Geometrie II – kuželosečky

(praktická část)

Jana Hromadová & Zdeněk Halas

sazba: Bc. Jiří Frantál

Obsah

1 Kuželosečky (opakování SŠ – klasifikace, množiny bodů daných vlastností)

Úloha 1.1 *Určete rovnici kružnice, která prochází body* $A = [1, 3]$ *a* $B = [-3, 1]$ *a její střed leží na přímce* $p:\ 2x - y - 8 = 0$.

Úloha 1.2 *Klasifikujte kuželosečky dané obecnou rovnicí, určete charakteristické prvky a načrtněte obrázek.*

a) $2x^2 + 3y^2 + 12x - 6y + 9 = 0$
b) $x^2 + 4x + 2y + 2 = 0$
c) $x^2 - 4y^2 + 4x - 4y + 2 = 0$

Úloha 1.3 *Napište rovnice tečen z bodu* M *k dané kuželosečce (určete souřadnice bodů dotyku).*

a) $M = [0, 0]$, $x^2 + 2y^2 - 8x + 4y + 12 = 0$
b) $M = [1, -1]$, $y^2 - 4x + 2y + 9 = 0$

Úloha 1.4 *Napište rovnici hyperboly, víte-li, že její asymptoty* a_1, a_2 *mají rovnice* $a_1:\ y = 2(x - 3)$ *a* $a_2:\ y = -2(x - 3)$ *a jedno ohnisko je* $E = [-2, 0]$.

Úloha 1.5 *Napište rovnici elipsy, která má osy rovnoběžné s osami souřadnic, dotýká se osy* x *i osy* y *a její střed je v bodě* $S = [4, -2]$.

Úloha 1.6 *Napište rovnici paraboly, která prochází bodem* $L = [4, 5]$, *její osa má rovnici* $x - 2 = 0$ *a tečna* t *ve vrcholu má rovnici* $y - 1 = 0$. *Určete také její ohnisko a rovnici řídící přímky.*

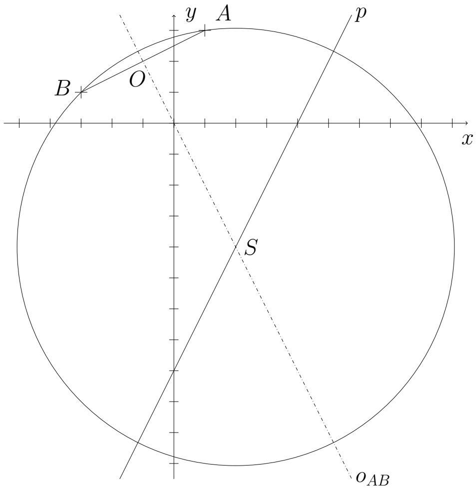
Úloha 1.7 *Je dána přímka* p *a bod* M, $|Mp| = 6$. *Vyšetřete množinu všech středů kružnic, které procházejí bodem* M *a dotýkají se přímky* p.

Úloha 1.8 *Vyšetřete množinu středů všech kružnic, které se dotýkají dané kružnice* $k(S, 5)$ *a procházejí bodem* M, *pro který platí* $|SM| = 3$.

Úloha 1.9 *Vyšetřete množinu středů všech kružnic, které se dotýkají dané kružnice* $k(S, 2)$ *a procházejí bodem* M, *pro který platí* $|SM| = 6$.

Řešení.

1.1



$$k: \quad (x-m)^2 + (y-n)^2 = r^2$$
$$S = [m,n],\ r = ?$$

1) Hledáme osu úsečky AB.

$$(B-A) = (-4,-2) \sim (2,1) = \vec{n}_{o_{AB}}$$

$$O = \frac{A+B}{2} = [-1,2]$$

$$\begin{aligned} o_{AB}: &\quad 2x + y + c = 0 \\ O \in o_{AB}: &\quad 2\cdot(-1) + 2 + c = 0 \\ &\quad c = 0 \end{aligned}$$

$$\underline{\underline{o_{AB}: \quad 2x + y = 0}}$$

2) Hledáme střed S.

$$\left.\begin{aligned} p: &\quad 2\,x - x - 8 = 0 \\ o_{AB}: &\quad 2\,x + y \quad\ = 0 \end{aligned}\right)+$$

$$4x = 8 \longrightarrow x = 2 \longrightarrow y = -4$$

$$\underline{\underline{S = [2,-4]}}$$

3) Hledáme poloměr kružnice.

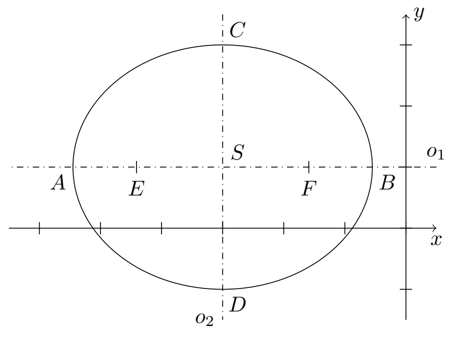
$$r = |SA| = [SB]$$
$$|SA| = \sqrt{(2-1)^2 + (-4-3)^2} = \underline{\underline{\sqrt{50}}}$$

$$\boxed{k: \quad (x-2)^2 + (y+4)^2 = 50}$$

1.2 a)

Upravíme na čtverec.

$$\begin{aligned} 2x^2 + 3y^2 + 12x - 6y + 9 &= 0 \\ 2\left(x^2 + 6x\right) + 3y^2 - 2y + 9 &= 0 \\ 2\,(x+3)^2 - 18 + 3\,(y-1)^2 - 3 + 9 &= 0 \\ 2\,(x+3)^2 + 3\,(y-1)^2 &= 12 \end{aligned}$$



$$\boxed{\frac{(x+3)^2}{6} + \frac{(y-1)^2}{4} = 1}$$

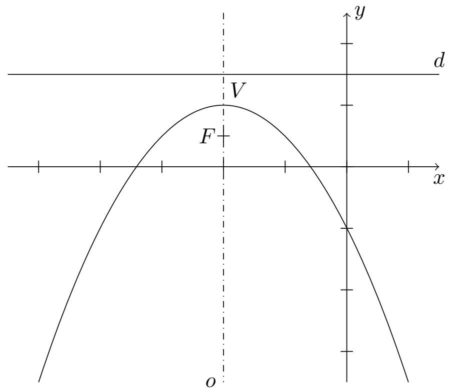
<u>elipsa</u>

$S = [-3, 1]$	$E = \left[-3 - \sqrt{2}, 1\right]$
$a = \sqrt{6}$	$F = \left[-3 + \sqrt{2}, 1\right]$
$b = 2$	$A = \left[-3 - \sqrt{6}, 1\right]$
$e^2 = a^2 - b^2 \implies e = \sqrt{2}$	$B = \left[-3 + \sqrt{6}, 1\right]$
$o_1 : \quad y = 1$	$C = [-3, 3]$
$o_2 : \quad x = -3$	$D = [-3, -1]$

b)

Upravíme na čtverec.

$$x^2 + 4x + 2y + 2 = 0$$
$$(x+2)^2 - 4 + 2y + 2 = 0$$
$$(x+2)^2 = -2y + 2$$



$$\boxed{(x+2)^2 = -2\,(y-1)}$$

parabola

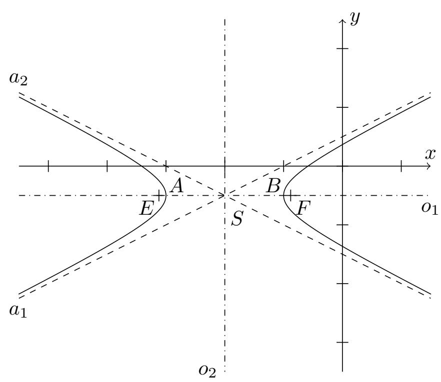
$$-2p = -2 \implies p = 1$$

$V = [-2, 1]$	$F = \left[-2, \frac{1}{2}\right]$
$o : \quad x = -2$	$d : \quad y = \frac{3}{2}$

c)

Upravíme na čtverec.

$$x^2 - 4y^2 + 4x - 4y + 2 = 0$$
$$(x+2)^2 - 4 - 4\left(y + \frac{1}{2}\right)^2 + 1 + 2 = 0$$
$$(x+2)^2 - 4\left(y + \frac{1}{2}\right)^2 = 1$$



$$\boxed{\frac{(x+2)^2}{1} - \frac{\left(y+\frac{1}{2}\right)^2}{\frac{1}{4}} = 1}$$

hyperbola

$$S = \left[-2, -\frac{1}{2}\right] \qquad E = \left[-2 - \frac{\sqrt{5}}{2}, -\frac{1}{2}\right]$$

$$a = 1 \qquad F = \left[-2 + \frac{\sqrt{5}}{2}, -\frac{1}{2}\right]$$

$$b = \frac{1}{2} \qquad A = \left[-3, -\frac{1}{2}\right]$$

$$e^2 = a^2 + b^2 \implies e = \frac{\sqrt{5}}{2} \qquad B = \left[-1, -\frac{1}{2}\right]$$

$$o_1 : \quad y = -\frac{1}{2}$$

$$o_2 : \quad x = -2$$

Asymptoty:

$$y = \pm\frac{b}{a}(x - m) + n$$

$$y = \pm\frac{1}{2}(x + 2) - \frac{1}{2}$$

$$y + \frac{1}{2} = \pm\frac{1}{2}(x + 2)$$

$$\underline{\underline{a_1 : \quad y = \frac{1}{2}x + \frac{1}{2}}} \qquad \underline{\underline{a_2 : \quad y = -\frac{1}{2}x - \frac{3}{2}}}$$

1.3 a) Upravíme na čtverec.

