ON THE EQUILIBRIUM OF PLANES

OR

THE CENTRES OF GRAVITY OF PLANES.

BOOK I.

"I POSTULATE the following:

1. Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance.

2. If, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium but incline towards that weight to which the addition was made.

3. Similarly, if anything be taken away from one of the weights, they are not in equilibrium but incline towards the weight from which nothing was taken.

4. When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide.

5. In figures which are unequal but similar the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

6. If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances.

7. In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure."

Proposition 1.

Weights which balance at equal distances are equal.

For, if they are unequal, take away from the greater the difference between the two. The remainders will then not balance [*Post.* 3]; which is absurd.

Therefore the weights cannot be unequal.

Proposition 2.

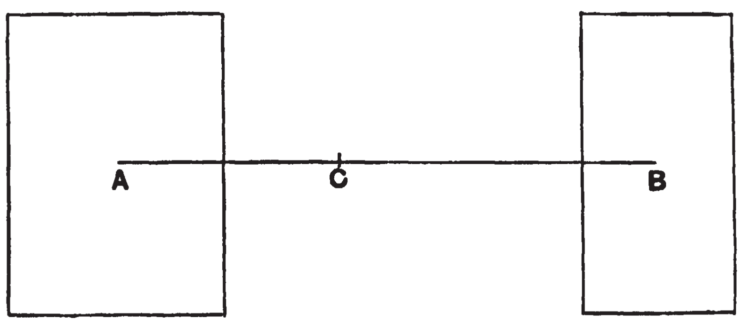
Unequal weights at equal distances will not balance but will incline towards the greater weight.

For take away from the greater the difference between the two. The equal remainders will therefore balance [*Post.* 1]. Hence, if we add the difference again, the weights will not balance but incline towards the greater [*Post.* 2].

Proposition 3.

Unequal weights will balance at unequal distances, the greater weight being at the lesser distance.

Let A, B be two unequal weights (of which A is the greater) balancing about C at distances AC, BC respectively.



Then shall AC be less than BC. For, if not, take away from A the weight $(A-B.)$ The remainders will then incline towards B [*Post.* 3]. But this is impossible, for (1) if $AC = CB$, the equal remainders will balance, or (2) if $AC > CB$, they will incline towards A at the greater distance [*Post.* 1].

Hence $AC < CB$.

Conversely, if the weights balance, and $AC < CB$, then $A > B$.

Proposition 4.

If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity.

[Proved from Prop. 3 by *reductio ad absurdum.* Archimedes assumes that the centre of gravity of both together is on the straight line joining the centres of gravity of each, saying that this had been proved before (προδέδεικται). The allusion is no doubt to the lost treatise *On levers* (περὶ ζυγῶν).]

Proposition 5.

If three equal magnitudes have their centres of gravity on a straight line at equal distances, the centre of gravity of the system will coincide with that of the middle magnitude.

[This follows immediately from Prop. 4.]

COR 1. *The same is true of any odd number of magnitudes if those which are at equal distances from the middle one are equal, while the distances between their centres of gravity are equal.*

COR. 2. *If there be an even number of magnitudes with their centres of gravity situated at equal distances on one straight line, and if the two middle ones be equal, while those which are equidistant from them (on each side) are equal respectively, the centre of gravity of the system is the middle point of the line joining the centres of gravity of the two middle ones.*

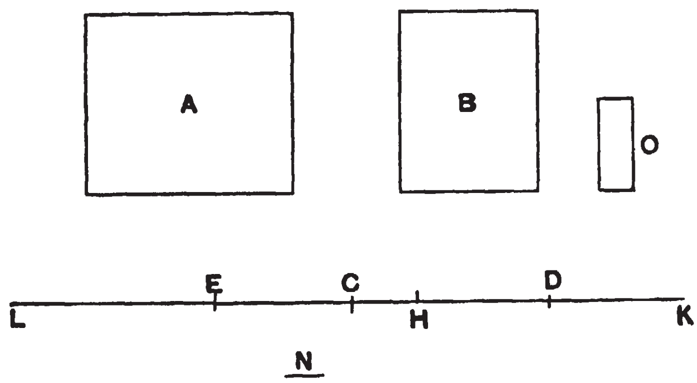
Propositions 6, 7.

