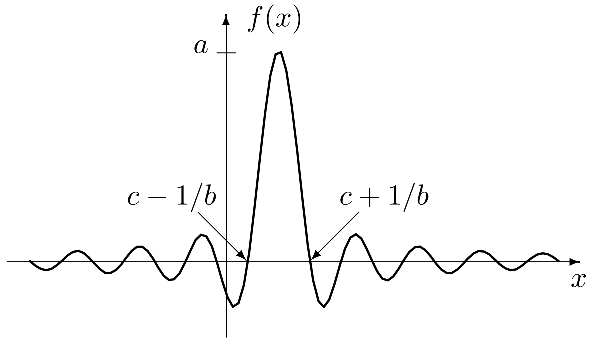
and the corresponding statement using the inverse Fourier transform,

$$f(0) = \int_{-\infty}^{\infty} e^{2\pi i s \cdot 0} \mathcal{F}f(s)\,ds = \int_{-\infty}^{\infty} \mathcal{F}f(s)\,ds\,.$$

Then we compute

$$\begin{aligned}
W_f &= \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t)\,dt \\
&= \frac{\mathcal{F}f(0)}{f(0)} \\
W_{\mathcal{F}f} &= \frac{1}{\mathcal{F}f(0)} \int_{-\infty}^{\infty} \mathcal{F}f(s)\,ds \\
&= \frac{f(0)}{\mathcal{F}f(0)} \\
\Rightarrow W_f W_{\mathcal{F}f} &= \frac{\mathcal{F}f(0)}{f(0)} \frac{f(0)}{\mathcal{F}f(0)} \\
\Rightarrow W_f W_{\mathcal{F}f} &= 1
\end{aligned}$$

2. (10 points) Find the Fourier transforms of the function shown in the graph (a shifted sinc)



Solution:

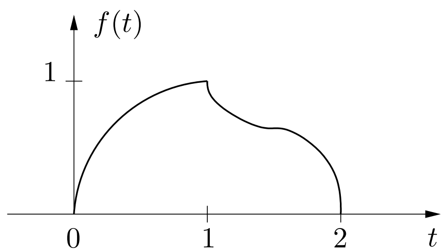
The sinc function is centered at c, stretched by $\frac{1}{b}$, scaled by a. Thus,

$$f(x) = a\ \mathrm{sinc}[b(x-c)].$$

Since $\mathcal{F}h[b(x-c)] = \frac{1}{|b|}e^{-2\pi ics}\hat{h}(\frac{s}{b})$,

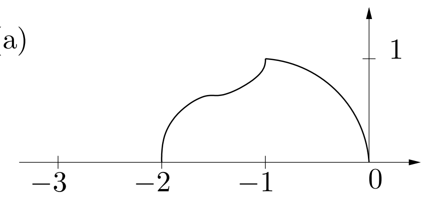
$$\hat{f}(s) = \frac{a}{|b|}e^{-2\pi ics}\Pi(\frac{s}{b})$$

3. (5 points each) The figures below show a signal $f(t)$ and six other signals derived from $f(t)$. Note the scales on the axes.

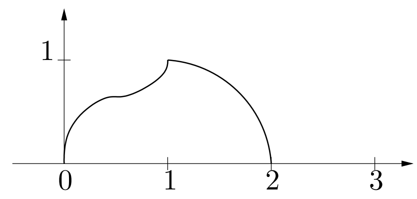


Suppose $f(t)$ has Fourier transform $F(s)$. Express the Fourier transforms of the other six signals in terms of $F(s)$.