$$x^2 + 2y^2 - 8x + 4y + 12 = 0$$

$$(x-4)^2 - 16 + 2(y+1)^2 - 2 + 12 = 0$$

$$(x-4)^2 + 2(y+1)^2 = 6 \quad / : 6$$

$$\frac{(x-4)^2}{6} + \frac{(y+1)^2}{3} = 1 \implies \text{elipsa}$$

Rovnice tečny t v bodě $T = [x_0, y_0]$: $\dfrac{(x-4)(x_0-4)}{6} + \dfrac{(y+1)(y_0+1)}{3} = 1$

$$M \in t : \qquad \frac{-4(x_0-4)}{6} + \frac{1(y_0+1)}{3} = 1$$

$$T \in t : \qquad \frac{(x_0-4)^2}{6} + \frac{(y_0+1)^2}{3} = 1$$

$$M : \quad -4x_0 + 16 + 2y_0 + 2 = 6 \qquad T : \quad x_0^2 + 2y_0^2 - 8x_0 + 4y_0 + 12 = 0$$

$$-4x_0 + 2y_0 + 12 = 0$$

$$y_0 = 2x_0 - 6$$

$$x_0^2 + 2(2x_0-6)^2 - 8x_0 + 4(2x_0-6) + 12 = 0$$

$$x_0^2 + 8x_0^2 - 48x_0 + 72 - 8x_0 + 8x_0 - 24 + 12 = 0$$

$$9x_0 - 48x_0 + 60 = 0 \quad / : 3$$

$$3x_0 - 16x_0 + 20 = 0$$

$$D = 256 - 240 = 16$$

$$x_{0_{1,2}} = \frac{16 \pm 4}{6} = \begin{cases} \frac{10}{3} \implies y_{0_1} = 2 \cdot \frac{10}{3} - 6 = \frac{2}{3} \\ 2 \implies y_{0_2} = 2 \cdot 2 - 6 = -2 \end{cases}$$

$$\underline{\underline{T_1 = \left[\frac{10}{3}, \frac{2}{3}\right]}} \qquad \underline{\underline{T_2 = [2, -2]}}$$

$$t_1: \quad \frac{(x-4)\left(\frac{10}{3} - 4\right)}{6} + \frac{(y+1)\left(\frac{2}{3} + 1\right)}{3} = 1$$

$$-\frac{2}{3}(x-4) + \frac{10}{3}(y+1) = 6$$

$$-\frac{2}{3}x + \frac{8}{3} + \frac{10}{3}y + \frac{10}{3} = 6 \quad / \cdot 3$$

$$-2x + 8 + 10y + 10 = 18$$

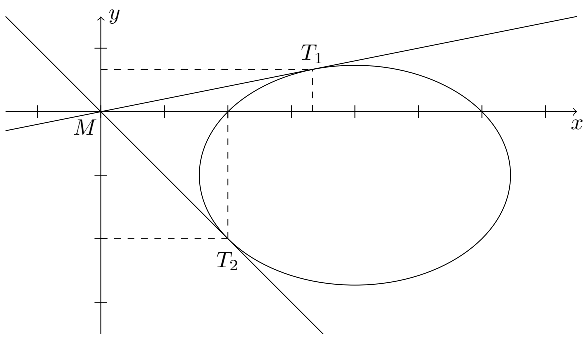
$$\underline{\underline{t_1: \quad x - 5y = 0}}$$

$$t_2: \quad \frac{(x-4)(2-4)}{6} + \frac{(y+1)(-2+1)}{3} = 1$$

$$-2(x-4) - 2(y+1) = 6$$

$$x - 4 + y + 1 = -3$$

$$\underline{\underline{t_2: \quad x + y = 0}}$$



Druhý způsob řešení:

Přímka procházející počátkem (až na osu y, ale lze snadno ověřit, že ta nebude tečnou) má rovnici $y = k \cdot x$.

Tečna má s kuželosečkou právě jeden společný bod. Dosadím-li do rovnice kuželosečky $y = k \cdot x$, hledám taková k, pro něž je diskriminant nulový – nalezneme hodnoty $k_1 = -1$ a $k_2 = \frac{1}{5}$.

b) Upravíme na čtverec.

$$y^2 - 4x + 2y + 9 = 0$$

$$(y+1)^2 - 1 - 4x + 9 = 0$$

$$(y+1)^2 = 4(x-2) \implies \text{parabola}$$

$$V = [2, -1]$$
$$p = 2$$
$$o: \quad y = -1$$

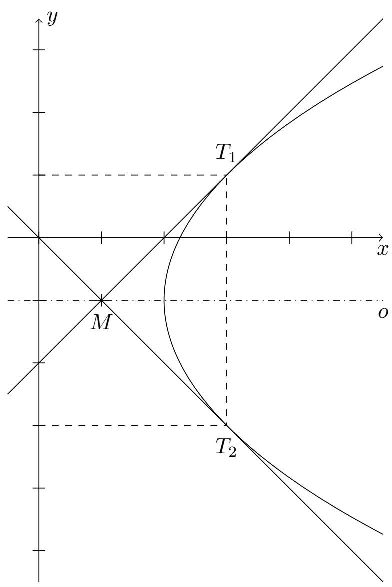
Rovnice tečny t v bodě $T = [x_0, y_0]$: $(y+1)(y_0+1) = 2(x-2) + 2(x_0-2)$

$$M \in t: \qquad (-1+1)(y_0+1) = 2(1-2) + 2(x_0-2)$$
$$0 = -2 + 2(x_0-2)$$
$$x_0 = 3$$

$$T \in t: \qquad y_0^2 - 4x_0 + 2y_0 + 9 = 0$$
$$y_0^2 - 12 + 2y_0 + 9 = 0$$
$$y_0^2 + 2y_0 - 3 = 0$$

$$D = 4 - 4 \cdot 1 \cdot (-3) = 16$$
$$y_{0_{1,2}} = \frac{-2 \pm 4}{2} = \begin{cases} 1 \\ -3 \end{cases}$$

$\underline{\underline{T_1 = [3, 1]}}$ $\qquad$ $\underline{\underline{T_2 = [3, -3]}}$



$$t_1: \quad (y+1)(1+1) = 2(x-2) + 2(3-2)$$
$$y + 1 = x - 2 + 1$$
$$\underline{\underline{t_1: \quad y = x - 2}}$$

$$t_2: \quad (y+1)(-3+1) = 2(x-2) + 2(3-2)$$
$$-(y+1) = x - 2 + 1$$
$$\underline{\underline{t_2: \quad y = -x}}$$

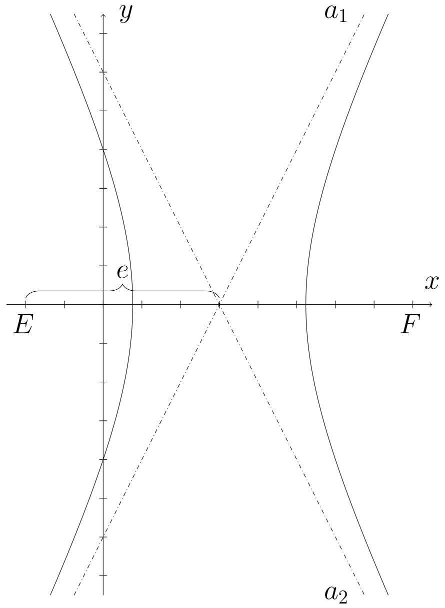
1.4

Obecná rovnice hyperboly:

$$\frac{(x-m)^2}{a^2} - \frac{(y-n)^2}{b^2} = 1$$

Obecné rovnice asymptot:

$$(y-n) = \pm\frac{b}{a}(x-m)$$



$$\frac{b}{a} = 2$$

$$m = 3, n = 0 \implies S = [3, 0]$$

$$e = |ES| = 5$$

$$e^2 = a^2 + b^2$$

$$5^2 = a^2 + b^2 \longrightarrow a^2 + b^2 = 25$$

$$\frac{b^2}{a^2} = 2^2 \longrightarrow b^2 = 4a^2$$

$$a^2 + 4a^2 = 25$$

$$b^2 = 20 \longleftarrow a^2 = 5$$

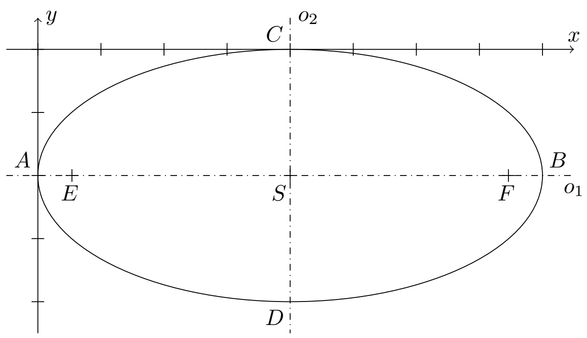
$$h: \quad \frac{(x-3)^2}{5} - \frac{y^2}{20} = 1$$

1.5

$$S = [4, -2]$$

$$\implies a = 4, b = 2$$

$$\frac{(x-4)^2}{16} + \frac{(y+2)^2}{4} = 1$$



$$e^2 = a^2 - b^2 \implies e = 2\sqrt{3}$$
$$E = \left[4 - 2\sqrt{3}, -2\right]$$
$$F = \left[4 + 2\sqrt{3}\right]$$
$$o_1 : \quad y = -2$$
$$o_2 : \quad x = 4$$

$$A = [0, -2]$$
$$B\,[8, -2]$$
$$C\,[4, 0]$$
$$D\,[4, -4]$$

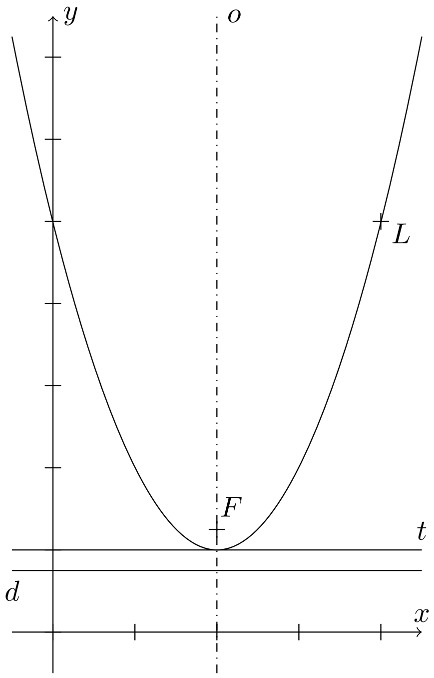
1.6

$$V \in t \cap o \implies V = [2, 1]$$

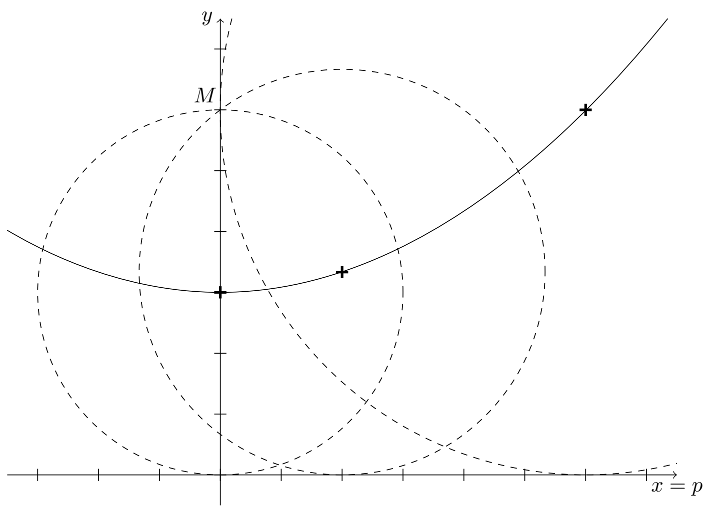
$$(x - m)^2 = 2p\,(y - n)$$
$$(x - 2)^2 = 2p\,(y - 1)$$
$$L \in p : \quad (4 - 2)^2 = 2p\,(5 - 1)$$
$$p = \frac{1}{2}$$

$$\boxed{p : \quad (x - 2)^2 = y - 1}$$

$$F = \left[2, \frac{5}{4}\right] \qquad d : \quad y = \frac{3}{4}$$



1.7



Zvolíme: $\quad p : \quad y = 0$

$$M = [0, 6]$$

$$X = [x, y]$$

$$\begin{aligned} |MX| &= |Xp| \\ \sqrt{x^2 + (y-6)^2} &= |y| \\ x^2 + (y-6)^2 &= y^2 \\ x^2 + y^2 - 12y + 36 &= y^2 \end{aligned}$$