Two magnitudes, whether commensurable [Prop. 6] *or incommensurable* [Prop. 7], *balance at distances reciprocally proportional to the magnitudes.*

I. Suppose the magnitudes A, B to be commensurable, and the points A, B to be their centres of gravity. Let DE be a straight line so divided at C that

$$A : B = DC : CE.$$

We have then to prove that, if A be placed at E and B at D, C is the centre of gravity of the two taken together.



Since A, B are commensurable, so are DC, CE. Let N be a common measure of DC, CE. Make DH, DK each equal to CE, and EL (on CE produced) equal to CD. Then $EH = CD$, since $DH = CE$. Therefore LH is bisected at E, as HK is bisected at D.

Thus LH, HK must each contain N an even number of times.

Take a magnitude O such that O is contained as many times in A as N is contained in LH, whence

$$A : O = LH : N.$$

But

$$B : A = CE : DC$$
$$= HK : LH.$$

Hence, *ex aequali*, $B : O = HK : N$, or O is contained in B as many times as N is contained in HK.

Thus O is a common measure of A, B.

Divide LH, HK into parts each equal to N, and A, B into parts each equal to O. The parts of A will therefore be equal in number to those of LH, and the parts of B equal in number to those of HK. Place one of the parts of A at the middle point of each of the parts N of LH, and one of the parts of B at the middle point of each of the parts N of HK.

Then the centre of gravity of the parts of A placed at equal distances on LH will be at E, the middle point of LH [Prop. 5, Cor. 2], and the centre of gravity of the parts of B placed at equal distances along HK will be at D, the middle point of HK.

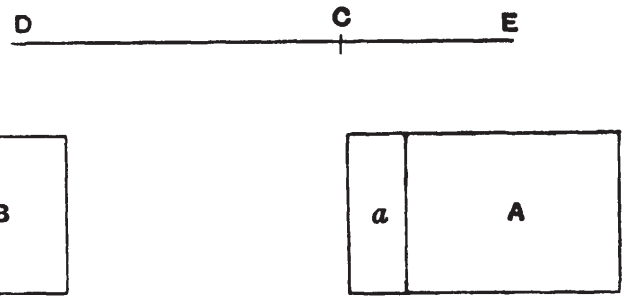
Thus we may suppose A itself applied at E, and B itself applied at D.

But the system formed by the parts O of A and B together is a system of equal magnitudes even in number and placed at equal distances along LK. And, since $LE = CD$, and $EC = DK$, $LC = CK$, so that C is the middle point of LK. Therefore C is the centre of gravity of the system ranged along LK.

Therefore A acting at E and B acting at D balance about the point C.

II. Suppose the magnitudes to be incommensurable, and let them be $(A+a)$ and B respectively. Let DE be a line divided at C so that

$$(A+a) : B = DC : CE.$$



Then, if $(A+a)$ placed at E and B placed at D do not balance about C, $(A+a)$ is either too great to balance B, or not great enough.

Suppose, if possible, that $(A+a)$ is too great to balance B. Take from $(A+a)$ a magnitude a smaller than the deduction which would make the remainder balance B, but such that the remainder A and the magnitude B are commensurable.

Then, since A, B are commensurable, and

$$A : B < DC : CE,$$

A and B will not balance [Prop. 6], but D will be depressed.

But this is impossible, since the deduction a was an insufficient deduction from $(A+a)$ to produce equilibrium, so that E was still depressed.

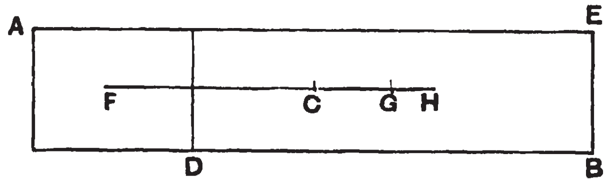
Therefore $(A+a)$ is not too great to balance B; and similarly it may be proved that B is not too great to balance $(A+a)$.