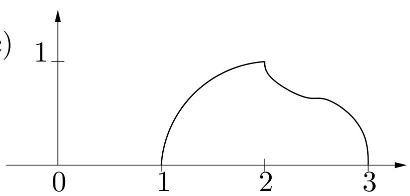
(a)



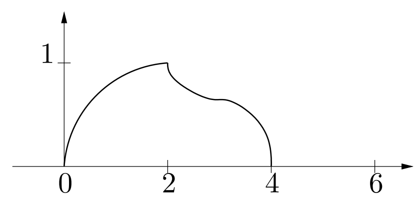
(b)



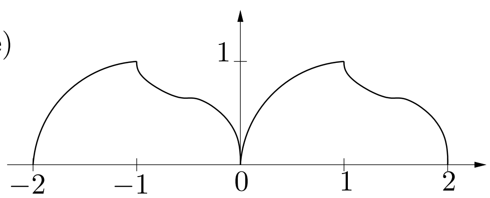
(c)



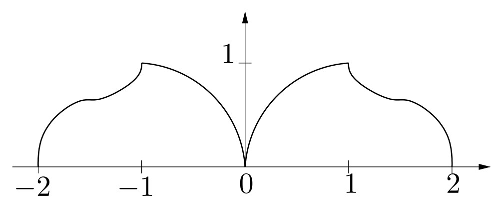
(d)



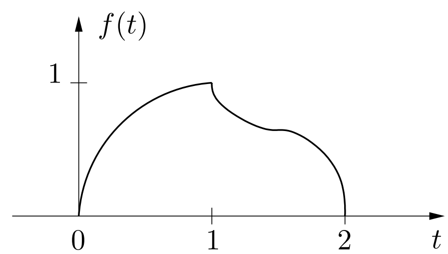
(e)



(f)



Solution:
The signal $f(t)$ is shown below. Denote its Fourier transform by $F(s)$.



(a) The first modified signal is given by $f(-t)$, or, without writing the variable, f^-. Its Fourier transform is F^-, or, writing the variable, $F(-s)$.

(b) To express next one in terms of f we first reverse f to f^- (which gives the preceding picture) and then shift to the right by 2, giving $f^-(t-2)$. The Fourier transform of this is $e^{-2\pi i 2s}\mathcal{F}(f^-)(s) = e^{-4\pi is}(\mathcal{F}f)^-(s) = e^{-4\pi is}F(-s)$.

(c) The next signal is just $f(t-1)$, and the Fourier transform is $e^{-2\pi is}F(s)$.

(d) The next one is twice as large in the horizontal direction, which is accomplished by forming $f(t/2)$. The Fourier transform is $2F(2s)$.

(e) This is given by $f(t) + f(t+2)$ so its Fourier transform is $F(s) + e^{4\pi is}F(s) = (1 + e^{4\pi is})F(s)$.

(f) Finally, we have two signals in opposite directions. That's just $f(t) + f(-t)$ so the Fourier transform is $F(s) + F(-s)$.

4. (35 points) *Some practice with convolution*
In this problem, we want you to have some practice with handling convolution and integration. So, for parts (a) to (c), explicitly evaluate the convolution integral. Also verify your results by applying the convolution theorem for Fourier transforms.

(a) What is $\Pi_a * \Pi_a$?

(b) Let $f(x) = e^{-|x|}$, $-\infty < x < \infty$. Find $(f * f)(x)$.

(c) Let $g(x) = e^{-\pi x^2}$, $-\infty < x < \infty$. Show that $(g * g)(x) = \frac{1}{\sqrt{2}} e^{-\pi x^2/2}$.

(d) From the result in part(c), deduce the result of the n-fold convolution of g, i.e., $g*g*...*g$ (with n factors of g).

Solution:

(a) This convolution is:

$$(\Pi_a * \Pi_a)(x) = \int_{-\infty}^{\infty} \Pi_a(y)\Pi_a(x-y)\,dy,$$

but since Π_a is zero outside the interval $(-a/2, a/2)$, we can expect the integration to be finite for any given x. We have to split this up into cases.

Case 1: $x < -a$ or $x > a$

In this case, we claim that $\Pi_a(y)\Pi_a(x-y) = 0$ for all y since the rectangular functions are non-overlapping.