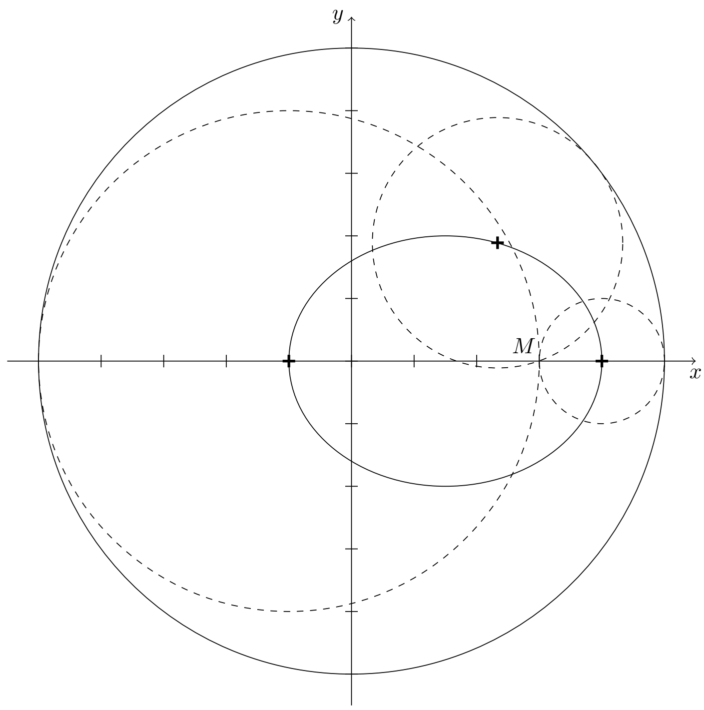
$$\boxed{x^2 = 12\,(y-3)}$$

parabola

$V = [0, 3]$ řídící přímka: $d = p$

$p = 6$ ohnisko: $F = M$

1.8



Zvolíme: $k: \quad x^2 + y^2 = 25$

$M = [3, 0]$

$$X = [x, y]$$

$$|MX| = 5 - |SX|$$

$$\sqrt{(x-3)^2 + y^2} = 5 - \sqrt{x^2+y^2} \quad /^2$$

$$(x-3)^2 + y^2 = 25 - 10\sqrt{x^2+y^2} + x^2 + y^2$$

$$-6x + 9 - 25 = -10\sqrt{x^2+y^2} \quad / : (-2)$$

$$3x + 8 = 5\sqrt{x^2+y^2} \quad /^2$$

$$9x^2 + 48x + 64 = 25x^2 + 25y^2$$

$$16x^2 - 48x + 25y^2 - 64 = 0$$

$$16\left(x - \frac{3}{2}\right)^2 - 36 + 25y^2 = 64 \quad / : 100$$

$$\boxed{\frac{\left(x-\frac{3}{2}\right)^2}{\frac{25}{4}} + \frac{y^2}{4} = 1}$$

elipsa

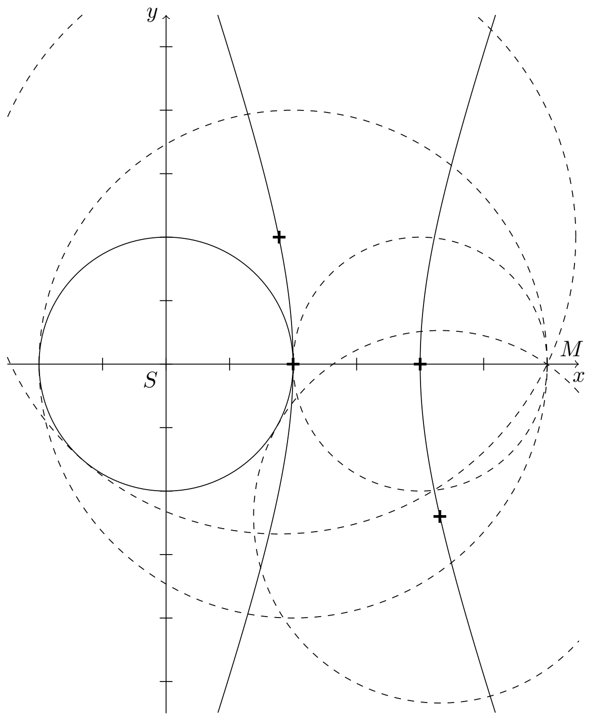
$$|MX| + |SX| = 5$$

$$a = \frac{5}{2} \qquad S_{\text{el.}} = \left[\frac{3}{2}, 0\right] = S_{SM}$$

$$b = 2 \qquad E = [0,0] = S$$

$$e = \frac{3}{2} \qquad F = [3,0] = M$$

1.9



$$\text{Zvolíme:} \qquad k: \quad x^2 + y^2 = 4$$
$$M = [6, 0]$$

$$X = [x, y]$$

$$(I): \quad |MX| = |SX| - 2 \ldots \text{vnější dotyk}$$
$$(II): \quad |MX| = |SX| + 2 \ldots \text{vnitřní dotyk}$$

$$\sqrt{(x-6)^2 + y^2} = \sqrt{x^2 + y^2} - 2 \quad /^2$$
$$(x-6)^2 + y^2 = x^2 + y^2 - 4\sqrt{x^2 + y^2} + 4$$
$$x^2 - 12x + 36 + y^2 = x^2 + y^2 + 4 - 4\sqrt{x^2 + y^2}$$
$$12x - 32 = 4\sqrt{x^2 + y^2}$$
$$3x - 8 = \sqrt{x^2 + y^2} \quad /^2$$
$$9x^2 - 48x + 64 = x^2 + y^2$$
$$8x^2 - y^2 - 48x + 64 = 0$$
$$8(x-3)^2 - 72 - y^2 = -64$$
$$8(x-3)^2 - y^2 = 8 \quad /:8$$

$$\boxed{\frac{(x-3)^2}{1} - \frac{y^2}{8} = 1}$$

hyperbola

$$||SX| - |MX|| = 2$$

$$a = 1 \qquad S_{\text{hyp.}} = [3, 0] = S_{SM}$$
$$b = 2\sqrt{2} \qquad E = [0, 0] = S$$
$$e = 3 \qquad F = [6, 0] = M$$

Poznámka: Při výpočtu jsme prováděli neekvivalentní úpravy. Snadno ale nahlédneme, že každé z rovnic (I) a (II) odpovídá jedna větev hyperboly.

□

2 Klasifikace kuželoseček pomocí otáčení soustavy souřadnic

Úloha 2.1 *Klasifikujte kuželosečky dané obecnou rovnicí, určete charakteristické prvky a načrtněte obrázek.*

a) $4x^2 + 4xy + y^2 - 100y + 500 = 0$
b) $5x^2 - 6xy + 5y^2 - 32 = 0$
c) $16x^2 - 24xy + 9y^2 - 90x - 120y + 525 = 0$
d) $7x^2 + 18xy + 7y^2 - 32x - 24y + 16 = 0$
e) $x^2 + xy + 2y^2 = 0$
f) $x^2 - 4xy + 4y^2 + 6x - 12y + 9 = 0$
g) $4x^2 - 4xy + y^2 + 4x - 2y - 3 = 0$

Úloha 2.2 *Určete množinu* $\{X \in E_2; |XF| = 2\,|Xp|\}$, *je-li* $p:\ x + 2y - 3 = 0$ *a* $F = [2, -1]$.

Řešení.

2.1 a) Určíme, o jaký úhel je třeba otočit soustavu souřadnic, aby osy kuželosečky byly rovnoběžné se souřadnicovými osami.

$$\begin{aligned} a &= 4 \\ 2b &= 4 \implies b = 2 \\ c &= 1 \end{aligned}$$

$$b \cdot \operatorname{tg}^2 \alpha + (a - c) \cdot \operatorname{tg} \alpha - b = 0$$

$$2 \cdot \operatorname{tg}^2 \alpha - 3 \cdot \operatorname{tg} \alpha - 2 = 0$$

$$D = 9 - 4 \cdot 2 \cdot (-2) = 25$$

$$\operatorname{tg} \alpha_{1,2} = \frac{-3 \pm 5}{4} = \begin{cases} \frac{1}{2} \\ -2 & \alpha \notin \left(0, \frac{\pi}{2}\right) \end{cases}$$

$$\operatorname{tg} \alpha = \frac{1}{2} \implies \frac{\sin \alpha}{\cos \alpha} = \frac{1}{2} \implies \frac{\sin^2 \alpha}{\cos^2 \alpha} = \frac{1}{4}$$

$$\begin{aligned} \sin^2 \alpha &= \frac{1}{4} \cos^2 \alpha \longrightarrow \cos^2 \alpha = \frac{4}{5} \\ \sin^2 \alpha &= \frac{1}{4} \left(1 - \sin^2 \alpha\right) \qquad \cos \alpha = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \\ 5 \sin^2 \alpha &= 1 \\ \sin^2 \alpha &= \frac{1}{5} \\ \sin \alpha &= \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \end{aligned}$$

Dosadíme do vzorce.

$$\begin{aligned} x &= x' \cos \alpha - y' \sin \alpha \\ y &= x' \sin \alpha + y' \cos \alpha \end{aligned} \tag{2.1}$$

$$\begin{aligned} x &= x' \cdot \frac{2\sqrt{5}}{5} - y' \cdot \frac{\sqrt{5}}{5} = \frac{\sqrt{5}}{5} \cdot \left(2x' - y'\right) \\ y &= x' \cdot \frac{\sqrt{5}}{5} + y' \cdot \frac{2\sqrt{5}}{5} = \frac{\sqrt{5}}{5} \cdot \left(x' + 2y'\right) \end{aligned} \tag{2.2}$$

Dosadíme do původní rovnice.

$$4 \cdot \frac{5}{25}\left(2x' - y\right)^2 + 4 \cdot \frac{5}{25}\left(2x' - y'\right)\left(x' + 2y'\right) + \frac{5}{25}\left(x' + 2y'\right)^2$$
$$-100 \cdot \frac{\sqrt{5}}{5}\left(x' + 2y'\right) + 500 = 0$$

$$\frac{4}{5}\left(4x'^2 - 4x'y' + y'^2\right) + \frac{4}{5}\left(2x'^2 + 3x'y' - 2y'^2\right) + \frac{1}{5}\left(x'^2 + 4x'y' + 4y'^2\right)$$
$$-20\sqrt{5}x' - 40\sqrt{5}y' + 500 = 0 \quad / \cdot 5$$

$$16x'^2 - 16x'y' + 4y'^2 + 8x'^2 + 12x'y' - 8y'^2 + x'^2 + 4x'y' + 4y'^2$$
$$-100\sqrt{5}x' - 200\sqrt{5}y' + 2500 = 0$$

$$25x'^2 - 100\sqrt{5}x' - 200\sqrt{5}y' + 2500 = 0 \quad / : 25$$

$$x'^2 - 4\sqrt{5}x' - 8\sqrt{y'} + 100 = 0$$

$$\left(x' - 2\sqrt{5}\right)^2 - 20 - 8\sqrt{5}y' + 100 = 0$$
$$\left(x' - 2\sqrt{5}\right)^2 = -80 + 8\sqrt{5}y'$$

$$\boxed{\left(x' - 2\sqrt{5}\right)^2 = 8\sqrt{5}\left(y' - 2\sqrt{5}\right)}$$

parabola

$$V' = \left[2\sqrt{5}; 2\sqrt{5}\right] \qquad F' = \left[2\sqrt{5}; 4\sqrt{5}\right]$$
$$2p = 8\sqrt{5} \implies p = 4\sqrt{5} \qquad d' : \quad y' = 0$$
$$o' \parallel y' \qquad o' : \quad x' = 2\sqrt{5}$$