Hence $(A+a)$, B taken together have their centre of gravity at C.

Proposition 8.

If AB be a magnitude whose centre of gravity is C, and AD a part of it whose centre of gravity is F, then the centre of gravity of the remaining part will be a point G on FC produced such that

$$GC : CF = (AD) : (DE).$$



For, if the centre of gravity of the remainder (DE) be not G, let it be a point H. Then an absurdity follows at once from Props. 6, 7.

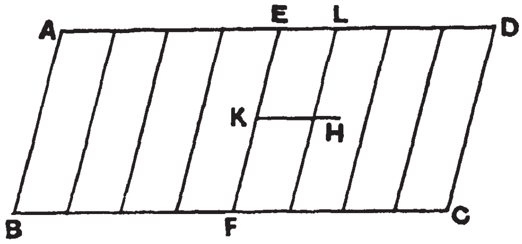
Proposition 9.

The centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides.

Let $ABCD$ be a parallelogram, and let EF join the middle points of the opposite sides AD, BC.

If the centre of gravity does not lie on EF, suppose it to be H, and draw HK parallel to AD or BC meeting EF in K.

Then it is possible, by bisecting *ED*, then bisecting the halves, and so on continually, to arrive at a length *EL* less



than *KH*. Divide both *AE* and *ED* into parts each equal to *EL*, and through the points of division draw parallels to *AB* or *CD*.

We have then a number of equal and similar parallelograms, and, if any one be applied to any other, their centres of gravity coincide [*Post.* 4]. Thus we have an even number of equal magnitudes whose centres of gravity lie at equal distances along a straight line. Hence the centre of gravity of the whole parallelogram will lie on the line joining the centres of gravity of the two middle parallelograms [Prop. 5, Cor. 2].

But this is impossible, for *H* is outside the middle parallelograms.

Therefore the centre of gravity cannot but lie on *EF*.

Proposition 10.

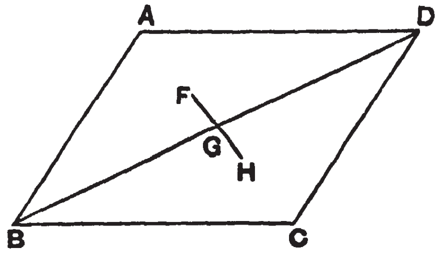
The centre of gravity of a parallelogram is the point of intersection of its diagonals.

For, by the last proposition, the centre of gravity lies on each of the lines which bisect opposite sides. Therefore it is at the point of their intersection; and this is also the point of intersection of the diagonals.

Alternative proof.

Let *ABCD* be the given parallelogram, and *BD* a diagonal. Then the triangles *ABD*, *CDB* are equal and similar, so that [*Post.* 4], if one be applied to the other, their centres of gravity will fall one upon the other.

Suppose F to be the centre of gravity of the triangle ABD. Let G be the middle point of BD. Join FG and produce it to H, so that $FG = GH$.



If we then apply the triangle ABD to the triangle CDB so that AD falls on CB and AB on CD, the point F will fall on H.

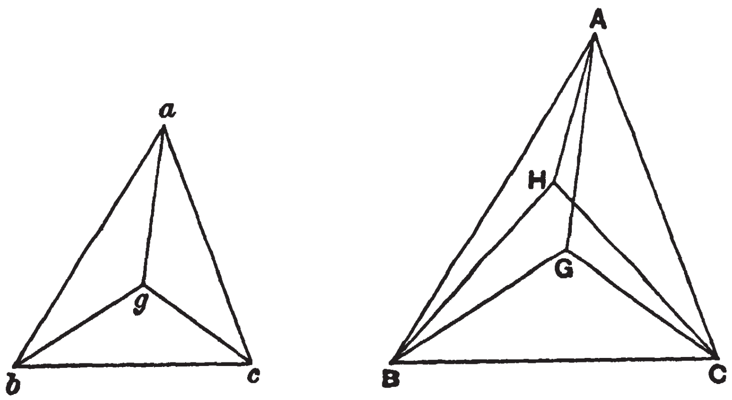
But [by *Post.* 4] F will fall on the centre of gravity of CDB. Therefore H is the centre of gravity of CDB.