Case 2: $-a \le x \le 0$

$$\begin{aligned}
(\Pi_a * \Pi_a)(x) &= \int_{-\infty}^{\infty} \Pi_a(y)\Pi_a(x-y)\,dy \\
&= \int_{-a/2}^{a/2} \Pi_a(x-y)\,dy \\
&= \int_{x-a/2}^{x+a/2} \Pi_a(u)\,du \qquad \text{(substituting } u = x-y\text{)} \\
&= \int_{-a/2}^{x+a/2} 1\,du \qquad \text{(since } x < 0 \text{ and } \Pi_a(u) \text{ is zero for } u < -a/2\text{)} \\
&= x + a.
\end{aligned}$$

Case 3: $0 \le x \le a$

This is handled very similarly to the previous case:

$$\begin{aligned}
(\Pi_a * \Pi_a)(x) &= \int_{-\infty}^{\infty} \Pi_a(y)\Pi_a(x-y)\,dy \\
&= \int_{-a/2}^{a/2} \Pi_a(x-y)\,dy \\
&= \int_{x-a/2}^{x+a/2} \Pi_a(u)\,du \qquad \text{(substituting } u = x-y\text{)} \\
&= \int_{x-a/2}^{a/2} 1\,du \qquad \text{(since } x > 0 \text{ and } \Pi_a(u) \text{ is zero for } u > a/2\text{)} \\
&= a - x.
\end{aligned}$$

Combining the cases we see that:

$$(\Pi_a * \Pi_a)(x) = \begin{cases} x+a, & -a \leq x \leq 0 \\ a-x, & 0 \leq x \leq a \\ 0, & \text{otherwise.} \end{cases}$$

We notice that this is the definition of $a\Lambda_a(x) = a\Lambda(x/a)$.
To verify the result using the convolution property of the Fourier transform, we take the Fourier transform of $(\Pi_a * \Pi_a)(x)$, which is $a^2\text{sinc}^2(as)$. This can be written as $a\,[a\,\text{sinc}(as)]$. Taking the inverse Fourier transform of this will yield our previous result of $a\Lambda_a(x)$.

(b) The convolution $(f * f)(x)$ is

$$\begin{aligned}(f*f)(x) &= \int_{-\infty}^{\infty} f(y)f(x-y)\,dy \\ &= \int_{-\infty}^{\infty} e^{-|y|}e^{-|x-y|}\,dy \\ &= \int_{-\infty}^{\infty} e^{-|y|}e^{-|y-x|}\,dy.\end{aligned}$$

Since the functions of interest are even, we only need to consider the case of $x < 0$. The result for $x > 0$ can be obtained using symmetry arguments. So, for the case where $x < 0$,

$$\begin{aligned}(f*f)(x) &= \int_{-\infty}^{\infty} e^{-|y|}e^{-|y-x|}\,dy \\ &= \int_{-\infty}^{x} e^{y}e^{y-x}\,dy + \int_{x}^{0} e^{y}e^{-y+x}\,dy + \int_{0}^{\infty} e^{-y}e^{-y+x}\,dy \\ &= \frac{e^x}{2} - xe^x + \frac{e^x}{2} \\ &= (1-x)e^x\end{aligned}$$

For $x \geq 0$,

$$(f*f)(x) = (1+x)e^{-x}$$

by symmetry. Combining these results, we get

$$(f*f)(x) = (1+|x|)e^{-|x|}.$$

(c) The convolution $(g * g)(x)$ is

$$\begin{aligned}(g*g)(x) &= \int_{-\infty}^{\infty} g(y)g(x-y)\,dy \\ &= \int_{-\infty}^{\infty} e^{-\pi y^2}e^{-\pi(x-y)^2}\,dy \\ &= \frac{1}{\sqrt{2}}e^{-\pi x^2/2} \qquad \text{(by integration by parts)}\end{aligned}$$

To verify the result using the convolution property of the Fourier transform, we take the Fourier transform of $(g * g)(x)$, which is $e^{-\pi s^2}.e^{-\pi s^2} = e^{-2\pi s^2}$. Taking the inverse Fourier transform of this will yield our previous result of $\frac{1}{\sqrt{2}}e^{-\pi x^2/2}$.

(d) It should be obvious that the n-fold convolution of $g(x)$ will yield $\frac{1}{\sqrt{n}}e^{-\pi x^2/n}$. We can show this using the Fourier transform. The Fourier transform of $g * g * ... * g$ is $e^{-n\pi s^2}$. Taking the inverse Fourier transform would be $\frac{1}{\sqrt{n}}e^{-\pi x^2/n}$.