Určíme prvky původní paraboly.

$$F' = \left[2\sqrt{5}; 4\sqrt{5}\right] \overset{(2.2)}{\Longrightarrow} \begin{aligned} x &= \frac{\sqrt{5}}{5}\left(2 \cdot 2\sqrt{5} - 4\sqrt{5}\right) = 0 \\ y &= \frac{\sqrt{5}}{5}\left(2\sqrt{5} + 2 \cdot 4\sqrt{5}\right) = 10 \end{aligned} \implies \underline{\underline{F = [0; 10]}}$$

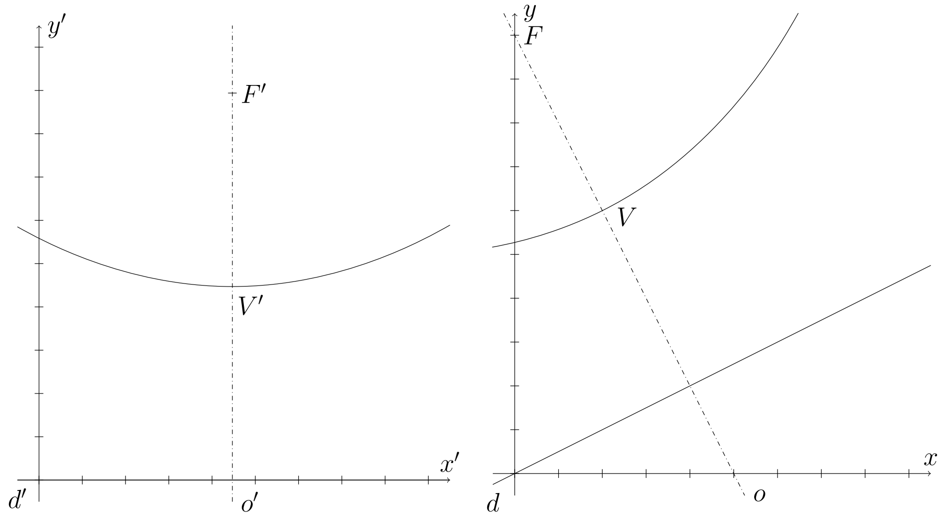
$$V' = \left[2\sqrt{5}; 2\sqrt{5}\right] \overset{(2.2)}{\Longrightarrow} \begin{aligned} x &= \frac{\sqrt{5}}{5}\left(2 \cdot 2\sqrt{5} - 2\sqrt{5}\right) = 2 \\ y &= \frac{\sqrt{5}}{5}\left(2\sqrt{5} + 2 \cdot 2\sqrt{5}\right) = 6 \end{aligned} \implies \underline{\underline{V = [2; 6]}}$$

$$\begin{aligned} x' &= \frac{\sqrt{5}}{5} \cdot (2x + y) \\ y' &= \frac{\sqrt{5}}{5} \cdot (-x + 2y) \end{aligned} \tag{2.3}$$

$$d': \quad y' = 0 \quad \overset{(2.3)}{\Longrightarrow} \quad 0 = \frac{\sqrt{5}}{5}(-x+2y) \quad \Longrightarrow \quad \underline{\underline{d: \quad -x+2y=0}}$$

$$o': \quad x' = 2\sqrt{5} \quad \overset{(2.3)}{\Longrightarrow} \quad 2\sqrt{5} = \frac{\sqrt{5}}{5}(2x+y) \quad \Longrightarrow \quad \underline{\underline{o: \quad 2x+y-10=0}}$$

Poznámka 2.3 *Rovnice (2.3) dostanu tak, že z (2.2) vyjádřím x', y' nebo si uvědomíme, že se jedná o otočení o úhel $-\alpha$.*



b) Určíme, o jaký úhel je třeba otočit soustavu souřadnic, aby osy kuželosečky byly rovnoběžné se souřadnicovými osami.

$$\begin{aligned} a &= 5 \\ 2b &= -6 \implies b = -3 \\ c &= 5 \end{aligned}$$

$$b \cdot \operatorname{tg}^2 \alpha + (a-c) \cdot \operatorname{tg} \alpha - b = 0$$

$$\begin{aligned} -3 \cdot \operatorname{tg}^2 \alpha + 0 \cdot \operatorname{tg} \alpha + 3 &= 0 \\ \operatorname{tg}^2 \alpha &= 1 \\ \operatorname{tg} \alpha &= \begin{cases} 1 & \\ -1 & \alpha \notin \left(0, \frac{\pi}{2}\right) \end{cases} \end{aligned}$$

$$\operatorname{tg} \alpha = 1 \implies \alpha = \frac{\pi}{4} \implies \sin \alpha = \frac{\sqrt{2}}{2} \quad \wedge \quad \cos \alpha = \frac{\sqrt{2}}{2}$$

Dosadíme do vzorce (2.1).

$$\begin{aligned} x &= \frac{\sqrt{2}}{2} \cdot (x' - y') \\ y &= \frac{\sqrt{2}}{2} \cdot (x' + y') \end{aligned} \tag{2.4}$$

Doadíme do původní rovnice.

$$
\begin{aligned}
5\cdot\frac{1}{2}\left(x'-y'\right)^2-6\cdot\frac{1}{2}\left(x'-y'\right)\left(x'+y'\right)+5\cdot\frac{1}{2}\left(x'+y'\right)^2-32&=0 \quad /\cdot 2\\
5\left(x'^2-2x'y'+y'^2\right)-6\left(x'^2-y'^2\right)+5\left(x'^2+2x'y'+y'^2\right)-64&=0\\
4x'^2+16y'^2-64&=0 \quad /:4\\
x'^2+4y'^2-16&=0
\end{aligned}
$$

$$x'^2+4y'^2=16 \quad /:16$$

$$\boxed{\frac{x'^2}{16}+\frac{y'^2}{4}=1}$$

elipsa

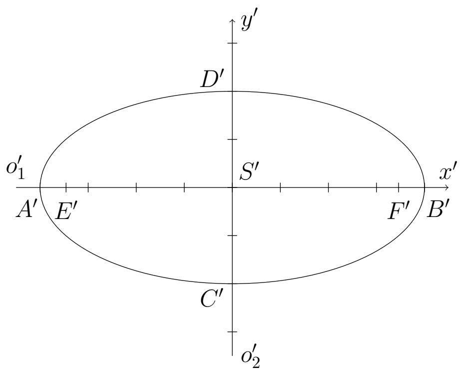
$$
\begin{aligned}
S'&=[0;0] & A'&=[-4;0]\\
a&=4 & B'&=[4;0]\\
b&=2 & C'&=[0;-2]\\
e&=\sqrt{a^2-b^2}=2\sqrt{3} & D'&=[0;2]\\
o_1':&\quad y'=0 & E'&=\left[-2\sqrt{3};0\right]\\
o_2':&\quad x'=0 & F'&=\left[2\sqrt{3};0\right]
\end{aligned}
$$

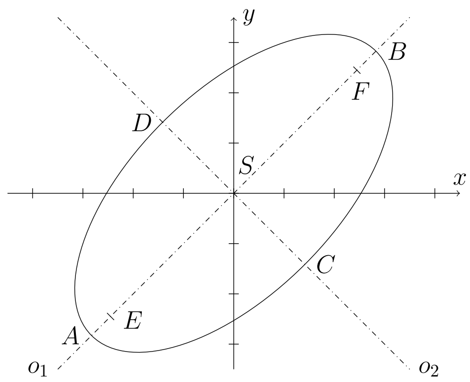
Určíme prvky původní elipsy.

$$
\begin{aligned}
S'&=[0;0] &\stackrel{(2.4)}{\Longrightarrow}&& S&=[0;0]\\
A'&=[-4;0] &\stackrel{(2.4)}{\Longrightarrow}&& A&=\left[-2\sqrt{2};-2\sqrt{2}\right]\\
B'&=[4;0] &\stackrel{(2.4)}{\Longrightarrow}&& B&=\left[2\sqrt{2};2\sqrt{2}\right]\\
C'&=[0;-2] &\stackrel{(2.4)}{\Longrightarrow}&& C&=\left[\sqrt{2};-\sqrt{2}\right]\\
D'&=[0;2] &\stackrel{(2.4)}{\Longrightarrow}&& D&=\left[-\sqrt{2};\sqrt{2}\right]\\
E'&=\left[-2\sqrt{3};0\right] &\stackrel{(2.4)}{\Longrightarrow}&& E&=\left[-\sqrt{6};-\sqrt{6}\right]\\
F'&=\left[2\sqrt{3};0\right] &\stackrel{(2.4)}{\Longrightarrow}&& F&=\left[\sqrt{6};\sqrt{6}\right]
\end{aligned}
$$

$$
\begin{aligned}
x'&=\frac{\sqrt{2}}{2}\cdot(x+y)\\
y'&=\frac{\sqrt{2}}{2}\cdot(-x+y)
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
o_1':&\quad y'=0 &\stackrel{(2.5)}{\Longrightarrow}&\quad 0=\frac{\sqrt{2}}{2}(-x+y) &\Longrightarrow&\quad o_1:\quad x-y=0\\
o_2':&\quad x'=0 &\stackrel{(2.5)}{\Longrightarrow}&\quad 0=\frac{\sqrt{2}}{2}(x+y) &\Longrightarrow&\quad o_2:\quad x+y=0
\end{aligned}
$$





c) Určíme, o jaký úhel je třeba otočit soustavu souřadnic, aby osy kuželosečky byly rovnoběžné se souřadnicovými osami.

$$\begin{aligned} a &= 16 \\ 2b &= -24 \implies b = -12 \\ c &= 9 \end{aligned}$$

$$b \cdot \operatorname{tg}^2 \alpha + (a - c) \cdot \operatorname{tg} \alpha - b = 0$$

$$-12 \cdot \operatorname{tg}^2 \alpha - 7 \cdot \operatorname{tg} \alpha + 12 = 0$$

$$D = 49 - 4 \cdot (-12) \cdot 12 = 625 = 25^2$$

$$\operatorname{tg} \alpha_{1,2} = \frac{-7 \pm 25}{-24} = \begin{cases} -\frac{18}{24} = -\frac{3}{4} & \alpha \notin \left(0, \frac{\pi}{2}\right) \\ \frac{32}{24} = \frac{4}{3} \end{cases}$$

$$\operatorname{tg} \alpha = \frac{4}{3} \implies \frac{\sin^2 \alpha}{\cos^2 \alpha} = \frac{16}{9}$$

$$\begin{aligned} \sin^2 \alpha &= \frac{16}{9} \left(1 - \sin^2 \alpha\right) & \longrightarrow \quad \cos^2 \alpha &= \frac{9}{25} \\ 25 \cdot \sin^2 \alpha &= 16 & \cos \alpha &= \frac{3}{5} \\ \sin^2 \alpha &= \frac{16}{25} \\ \sin \alpha &= \frac{4}{5} \end{aligned}$$