Hence, since F, H are the centres of gravity of the two equal triangles, the centre of gravity of the whole parallelogram is at the middle point of FH, i.e. at the middle point of BD, which is the intersection of the two diagonals.

Proposition 11.

If abc, ABC be two similar triangles, and g, G two points in them similarly situated with respect to them respectively, then, if g be the centre of gravity of the triangle abc, G must be the centre of gravity of the triangle ABC.

Suppose $ab : bc : ca = AB : BC : CA$.



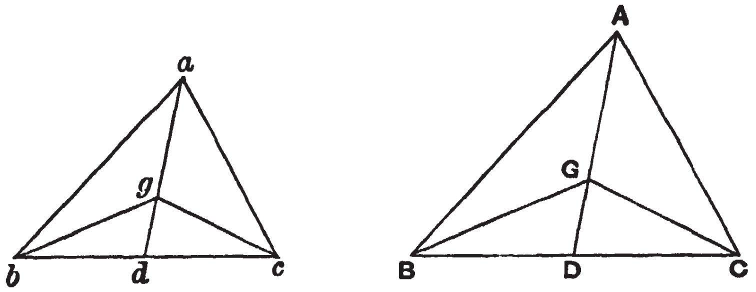
The proposition is proved by an obvious *reductio ad absurdum*. For, if G be not the centre of gravity of the triangle ABC, suppose H to be its centre of gravity.

Post. 5 requires that g, H shall be similarly situated with respect to the triangles respectively; and this leads at once to the absurdity that the angles HAB, GAB are equal.

Proposition 12.

Given two similar triangles abc, ABC, and d, D the middle points of bc, BC respectively, then, if the centre of gravity of abc lie on ad, that of ABC will lie on AD.

Let g be the point on ad which is the centre of gravity of abc.



Take G on AD such that

$$ad : ag = AD : AG,$$

and join gb, gc, GB, GC.

Then, since the triangles are similar, and bd, BD are the halves of bc, BC respectively,

$$ab : bd = AB : BD,$$

and the angles abd, ABD are equal.

Therefore the triangles abd, ABD are similar, and

$$\angle bad = \angle BAD.$$

Also $\qquad ba : ad = BA : AD,$

while, from above, $\qquad ad : ag = AD : AG.$

Therefore $ba : ag = BA : AG$, while the angles bag, BAG are equal.

Hence the triangles bag, BAG are similar, and

$$\angle abg = \angle ABG.$$

And, since the angles abd, ABD are equal, it follows that

$$\angle gbd = \angle GBD.$$

In exactly the same manner we prove that

$$\angle gac = \angle GAC,$$

$$\angle acg = \angle ACG,$$

$$\angle gcd = \angle GCD.$$

Therefore g, G are similarly situated with respect to the triangles respectively; whence [Prop. 11] G is the centre of gravity of ABC.

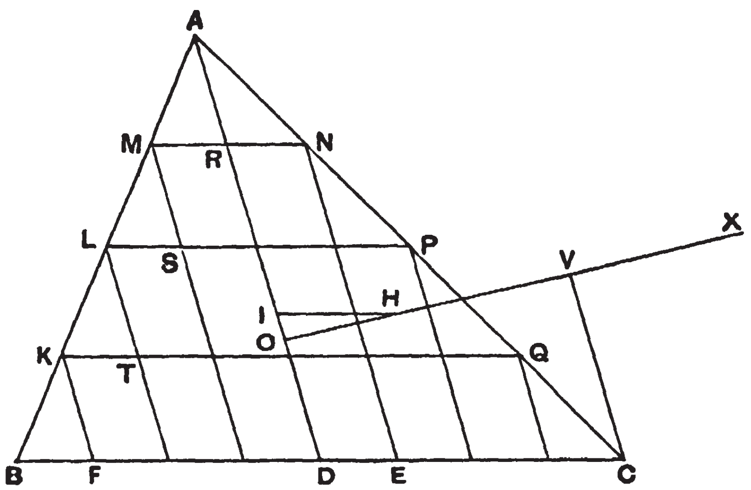
Proposition 13.

