The theorem of Lyndon and Schützenberger

Theorem. If $x^n y^m = z^p$, with $x, y, z \in \Sigma^+$ and $n, m, p \geq 2$, then the word x, y a z commute.

Proof. By symmetry, assume $|x^n| \geq |y^m|$.

The word x^n has periods $|x|$ a $|z|$. If $|x^n| \geq |z| + |x|$, then the Periodicity lemma implies that x and z have a period dividing $|x|$ a $|z|$, which easily yields that they commute. Similarly if $|y^m| \geq |z| + |y|$.

Suppose therefore that x^{n-1} is a proper prefix of z and y^{m-1} a proper suffix of z. Then $|x^n| < 2|z|$ and $|y^m| < 2|z|$, hence $p < 4$.

Let $p = 3$. If $n \geq 3$, then $|x^2| < |z|$ implies $|x^3| < \frac{3}{2}|z|$, contradicting the assumption $|x^n| \geq |y^m|$. Therefore $n = 2$ and $|x| > |y|$. There are words u, v, w such that $x = uw = wv$, $z = xu = wvu$ and $y^m = vuwvu$. The word $uwv = xv = ux$ has periods $|u|$ and $|y|$. Note that $|uwv| = |u| + |x| > |u| + |y|$ holds. By the Periodicity lemma, the word uwv has a period d dividing both $|u|$ and $|y|$. Therefore u and wv commute, and also z has a period d. Hence, both y and z are powers of their common suffix of length d, which yields the claim.

The case $p = 2$ remains. We have $z = x^{n-1}u = wy^m$, where $uw = x$. Then $wz = (wu)^n = w^2 y^m$, where wu is shorter than z. The claim clearly holds if $|z| = 1$ and the proof is completed by induction. $\square$