Dosadíme do vzorce (2.1).

$$\begin{aligned} x &= \frac{1}{5} \cdot \left(3x' - 4y'\right) \\ y &= \frac{1}{5} \cdot \left(4x' + 3y'\right) \end{aligned} \tag{2.6}$$

Dosadíme do původní rovnice.

$$\begin{aligned}16\cdot\frac{1}{25}\left(3x'-4y'\right)^2-24\cdot\frac{1}{25}\left(3x'-4y'\right)\left(4x'+3y'\right)\\+9\cdot\frac{1}{25}\left(4x'+3y'\right)^2-90\cdot\frac{1}{5}\left(3x'-4y'\right)-120\cdot\frac{1}{5}\left(4x'+3y'\right)+525=0 \quad /\cdot 25\\16\left(9x'^2-24x'y'+16y'^2\right)-24\left(12x'^2-7x'y'-12y'^2\right)\\+9\left(16x'^2+24x'y'+9y'^2\right)^2-450\left(3x'-4y'\right)-600\left(4x'+3y'\right)+13\ 125=0\\144x'^2-384x'y'+256y'^2-288x'^2+168x'y'+288y'^2\\+144x'^2+216x'y'+81y'^2-1\ 350x'+1\ 800y'-2\ 400x'-1\ 800y'+13\ 125=0\\625y'^2-3\ 750x'+13\ 125=0 \quad /:625\\y'^2-6x'+21=0\end{aligned}$$

$$y'^2=6\left(x'-\frac{21}{6}\right)$$

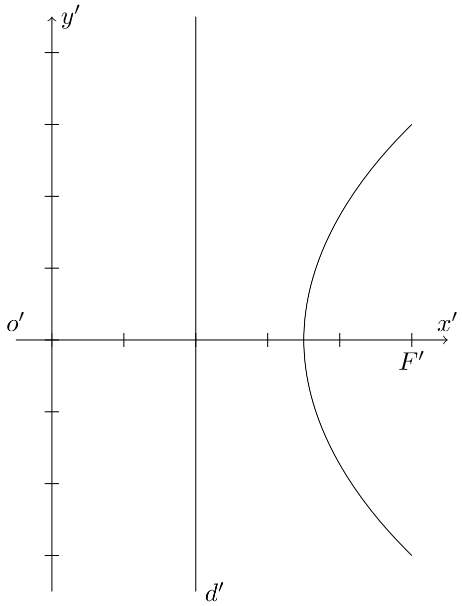
$$\boxed{y'^2=6\left(x'-\frac{7}{2}\right)}$$

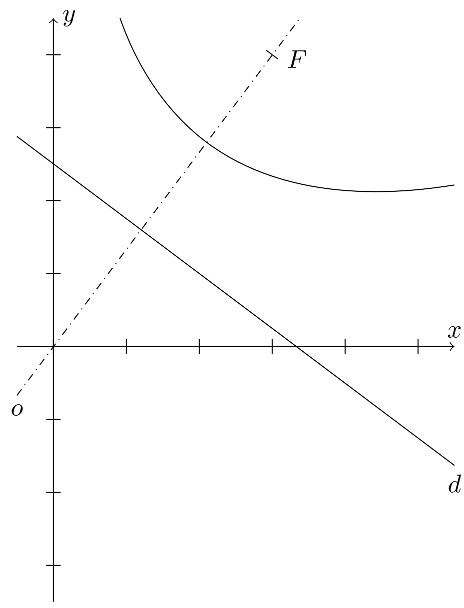
parabola

$$V'=\left[\frac{7}{2};0\right] \qquad F'=[5;0]$$

$$2p=6 \implies p=3 \qquad d':\quad x'=2$$

$$o'\parallel x' \qquad o':\quad y'=0$$





Určíme prvky původní paraboly.

$$F' = [5;5] \quad \overset{(2.6)}{\Longrightarrow} \quad F = [3;4]$$

$$V' = \left[\frac{7}{2}\right] \quad \overset{(2.6)}{\Longrightarrow} \quad V = \left[\frac{21}{10}; \frac{14}{5}\right]$$

$$\begin{aligned} x' &= \frac{1}{5} \cdot (3x + 4y) \\ y' &= \frac{1}{5} \cdot (-4x + 3y) \end{aligned} \tag{2.7}$$

$$d': \quad x' = 2 \quad \overset{(2.7)}{\Longrightarrow} \quad d: \quad 3x + 4y - 10 = 0$$

$$o': \quad y' = 0 \quad \overset{(2.7)}{\Longrightarrow} \quad o: \quad -4x + 3y = 0$$

d) Určíme, o jaký úhel je třeba otočit soustavu souřadnic, aby osy kuželosečky byly rovnoběžné se souřadnicovými osami.

$$\begin{aligned} a &= 7 \\ 2b &= 18 \implies b = 9 \\ c &= 7 \end{aligned}$$

$$b \cdot \operatorname{tg}^2 \alpha + (a - c) \cdot \operatorname{tg} \alpha - b = 0$$

$$\begin{aligned} 9 \cdot \operatorname{tg}^2 \alpha + 0 \cdot \operatorname{tg} \alpha - 9 &= 0 \\ \operatorname{tg}^2 \alpha &= 1 \\ \operatorname{tg} \alpha &= \begin{cases} 1 \\ -1 & \alpha \notin \left(0, \frac{\pi}{2}\right) \end{cases} \end{aligned}$$

$$\operatorname{tg} \alpha = 1 \implies \alpha = \frac{\pi}{4} \implies \sin \alpha = \frac{\sqrt{2}}{2} \wedge \cos \alpha = \frac{\sqrt{2}}{2}$$

Dosadíme do vzorce (2.1).

$$\begin{aligned} x &= \frac{\sqrt{2}}{2} \cdot (x' - y') \\ y &= \frac{\sqrt{2}}{2} \cdot (x' + y') \end{aligned} \tag{2.8}$$

Dosadíme do původní rovnice.

$$7 \cdot \frac{1}{2} \left(x' - y'\right)^2 + 18 \cdot \frac{1}{2} \left(x' - y'\right) \left(x' + y'\right) + 7 \cdot \frac{1}{2} \left(x' + y'\right)^2$$
$$-32 \cdot \frac{\sqrt{2}}{2} \left(x' - y'\right) - 24 \cdot \frac{\sqrt{2}}{2} \left(x' + y'\right) + 16 = 0 \quad / \cdot 2$$

$$7 \left(x'^2 - 2x'y' + y'^2\right)^2 + 18 \left(x'^2 - y'^2\right) + 7 \left(x'^2 + 2x'y' + y'^2\right)^2$$
$$-32\sqrt{2} \left(x' - y'\right) - 24\sqrt{2} \left(x' + y'\right) + 32 = 0$$

$$7x'^2 - 14x'y' + 7y'^2 + 18x'^2 - 18y'^2 + 7x'^2 + 14x'y' + 7y'^2$$
$$-32\sqrt{2}x' + 32\sqrt{2}y' - 24\sqrt{2}x' - 24\sqrt{2}y' + 32 = 0$$

$$32x'^2 - 4y'^2 - 56\sqrt{2}x' + 8\sqrt{2}y' + 32 = 0 \quad / : 4$$

$$8x'^2 - y'^2 - 14\sqrt{2}x' + 2\sqrt{2}y' + 8 = 0$$

$$8\left(x'^2-\frac{7}{4}\sqrt{2}x'\right)-\left(y'^2-2\sqrt{2}y'\right)+8=0$$

$$8\left(x'-\frac{7}{8}\sqrt{2}\right)^2-\frac{49\cdot 2}{8}-\left(y'-\sqrt{2}\right)^2+2+8=0$$

$$8\left(x'-\frac{7}{8}\sqrt{2}\right)^2-\left(y'-\sqrt{2}\right)^2=\frac{9}{4}$$

$$\boxed{\frac{\left(x'-\frac{7}{8}\sqrt{2}\right)^2}{\frac{9}{32}}-\frac{\left(y'-\sqrt{2}\right)^2}{\frac{9}{4}}=1}$$

hyperbola

$$S'=\left[\frac{7\sqrt{2}}{8};\sqrt{2}\right] \qquad A'=\left[\frac{\sqrt{2}}{2};\sqrt{2}\right]$$

$$a=\frac{3\sqrt{2}}{8} \qquad B'=\left[\frac{5\sqrt{2}}{4};\sqrt{2}\right]$$

$$b=\frac{3}{2} \qquad E'=\left[-\frac{\sqrt{2}}{4};\sqrt{2}\right]$$

$$e^2=a^2+b^2=\frac{81}{32} \implies e=\frac{9\sqrt{2}}{8} \qquad F'=\left[2\sqrt{2};\sqrt{2}\right]$$

$$o_1':\quad y'=\sqrt{2} \qquad o_2':\quad x'=\frac{7\sqrt{2}}{8}$$

asymptoty: $$y'-n=\pm\frac{b}{a}(x-m)$$

$$y'-\sqrt{2}=\pm\frac{\frac{3}{2}}{\frac{3\sqrt{2}}{8}}\left(x-\frac{7}{8}\sqrt{2}\right)$$

$$u_{1,2}':\quad y'-\sqrt{2}=\pm 2\sqrt{2}\left(x'-\frac{7}{8}\sqrt{2}\right)$$

Určíme prvky původní hyperboly.