5. (30 points) *Convolution: Reversals, Shifts and Stretches* Let $f(t)$ and $g(t)$ be signals.

   (a) If both $f(t)$ and $g(t)$ are reversed, what happens to their convolution? If one of $f(t)$ and $g(t)$ is reversed what happens to their convolution?

Define the shift operator $\tau_b f$ and the stretch operator $\sigma_a f$ by

$$\tau_b f(t) = f(t-b)\,, \quad \sigma_a f(t) = f(at)\,.$$

(b) Show that

$$(\tau_b f) * g = \tau_b(f * g) = f * (\tau_b g)\,.$$

Write out in words what this says. Use this result to deduce that if either f or g is periodic of period T then $f * g$ is periodic of period T. (However, see Problem 6 below!)

(c) Show that

$$(\sigma_a f) * g = \frac{1}{|a|}\sigma_a(f * (\sigma_{1/a} g))\,, \quad (\sigma_a f) * (\sigma_a g) = \frac{1}{|a|}\sigma_a(f * g)\,.$$

Write out in words what these identities say.

Solution:

(a) If both signals are reversed we have

$$\begin{aligned}
(f^- * g^-)(x) &= \int_{-\infty}^{\infty} f^-(y)g^-(x-y)\,dy \\
&= \int_{-\infty}^{\infty} f(-y)g(-(x-y))\,dy = \int_{-\infty}^{\infty} f(-y)g(-x+y)\,dy \\
&= \int_{-\infty}^{\infty} f(u)g(-x-u)\,du \quad \text{(substituting } u = -y\text{)} \\
&= (f * g)(-x) \\
&= (f * g)^-(x)
\end{aligned}$$

Thus we conclude that

$$f^- * g^- = (f * g)^-\,.$$

Suppose now that just one signal is reversed, say f. Then

$$\begin{aligned}
(f^- * g)(x) &= \int_{-\infty}^{\infty} f^-(y)g(x-y)\,dy \\
&= \int_{-\infty}^{\infty} f(-y)g(x-y)\,dy \\
&= \int_{-\infty}^{\infty} f(u)g(x+u)\,du \quad \text{(substituting } u=-y\text{)} \\
&= \int_{-\infty}^{\infty} f(u)g(-(-x-u))\,du \\
&= \int_{-\infty}^{\infty} f(u)g^-(-x-u)\,du \\
&= (f * g^-)(-x)
\end{aligned}$$

We can write this without the variable as

$$f^- * g = (f * g^-)^- .$$

Similarly

$$f * g^- = (f^- * g)^- .$$

Note how the earlier formula follows from either of these. Say we establish $f^- * g = (f * g^-)^-$. Then

$$f^- * g^- = (f * (g^-)^-)^- = (f * g)^- .$$

(b) We compute

$$\begin{aligned}
((\tau_b f) * g)(x) &= \int_{-\infty}^{\infty} \tau_b f(y)g(x-y)\,dy \\
&= \int_{-\infty}^{\infty} f(y-b)g(x-y)\,dy \\
&= \int_{-\infty}^{\infty} f(u)g(x-(u+b))\,dy \quad \text{(substituting } u=y-b\text{)} \\
&= \int_{-\infty}^{\infty} f(u)g(x-b-u)\,dy \\
&= \int_{-\infty}^{\infty} f(u)(\tau_b g)(x-u)\,du \\
&= (f * (\tau_b g))(x)
\end{aligned}$$

This establishes one of the identities, that

$$(\tau_b f) * g = f * (\tau_b g) ,$$

But notice that we can also write the penultimate integral above as

$$\int_{-\infty}^{\infty} f(u)g(x-b-u)\,dy = (f * g)(x-b) ,$$

and so

$$(\tau_b f) * g = \tau_b (f * g)$$

as well.

In words, the identities $(\tau_b f) * g = \tau_b(f * g) = f * (\tau_b g)$ say that a delay of b in either signal f or g also delays their convolution by b.