In any triangle the centre of gravity lies on the straight line joining any angle to the middle point of the opposite side.

Let ABC be a triangle and D the middle point of BC. Join AD. Then shall the centre of gravity lie on AD.

For, if possible, let this not be the case, and let H be the centre of gravity. Draw HI parallel to CB meeting AD in I.

Then, if we bisect DC, then bisect the halves, and so on, we shall at length arrive at a length, as DE, less than HI.



Divide both BD and DC into lengths each equal to DE, and through the points of division draw lines each parallel to DA meeting BA and AC in points as K, L, M and N, P, Q respectively.

Join MN, LP, KQ, which lines will then be each parallel to BC.

We have now a series of parallelograms as FQ, TP, SN, and AD bisects opposite sides in each. Thus the centre of gravity of each parallelogram lies on AD [Prop. 9], and therefore the centre of gravity of the figure made up of them all lies on AD.

Let the centre of gravity of all the parallelograms taken together be O. Join OH and produce it; also draw CV parallel to DA meeting OH produced in V.

Now, if n be the number of parts into which AC is divided,

$$\triangle ADC : (\text{sum of triangles on } AN, NP, \ldots)$$
$$= AC^2 : (AN^2 + NP^2 + \ldots)$$
$$= n^2 : n$$
$$= n : 1$$
$$= AC : AN.$$

Similarly

$$\triangle ABD : (\text{sum of triangles on } AM, ML, \ldots) = AB : AM.$$

And $$AC : AN = AB : AM.$$

It follows that

$$\triangle ABC : (\text{sum of all the small } \triangle\text{s}) = CA : AN$$
$$> VO : OH, \text{ by parallels.}$$

Suppose OV produced to X so that

$$\triangle ABC : (\text{sum of small } \triangle\text{s}) = XO : OH,$$

whence, *dividendo*,

$$(\text{sum of parallelograms}) : (\text{sum of small } \triangle\text{s}) = XH : HO.$$

Since then the centre of gravity of the triangle ABC is at H, and the centre of gravity of the part of it made up of the parallelograms is at O, it follows from Prop. 8 that the centre of gravity of the remaining portion consisting of all the small triangles taken together is at X.

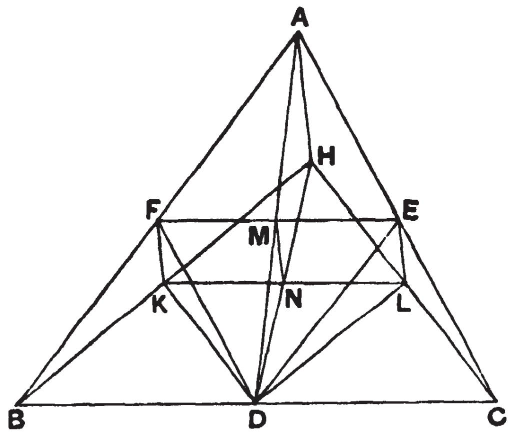
But this is impossible, since all the triangles are on one side of the line through X parallel to AD.

Therefore the centre of gravity of the triangle cannot but lie on AD.

Alternative proof.

Suppose, if possible, that H, not lying on AD, is the centre of gravity of the triangle ABC. Join AH, BH, CH. Let E, F be the middle points of CA, AB respectively, and join DE, EF, FD. Let EF meet AD in M.

Draw FK, EL parallel to AH meeting BH, CH in K, L respectively. Join KD, HD, LD, KL. Let KL meet DH in N, and join MN.



Since DE is parallel to AB, the triangles ABC, EDC are similar.

And, since $CE = EA$, and EL is parallel to AH, it follows that $CL = LH$. And $CD = DB$. Therefore BH is parallel to DL.