$$S'=\left[\frac{7\sqrt{2}}{8};\sqrt{2}\right] \quad \overset{(2.8)}{\Longrightarrow} \quad S=\left[-\frac{1}{8};\frac{15}{8}\right]$$

$$E'=\left[-\frac{\sqrt{2}}{4};\sqrt{2}\right] \quad \overset{(2.8)}{\Longrightarrow} \quad E=\left[-\frac{5}{4};\frac{3}{4}\right]$$

$$F'=\left[2\sqrt{2};\sqrt{2}\right] \quad \overset{(2.8)}{\Longrightarrow} \quad F=[1;3]$$

$$A'=\left[\frac{5\sqrt{2}}{4};\sqrt{2}\right] \quad \overset{(2.8)}{\Longrightarrow} \quad A=\left[\frac{1}{4};\frac{9}{4}\right]$$

$$B'=\left[\frac{\sqrt{2}}{2};\sqrt{2}\right] \quad \overset{(2.8)}{\Longrightarrow} \quad B=\left[-\frac{1}{2};\frac{3}{2}\right]$$

$$\begin{aligned} x' &= \frac{\sqrt{2}}{2} \cdot (x+y) \\ y' &= \frac{\sqrt{2}}{2} \cdot (-x+y) \end{aligned} \tag{2.9}$$

$$o_1': \quad y' = \sqrt{2} \quad \stackrel{(2.9)}{\Longrightarrow} \quad \frac{\sqrt{2}}{2}(-x+y) = \sqrt{2} \quad \Longrightarrow \quad o_1: \quad x - y + 2 = 0$$

$$o_2': \quad x' = \frac{7\sqrt{2}}{8} \quad \stackrel{(2.9)}{\Longrightarrow} \quad \frac{\sqrt{2}}{2}(x+y) = \frac{7\sqrt{2}}{8} \quad \Longrightarrow \quad o_2: \quad 4x + 4y - 7 = 0$$

$$u_{1,2}': \quad y' - \sqrt{2} = \pm 2\sqrt{2}\left(x' - \frac{7\sqrt{2}}{8}\right)$$

$$\Downarrow (2.9)$$

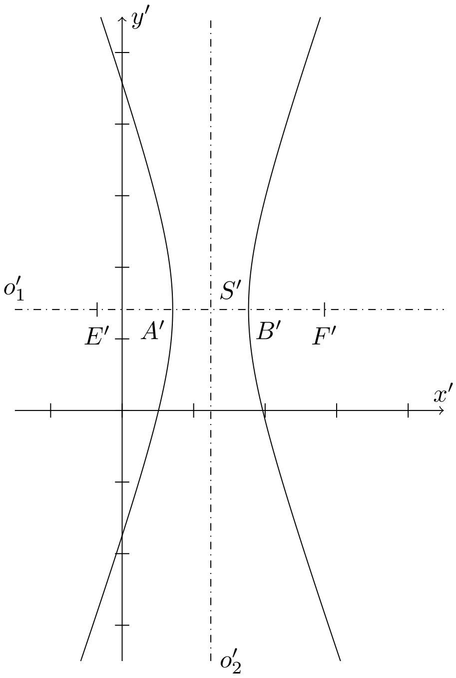
$$\frac{\sqrt{2}}{2}(-x+y) - \sqrt{2} = \pm 2\sqrt{2}\left(\frac{\sqrt{2}}{2}(x+y) - \frac{7\sqrt{2}}{8}\right) \quad / \cdot 2$$

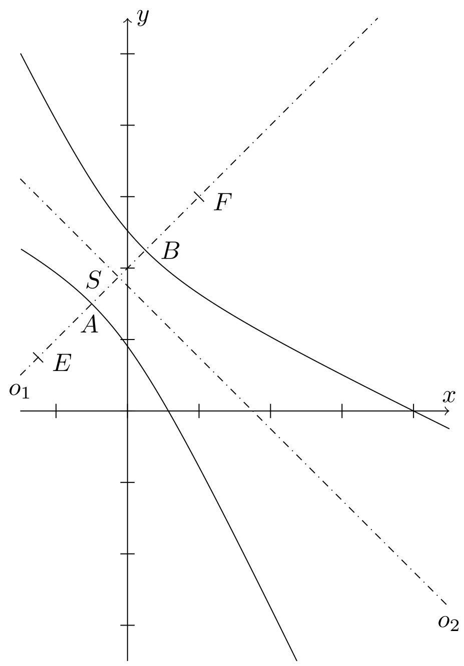
$$\sqrt{2}\,(-x+y) - 2\sqrt{2} = 4\,(x+y) - 7$$

$$u_1: \quad \left(4+\sqrt{2}\right)x + \left(4-\sqrt{2}\right)y + \left(-7+2\sqrt{2}\right) = 0$$

$$\sqrt{2}\,(-x+y) - 2\sqrt{2} = -4\,(x+y) + 7$$

$$u_2: \quad \left(-4+\sqrt{2}\right)x + \left(-4-\sqrt{2}\right)y + \left(7+2\sqrt{2}\right) = 0$$





e) Určíme, o jaký úhel je třeba otočit soustavu souřadnic, aby osy kuželosečky byly rovnoběžné se souřadnicovými osami.

$$a = 1$$

$$2b = 2 \Longrightarrow b = \frac{1}{2}$$

$$c = 2$$

$$b \cdot \operatorname{tg}^2 \alpha + (a-c) \cdot \operatorname{tg} \alpha - b = 0$$

$$\frac{1}{2} \cdot \operatorname{tg}^2 \alpha - 1 \cdot \operatorname{tg} \alpha - \frac{1}{2} = 0 \quad / \cdot 2$$
$$\operatorname{tg}^2 \alpha - 2 \cdot \operatorname{tg} \alpha - 1 = 0$$

$$D = 4 - 4 \cdot 1 \cdot (-1) = 8$$
$$\operatorname{tg} \alpha_{1,2} = \frac{2 \pm 2\sqrt{2}}{2} = \begin{cases} 1+\sqrt{2} & \\ 1-\sqrt{2} & \alpha \notin \left(0, \frac{\pi}{2}\right) \end{cases}$$

$$\operatorname{tg} \alpha = 1 + \sqrt{2} \implies \frac{\sin^2 \alpha}{\cos^2 \alpha} = \left(1+\sqrt{2}\right)^2$$

$$\sin^2 \alpha = \left(1+\sqrt{2}\right)^2 \left(1 - \sin^2 \alpha\right) \longrightarrow \cos^2 \alpha = \frac{2-\sqrt{2}}{4}$$
$$\left(4+2\sqrt{2}\right) \cdot \sin^2 \alpha = 3 + 2\sqrt{2} \qquad \cos \alpha = \sqrt{\frac{2-\sqrt{2}}{4}}$$
$$\sin^2 \alpha = \frac{3+2\sqrt{2}}{4+2\sqrt{2}} = \frac{2+\sqrt{2}}{4}$$
$$\sin \alpha = \sqrt{\frac{2+\sqrt{2}}{4}}$$

Dosadíme do vzorce (2.1).

$$\begin{aligned} x &= \sqrt{\frac{2-\sqrt{2}}{4}} x' - \sqrt{\frac{2+\sqrt{2}}{4}} y' \\ y &= \sqrt{\frac{2+\sqrt{2}}{4}} x' + \sqrt{\frac{2-\sqrt{2}}{4}} y' \end{aligned} \tag{2.10}$$

Dosadíme do původní rovnice.

$$\begin{aligned}
&\left(\sqrt{\frac{2-\sqrt{2}}{4}}x' - \sqrt{\frac{2+\sqrt{2}}{4}}y'\right)^2 \\
+&\left(\sqrt{\frac{2-\sqrt{2}}{4}}x' - \sqrt{\frac{2+\sqrt{2}}{4}}y'\right)\cdot\left(\sqrt{\frac{2+\sqrt{2}}{4}}x' + \sqrt{\frac{2-\sqrt{2}}{4}}y'\right) \\
&+2\cdot\left(\sqrt{\frac{2+\sqrt{2}}{4}}x' + \sqrt{\frac{2-\sqrt{2}}{4}}y'\right)^2 = 0
\end{aligned}$$

$$\begin{aligned}
\frac{2-\sqrt{2}}{4}x'^2 - 2\cdot\sqrt{\frac{4-2}{16}}x'y' + \frac{2+\sqrt{2}}{4}y'^2 \\
+\sqrt{\frac{2-\sqrt{2}}{4}\cdot\frac{2+\sqrt{2}}{4}}x'^2 - \frac{2\sqrt{2}}{4} - \sqrt{\frac{2+\sqrt{2}}{4}\cdot\frac{2-\sqrt{2}}{4}}y'^2 \\
+2\cdot\frac{2-\sqrt{2}}{4}x'^2 + 4\cdot\sqrt{\frac{4-2}{16}}x'y' + 2\cdot\frac{2+\sqrt{2}}{4}y'^2 = 0 \\
\frac{6+\sqrt{2}}{4}x'^2 + \frac{6-\sqrt{2}}{4}y'^2 + \sqrt{\frac{4-2}{16}}x'^2 + \sqrt{\frac{4-2}{16}}y'^2 = 0 \quad / \cdot 4 \\
\left(3+\sqrt{2}\right)x'^2 + \left(3-\sqrt{2}\right)y'^2 = 0
\end{aligned}$$

$$A^2 + B^2 = (A+iB)\cdot(A-iB)$$

$$\boxed{\left(x'\sqrt{3+\sqrt{2}} + iy'\sqrt{3-\sqrt{2}}\right)\cdot\left(x'\sqrt{3+\sqrt{2}} - iy'\sqrt{3-\sqrt{2}}\right) = 0}$$

dvě imaginární přímky s reálným průsečíkem

$$\begin{aligned}
p_1' : \quad & x'\sqrt{3+\sqrt{2}} + iy'\sqrt{3-\sqrt{2}} = 0 \qquad S' = [0;0] \\
p_2' : \quad & x'\sqrt{3+\sqrt{2}} - iy'\sqrt{3-\sqrt{2}} = 0
\end{aligned}$$

Určíme původní přímky a jejich průsečík.

$$S' = [0;0] \quad \overset{(2.10)}{\Longrightarrow} \quad S = [0;0]$$

$$\begin{aligned}
x' &= \sqrt{\frac{2-\sqrt{2}}{4}}x + \sqrt{\frac{2+\sqrt{2}}{4}}y \\
y' &= -\sqrt{\frac{2+\sqrt{2}}{4}}x + \sqrt{\frac{2-\sqrt{2}}{4}}y
\end{aligned} \tag{2.11}$$

$$p_1': \quad x'\sqrt{3+\sqrt{2}} + iy'\sqrt{3-\sqrt{2}} = 0$$
$$\Downarrow (2.11)$$
$$p_1: \quad x\left(\sqrt{4-\sqrt{2}} - i\sqrt{4+\sqrt{2}}\right) + y\left(\sqrt{8+5\sqrt{2}} + i\sqrt{8-5\sqrt{2}}\right) = 0$$

$$p_2': \quad x'\sqrt{3+\sqrt{2}} - iy'\sqrt{3-\sqrt{2}} = 0$$
$$\Downarrow (2.11)$$
$$p_2: \quad x\left(\sqrt{4-\sqrt{2}} + i\sqrt{4+\sqrt{2}}\right) + y\left(\sqrt{8+5\sqrt{2}} - i\sqrt{8-5\sqrt{2}}\right) = 0$$

f) Určíme, o jaký úhel je třeba otočit soustavu souřadnic, aby osy kuželosečky byly rovnoběžné se souřadnicovými osami.

$$\begin{aligned} a &= 1 \\ 2b &= -4 \implies b = -2 \\ c &= 4 \end{aligned}$$

$$b \cdot \operatorname{tg}^2 \alpha + (a-c)\cdot \operatorname{tg}\alpha - b = 0$$

$$-2 \cdot \operatorname{tg}^2 \alpha - 3 \cdot \operatorname{tg}\alpha + 2 = 0$$

$$D = 9 - 4\cdot(-2)\cdot 2 = 25$$
$$\operatorname{tg}\alpha_{1,2} = \frac{3 \pm 5}{-4} = \begin{cases} -2 & \alpha \notin \left(0, \frac{\pi}{2}\right) \\ \frac{1}{2} \end{cases}$$

$$\operatorname{tg}^2\alpha = \frac{1}{2} \implies \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{4}$$

$$\begin{aligned} \sin^2\alpha &= \frac{1}{4}\left(1 - \sin^2\alpha\right) \\ \frac{5}{4}\cdot\sin^2\alpha &= \frac{1}{4} \\ \sin^2\alpha &= \frac{1}{5} \\ \sin\alpha &= \frac{\sqrt{5}}{5} \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \cos^2\alpha &= \frac{4}{5} \\ \cos\alpha &= \frac{2\sqrt{5}}{5} \end{aligned}$$