Now suppose that $f(t)$ is periodic of period T. Then $\tau_T f = f$, i.e., a delay by T doesn't change the signal. For the convolution of f with any g we then have

$$\tau_T(f * g) = (\tau_T f) * g = f * g\,.$$

But this says that the convolution is also unchanged if delayed by T, and so $f * g$ is periodic of period T. We could draw the same conclusion if we started with g being periodic. However, in anticipation of addressing the issue raised in the 'Conversation on convolution' all of this holds *provided* the integrals defining the convolutions converge

(c) For the first identity, $(\sigma_a f) * g = 1/|a|\sigma_a(f * (\sigma_{1/a} g))$, we compute:

$$\begin{aligned}
((\sigma_a f) * g)(x) &= \int_{-\infty}^{\infty} (\sigma_a f)(y) g(x-y)\,dy \\
&= \int_{-\infty}^{\infty} f(ay) g(x-y)\,dy \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(u) g(x-\frac{u}{a})\,du \quad \text{(substituting } u = ay) \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(u) g(\frac{1}{a}(ax-u))\,du \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)(\sigma_{1/a} g)(ax-u)\,du \\
&= \frac{1}{|a|}(f * (\sigma_{1/a} g))(ax) \\
&= \frac{1}{|a|}\sigma_a(f * (\sigma_{1/a} g))(x)\,.
\end{aligned}$$

The second identity, $(\sigma_a f) * (\sigma_a g) = (1/|a|)\sigma_a(f * g)$, actually follows from this. For according to the first identity

$$(\sigma_a f) * (\sigma_a g) = \frac{1}{|a|}\sigma_a(f * \sigma_{1/a}\sigma_a g)\,,$$

and

$$\sigma_{1/a}\sigma_a g = g\,.$$

In words, the second identity (a little easier) says that if each of $f(t)$ and $g(t)$ are operated on by σ_a, so scaled to $f(at)$ and $g(at)$, then their convolution is also operated on by σ_a and multiplied by $1/|a|$. The first identity says that if σ_a operates on f then in convolution with g the operation becomes $\sigma 1/a$ operating on g, and the whole convolution is multiplied by $1/|a|$.

6. (10 points) Rajiv and Lykomidis are arguing about convolution over dinner one night:

Rajiv: You know, convolution really is a remarkable operation, the way it imparts properties of one function onto the convolution with another. Take periodicity – if $f(t)$ is periodic then $(f * g)(t)$ is periodic with the same period. I think that was a homework problem we gave the class.

Lykomidis: There's a problem with that statement. You want to say that if $f(t)$ is a periodic function of period T then $(f * g)(t)$ is also periodic of period T.

Rajiv: Right.

Lykomidis: What if $g(t)$ is also periodic, say of period R? Then doesn't $(f * g)(t)$ have two periods, T and R?

Rajiv: I suppose so.

Lykomidis: But wouldn't this lead right to a contradiction? I mean, for example, you can't have a function with two periods, can you?

The conversation continues. They are joined by Thomas:

Rajiv and Lykomidis: We think we've found a fundamental contradiction in mathematics.

Thomas: Why don't you look at a simple, special case first. What happens if you convolve $\sin 2\pi t$ with itself?

Rajiv: OK, both functions have period 1 so for the convolution you get a function that's periodic of period 1, no problem.

Lykomidis: No, you don't. Something goes wrong.

What's going on? With whom do you agree and why?

Solution:
The problem is that the integral defining the convolution of two periodic functions won't converge (maybe it will for some values, though I doubt it, but certainly not in general). Taking $f(t) = \sin 2\pi t$ as an example, the convolution is

$$(f * f)(t) = \int_{-\infty}^{\infty} \sin 2\pi\tau \sin 2\pi(t-\tau)\, d\tau\,.$$

Let $t = 1$. Then because of periodicity

$$(f * f)(1) = \int_{-\infty}^{\infty} \sin 2\pi\tau \sin 2\pi(1-\tau)\, d\tau = \int_{-\infty}^{\infty} \sin 2\pi\tau(-\sin 2\pi\tau)\, d\tau = -\int_{-\infty}^{\infty} \sin^2 2\pi\tau\, d\tau\,,$$

and this integral is infinite.