Thus in the similar and similarly situated triangles ABC, EDC the straight lines AH, BH are respectively parallel to EL, DL; and it follows that H, L are similarly situated with respect to the triangles respectively.

But H is, by hypothesis, the centre of gravity of ABC. Therefore L is the centre of gravity of EDC. [Prop. 11]

Similarly the point K is the centre of gravity of the triangle FBD.

And the triangles FBD, EDC are equal, so that the centre of gravity of both together is at the middle point of KL, i.e. at the point N.

The remainder of the triangle ABC, after the triangles FBD, EDC are deducted, is the parallelogram $AFDE$, and the centre of gravity of this parallelogram is at M, the intersection of its diagonals.

It follows that the centre of gravity of the whole triangle ABC must lie on MN; that is, MN must pass through H, which is impossible (since MN is parallel to AH).

Therefore the centre of gravity of the triangle ABC cannot but lie on AD.

Proposition 14.

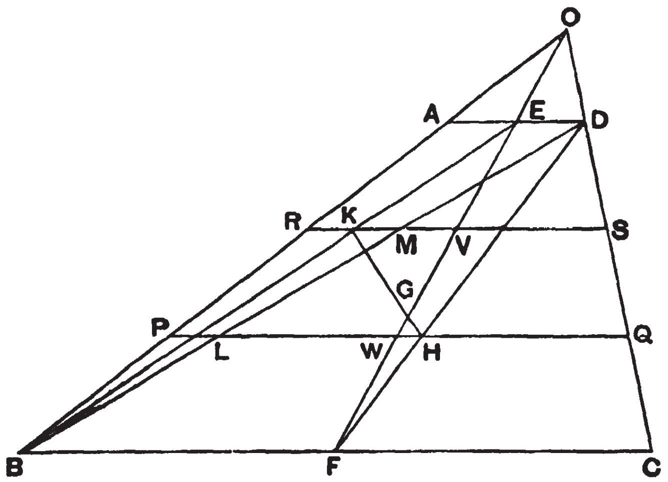
It follows at once from the last proposition that *the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.*

Proposition 15.

If AD, BC be the two parallel sides of a trapezium $ABCD$, AD being the smaller, and if AD, BC be bisected at E, F respectively, then the centre of gravity of the trapezium is at a point G on EF such that

$$GE : GF = (2BC + AD) : (2AD + BC).$$

Produce BA, CD to meet at O. Then FE produced will also pass through O, since $AE = ED$, and $BF = FC$.



Now the centre of gravity of the triangle OAD will lie on OE, and that of the triangle OBC will lie on OF. [Prop. 13]

It follows that the centre of gravity of the remainder, the trapezium $ABCD$, will also lie on OF. [Prop. 8]

Join BD, and divide it at L, M into three equal parts. Through L, M draw PQ, RS parallel to BC meeting BA in P, R, FE in W, V, and CD in Q, S respectively.

Join DF, BE meeting PQ in H and RS in K respectively.

Now, since $BL = \frac{1}{3} BD$,

$$FH = \frac{1}{3} FD.$$

Therefore H is the centre of gravity of the triangle DBC*.

Similarly, since $EK = \frac{1}{3} BE$, it follows that K is the centre of gravity of the triangle ADB.

Therefore the centre of gravity of the triangles DBC, ADB together, i.e. of the trapezium, lies on the line HK.

But it also lies on OF.

Therefore, if OF, HK meet in G, G is the centre of gravity of the trapezium.

Hence [Props. 6, 7]

$$\triangle DBC : \triangle ABD = KG : GH$$
$$= VG : GW.$$

But
$$\triangle DBC : \triangle ABD = BC : AD.$$

Therefore
$$BC : AD = VG : GW.$$

It follows that

$$(2BC + AD) : (2AD + BC) = (2VG + GW) : (2GW + VG)$$
$$= EG : GF.$$

Q. E. D.

* This easy deduction from Prop. 14 is assumed by Archimedes without proof.