Dosadíme do vzorce (2.1).

$$\begin{aligned} x &= \frac{\sqrt{5}}{5}\left(2x' - y'\right) \\ y &= \frac{\sqrt{5}}{5}\left(x' + 2y'\right) \end{aligned} \tag{2.12}$$

Dosadíme do původní rovnice.

$$\frac{1}{5}\left(2x'-y'\right)^2-4\cdot\frac{1}{5}\left(2x'-y'\right)\left(x'+2y'\right)+4\cdot\frac{1}{5}\left(x'+2y'\right)^2$$
$$+6\cdot\frac{\sqrt{5}}{5}\left(2x'-y'\right)-12\cdot\frac{\sqrt{5}}{5}\left(x'+2y'\right)+9=0 \quad /\cdot 5$$
$$\left(4x'^2-4x'y'+y'^2\right)-4\left(2x'^2+3x'y'-2y'^2\right)+4\left(x'^2+4x'y'+4y'^2\right)$$
$$+6\sqrt{5}\left(2x'-y'\right)-12\sqrt{5}\left(x'+2y'\right)+45=0$$
$$4x'^2-4x'y'+y'^2-8x'^2-12x'y'+8y'^2+4x'^2+16x'y'+16y'^2$$
$$+12\sqrt{5}x'-6\sqrt{5}y'-12\sqrt{5}x'-24\sqrt{5}y'+45=0$$
$$25y'^2-30\sqrt{5}y'+45=0 \quad /\cdot 5$$
$$5y'^2-6\sqrt{5}y'+9=0$$

$$D=36\cdot 5-4\cdot 5\cdot 9=180-180=0$$
$$y'_{1,2}=\frac{6\sqrt{5}}{10}=\frac{3\sqrt{5}}{5}$$

$$\boxed{5\left(y'-\frac{3\sqrt{5}}{5}\right)^2=0}$$

dvojnásob počítaná přímka

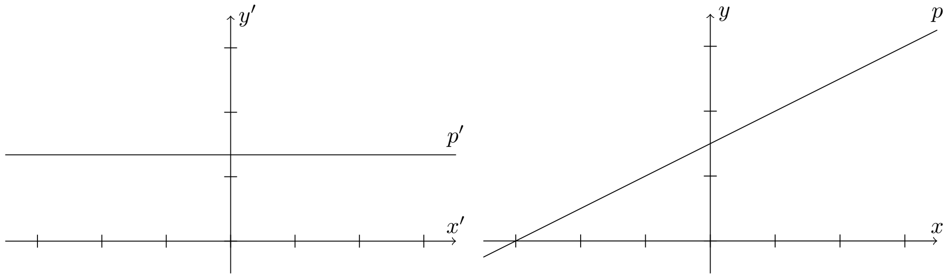
$$p':\quad y'=\frac{3\sqrt{5}}{5}$$

Určíme původní přímku.

$$\begin{aligned} x'&=\frac{\sqrt{5}}{5}\cdot(2x+y)\\ y'&=\frac{\sqrt{5}}{5}\cdot(-x+2y)\end{aligned} \tag{2.13}$$

$$p':\quad y'=\frac{3\sqrt{5}}{5} \quad \overset{(2.13)}{\Longrightarrow} \quad \frac{3\sqrt{5}}{5}=\frac{\sqrt{5}}{5}\cdot(-x+2y) \quad \Longrightarrow \quad p:\quad x-2y+3=0$$

$$\text{původní rovnice:}\qquad (x-2y+3)^2=0$$



g) Určíme, o jaký úhel je třeba otočit soustavu souřadnic, aby osy kuželosečky byly rovnoběžné se souřadnicovými osami.

$$\begin{aligned} a &= 4 \\ 2b &= -4 \implies b = -2 \\ c &= 1 \end{aligned}$$

$$b \cdot \operatorname{tg}^2 \alpha + (a - c) \cdot \operatorname{tg} \alpha - b = 0$$

$$-2 \cdot \operatorname{tg}^2 \alpha + 3 \cdot \operatorname{tg} \alpha + 2 = 0$$

$$D = 9 - 4 \cdot (-2) \cdot 2 = 25$$

$$\operatorname{tg} \alpha_{1,2} = \frac{-3 \pm 5}{-4} = \begin{cases} -\frac{1}{2} & \alpha \notin \left(0, \frac{\pi}{2}\right) \\ 2 \end{cases}$$

$$\operatorname{tg}^2 \alpha = 2 \implies \frac{\sin^2 \alpha}{\cos^2 \alpha} = 4$$

$$\begin{aligned} \sin^2 \alpha &= 4\left(1 - \sin^2 \alpha\right) \\ 5 \cdot \sin^2 \alpha &= 4 \\ \sin^2 \alpha &= \frac{4}{5} \\ \sin \alpha &= \frac{2\sqrt{5}}{5} \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \cos^2 \alpha &= \frac{1}{5} \\ \cos \alpha &= \frac{\sqrt{5}}{5} \end{aligned}$$

Dosadíme do vzorce (2.1).

$$\begin{aligned} x &= \frac{\sqrt{5}}{5}\left(x' - 2y'\right) \\ y &= \frac{\sqrt{5}}{5}\left(2x' + y'\right) \end{aligned} \tag{2.14}$$

Dosadíme do původní rovnice.

$$\begin{aligned} 4 \cdot \frac{1}{5}\left(x' - 2y'\right)^2 - 4 \cdot \frac{1}{5}\left(x' - 2y'\right)\left(2x' + y'\right) + \frac{1}{5}\left(2x' + y'\right)^2 & \\ +4 \cdot \frac{\sqrt{5}}{5}\left(x' - 2y'\right) - 2 \cdot \frac{\sqrt{5}}{5}\left(2x' + y'\right) - 3 = 0 & \quad / \cdot 5 \\ 4\left(x'^2 - 4x'y' + 4y'^2\right) - 4\left(2x'^2 - 3x'y' - 2y'^2\right) + \left(4x'^2 + 4x'y' + y'^2\right) & \\ +4\sqrt{5}\left(x' - 2y'\right) - 2\sqrt{5}\left(2x' + y'\right) - 15 = 0 & \\ 4x'^2 - 16x'y' + 16y'^2 - 8x'^2 + 12x'y' + 8y'^2 + 4x'^2 + 4x'y' + y'^2 & \\ +4\sqrt{5}x' - 8\sqrt{5}y' - 4\sqrt{5}x' - 2\sqrt{5}y' - 15 = 0 & \\ 25y'^2 - 10\sqrt{5}y' - 15 = 0 & \\ 5y'^2 - 2\sqrt{5}y' - 3 = 0 & \end{aligned}$$

$$D = 4 \cdot 5 - 4 \cdot 5 \cdot (-3) = 80$$

$$y'_{1,2} = \frac{2\sqrt{5} \pm 4\sqrt{5}}{10} = \begin{cases} \frac{3\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \end{cases}$$

$$\boxed{5\left(y' - \frac{3\sqrt{5}}{5}\right)\left(y' + \frac{\sqrt{5}}{5}\right) = 0}$$

dvě rovnoběžné přímky

$$p'_1: \quad y' = \frac{3\sqrt{5}}{5} \qquad\qquad p'_2: \quad y' = -\frac{\sqrt{5}}{5}$$

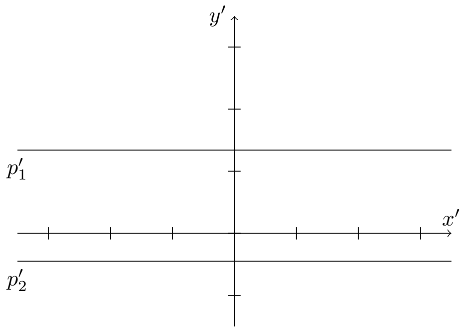
Určíme rovnice původních přímek.

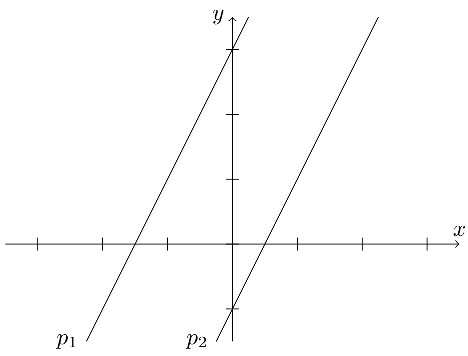
$$\begin{aligned} x' &= \frac{\sqrt{5}}{5} \cdot (x + 2y) \\ y' &= \frac{\sqrt{5}}{5} \cdot (-2x + y) \end{aligned} \tag{2.15}$$

$$p'_1: \quad y' = \frac{3\sqrt{5}}{5} \quad \overset{(2.15)}{\Longrightarrow} \quad \frac{3\sqrt{5}}{5} = \frac{\sqrt{5}}{5} \cdot (-2x + y) \quad \Longrightarrow \quad p_1: \quad 2x - y + 3 = 0$$

$$p'_2: \quad y' = -\frac{\sqrt{5}}{5} \quad \overset{(2.15)}{\Longrightarrow} \quad -\frac{\sqrt{5}}{5} = \frac{\sqrt{5}}{5} \cdot (-2x + y) \quad \Longrightarrow \quad p_2: \quad 2x - y - 1 = 0$$

$$\text{původní rovnice:} \qquad (2x - y + 3)(2x - y - 1) = 0$$





2.2 Určíme obecnou rovnici.

$$X = [x; y]$$

$$|XF| = 2 \cdot |Xp|$$

$$\sqrt{(x-2)^2 + (y+1)^2} = 2 \cdot \frac{|x + 2y - 3|}{\sqrt{1+4}}$$

$$(x-2)^2 + (y+1)^2 = 2^2 \cdot \frac{(x + 2y - 3)^2}{5}$$

$$5\left(x^2 + 4x + 4 + y^2 + 2y + 1\right) = 4\left(x^2 + 4y^2 + 9 + 4xy - 6x - 12y\right)$$

$$5x^2 - 20x + 20 + 5y^2 + 10y + 5 = 4x^2 + 16y^2 + 36 + 16xy - 24x - 48y$$

$$x^2 - 16xy - 11y^2 + 4x + 58y - 11 = 0$$

Určíme, o jaký úhel je třeba otočit soustavu souřadnic, aby osy kuželosečky byly rovnoběžné se souřadnicovými osami.