7. (25 points)
Probability Distributions, Convolution and MATLAB. You have three six-sided dice, one white, one red and one black. The white one is fair but the red and the black are not, in particular:

	Prob (1)	Prob (2)	Prob (3)	Prob (4)	Prob (5)	Prob (6)
white	1/6	1/6	1/6	1/6	1/6	1/6
red	1/12	1/12	2/12	2/12	3/12	3/12
black	4/12	3/12	2/12	1/12	1/12	1/12

(a) (10 points) You roll each die once. What is the probability that the sum of all the three rolls is equal to 7 or 8 or 12. (Hint: You may want to use matlab)

(b) (5 points) You repeat rolling all three dice many times. (Each time your roll each die once.) Let X_i, Y_i and Z_i be the random variable denoting the i-th roll of the white, red and black die correspondingly. As the number of trials N increases, what do you think the empirical average of the numbers

$$\frac{1}{3N}\sum_{i=1}^{N}(X_i + Y_i + Z_i)$$

should be.

(c) (10 points) Use MATLAB to plot the distribution of the empirical average, for $N = 2, 10, 100$ and 1000.

Solution:

(a) We know from class that if you add two independent random variables X and Y ($Z = X + Y$) whose probability density functions are p_x and p_y, then you get a new random variable whose probability density function is equal to $p_x * p_y = \int_{-\infty}^{\infty} p_x(z-u)p_y(u)du$. If you add two independent *discrete* random variables X and Y, you get a similar result except that instead of integrating you sum:

$$\begin{aligned}\text{Prob}\{X+Y=z\} &= \sum_u \text{Prob}\{X=z-u\}\,\text{Prob}\{X_2=u\} \\ &= \sum_u p_x(z-u)p_y(u).\end{aligned}$$

See the section on the Central Limit Theorem in the course reader for a discussion of this and the connection with convolution. In fact, since MATLAB works with vectors and not with continuous functions, the convolution command **conv** actually gives you the summation described above. This is perfect since it's exactly what we need to calculate the distribution of the sum of the rolls from the three dice.
A sample MATLAB code for finding this purpose is:

```
white_die = [ 1/6  1/6  1/6  1/6  1/6  1/6 ];
red_die   = [ 1/12 1/12 2/12 2/12 3/12 3/12];
black_die = [ 4/12 3/12 2/12 1/12 1/12 1/12];
% Observe that the sum of the three dice must be between 3 and 18
dice_sum     = 3:18
pr_dice_sum  = conv(white_die,red_die);
pr_dice_sum  = conv(pr_dice_sum,black_die);

disp('Prob(three dice sum equal to 7 or 8 or 12)');
pr_dice_sum( find(dice_sum==7) )+pr_dice_sum( find(dice_sum==8) )+pr_dice_sum( find(dice_sum==12) )
```

The desired probability turns out to be 0.2998.

(b) If you take enough trials, the sample average will converge to the mean value of a random roll of the die,

```
sum(dice_sum.*pr_dice_sum)/3
```

that is 3.4167. We did not ask for any sort of proof, and we won't give any, other than the claim that "probability theory works".

(c) A sample MATLAB code for finding the distribution for N rolls (in this case $N = 10$) is:

```
average = 1;
N = 2;
for i=1:N
  average = conv(average,pr_dice_sum);
end
xaxis = [0:length(average)-1] + 3*N;
xaxis = xaxis/(3*N);

figure();
set(gca, 'Linewidth',1.6);
set(gca, 'FontSize',18);
bar(xaxis, average);
axis([1 6 -inf inf]);
title('Distribution of the Empirical Average, N=2');
print('-depsc','EmpAverage2.eps');
```

The MATLAB plots themselves are below. It's technically not correct, but no points should be taken off if they used the **plot** command instead of the **bar** command (the difference being that **plot** connects the points and **bar** leaves them separate).