$$\begin{aligned} a &= 1 \\ 2b &= -16 \implies b = -8 \\ c &= -11 \end{aligned}$$

$$b \cdot \operatorname{tg}^2 \alpha + (a - c) \cdot \operatorname{tg} \alpha - b = 0$$

$$\begin{aligned} -8 \cdot \operatorname{tg}^2 \alpha + 12 \cdot \operatorname{tg} \alpha + 8 &= 0 \quad / : (-4) \\ 2 \cdot \operatorname{tg}^2 \alpha - 3 \cdot \operatorname{tg} \alpha - 2 &= 0 \end{aligned}$$

$$\operatorname{tg} \alpha = 2 \implies \sin \alpha = \frac{2\sqrt{5}}{5} \quad \wedge \quad \cos \alpha = \frac{\sqrt{5}}{5}$$

Dosadíme do vzorce (2.1).

$$\begin{aligned} x &= \frac{\sqrt{5}}{5} \left(x' - 2y'\right) \\ y &= \frac{\sqrt{5}}{5} \left(2x' + y'\right) \end{aligned} \tag{2.16}$$

Dosadíme do obecné rovnice.

$$\begin{aligned} \frac{1}{5} \left(x' - 2y'\right)^2 - 16 \cdot \frac{1}{5} \left(x' - 2y'\right) \left(2x' + y'\right) - 11 \cdot \frac{1}{5} \left(2x' + y'\right)^2 \\ +4 \cdot \frac{\sqrt{5}}{5} \left(x' - 2y'\right) + 58 \cdot \frac{\sqrt{5}}{5} \left(2x' + y'\right) - 11 = 0 \quad / \cdot 5 \\ \left(x'^2 - 4x'y' + 4y'^2\right) - 16 \left(2x'^2 - 3x'y' - 2y'^2\right) - 11 \left(4x'^2 + 4x'y' + y'^2\right) \\ +4\sqrt{5} \left(x' - 2y'\right) + 58\sqrt{5} \left(2x' + y'\right) - 55 = 0 \\ x'^2 - 4x'y' + 4y'^2 - 32x'^2 + 48x'y' + 32y'^2 - 44x'^2 - 44x'y' - 11y'^2 \\ +4\sqrt{5}x' - 8\sqrt{5}y' + 116\sqrt{5}x' + 58\sqrt{5}y' - 55 = 0 \\ -75x'^2 + 25y'^2 + 120\sqrt{5}x' + 50\sqrt{5}y' - 55 = 0 \quad / : 5 \\ -15x'^2 + 5y'^2 + 24\sqrt{5}x' + 10\sqrt{5}y' - 11 = 0 \end{aligned}$$

$$\begin{aligned} -15 \left(x' - \frac{4\sqrt{5}}{5}\right)^2 + 15 \cdot \frac{16 \cdot 5}{25} + 5 \left(y' + \sqrt{5}\right)^2 - 25 - 11 = 0 \\ -15 \left(x' - \frac{4\sqrt{5}}{5}\right)^2 + 5 \left(y' + \sqrt{5}\right)^2 = -12 \quad / : (-12) \end{aligned}$$

$$\boxed{\frac{\left(x' - \frac{4\sqrt{5}}{5}\right)^2}{\frac{4}{5}} - \frac{\left(y' + \sqrt{5}\right)^2}{\frac{12}{5}} = 1}$$

hyperbola

$$S' = \left[\frac{4\sqrt{5}}{5}; -\sqrt{5}\right] \qquad A' = \left[\frac{2\sqrt{5}}{5}; -\sqrt{5}\right]$$

$$a = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad B' = \left[\frac{6\sqrt{5}}{5}; -\sqrt{5}\right]$$

$$b = \frac{2\sqrt{3}}{\sqrt{5}} = \frac{2\sqrt{15}}{5} \qquad E' = \left[0; -\sqrt{5}\right]$$

$$e = \sqrt{a^2 + b^2} = \frac{4\sqrt{5}}{5} \qquad F' = \left[\frac{8\sqrt{5}}{5}; -\sqrt{5}\right]$$

$$o_1' : \quad y' = -\sqrt{5} \qquad o_2' : \quad x' = \frac{4\sqrt{5}}{5}$$

Určíme prvky původní hyperboly.

$$S' = \left[\frac{4\sqrt{5}}{5}; -\sqrt{5}\right] \overset{(2.16)}{\Longrightarrow} S = \left[\frac{14}{5}; \frac{3}{5}\right]$$

$$A' = \left[\frac{2\sqrt{5}}{5}; -\sqrt{5}\right] \overset{(2.16)}{\Longrightarrow} A = \left[\frac{12}{5}; -\frac{1}{5}\right]$$

$$B' = \left[\frac{6\sqrt{5}}{5}; -\sqrt{5}\right] \overset{(2.16)}{\Longrightarrow} B = \left[\frac{16}{5}; \frac{7}{5}\right]$$

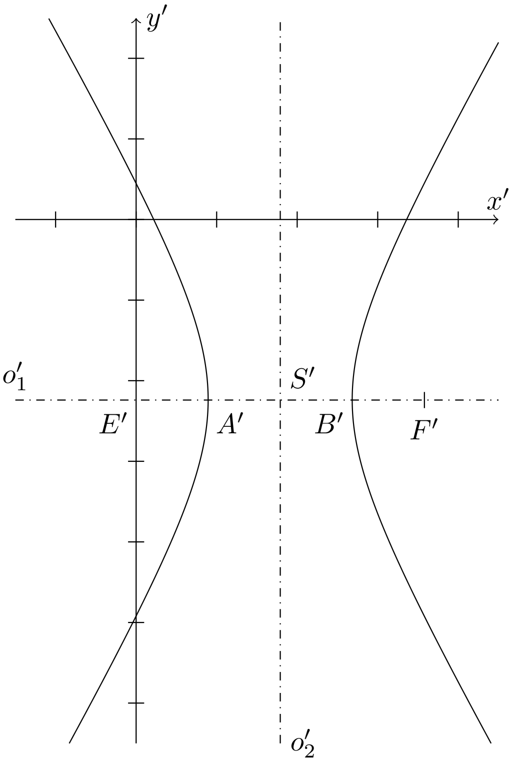
$$E' = \left[0; -\sqrt{5}\right] \overset{(2.16)}{\Longrightarrow} E = [2; -1]$$

$$F' = \left[\frac{8\sqrt{5}}{5}; -\sqrt{5}\right] \overset{(2.16)}{\Longrightarrow} F = \left[\frac{18}{5}; \frac{11}{5}\right]$$

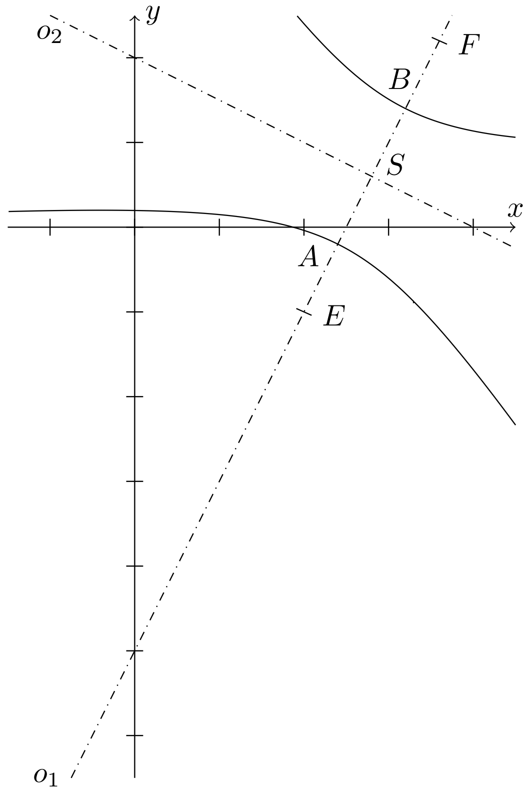
$$\begin{aligned} x' &= \frac{\sqrt{5}}{5} \cdot (x + 2y) \\ y' &= \frac{\sqrt{5}}{5} \cdot (-2x + y) \end{aligned} \tag{2.17}$$

$$o_1' : \quad y' = -\sqrt{5} \qquad \overset{(2.17)}{\Longrightarrow} \qquad o_1 : \quad 2x - y - 5 = 0$$

$$o_2' : \quad x' = \frac{4\sqrt{5}}{5} \qquad \overset{(2.17)}{\Longrightarrow} \qquad o_1 : \quad x + 2y - 4 = 0$$



y'
x'
o'_1
S'
E'
A'
B'
F'
o'_2



y
o_2
F
B
S
x
A
E
o_1

□

3 Klasifikace kuželoseček metodou invariantů

Úloha 3.1 *Metodou invariantů určete osy elipsy* $9x^2 - 4xy + 6y^2 + 6x - 8y + 2 = 0$.

Úloha 3.2 *Určete asymptoty hyperboly* $5x^2 + 12xy - 22x - 12y - 19 = 0$.

Úloha 3.3 *Nalezněte vrchol a osu paraboly* $12x^2 - 12xy + 3y^2 - 26x + 8y - \sqrt{5} = 0$.

Úloha 3.4 *Metodou invariantů klasifikujte kuželosečku zadanou obecnou rovnicí a určete její charakteristické prvky.*

a) $13x^2 + 13y^2 + 10xy + 42x - 6y - 27 = 0$
b) $11x^2 + 4y^2 + 24xy + 46x + 32y + 19 = 0$
c) $9x^2 + y^2 + 6xy + 6x + 22y - 20 = 0$

Řešení.

3.1

$$K = \begin{pmatrix} 9 & -2 & 3 \\ -2 & 6 & -4 \\ 3 & -4 & 2 \end{pmatrix} \qquad \det K = -50 \neq 0 \implies \text{regulární kuželosečka}$$

$$A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \qquad \det A = 50 > 0 \implies \text{elipsa}$$

Střed je řešením soustavy rovnic sestavené z prvních dvou řádků matice K.[1]

$$\left.\begin{aligned} 9x - 2y + 3 &= 0 \quad / \cdot 3 \\ -2x + 6y - 4 &= 0 \end{aligned}\right) + \implies 25x + 5 = 0 \implies \begin{aligned} x &= -\frac{1}{5} \\ y &= \frac{3}{5} \end{aligned}$$

$$S = \left[-\frac{1}{5}; \frac{3}{5}\right]$$

Osy elipsy procházejí středem, jejich směry určují vlastní vektory.

Vlastní čísla:

$$|A - \lambda E| = 0$$

$$\begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = (9-\lambda)(6-\lambda) - 4 = \lambda^2 - 15\lambda + 50$$
$$= (\lambda - 5)(\lambda - 10) = 0$$

$$\lambda_1 = 5$$
$$\lambda_2 = 10$$

Vlastní vektory:

$$\lambda_1 = 5: \quad \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{v_1} = (v_x; v_y) = (1; 2)$$

$$\lambda_2 = 10: \quad \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \qquad \vec{v_2} = (2; -1)$$

[1] Poznámka pro studenty deskriptivní geometrie: Střed S je vlastní pól nevlastní přímky, tj. je polárně sdružen s každým nevlastním bodem, tedy například i s body $(1;0;0) = X$ a $(0;1;0) = Y$.
Platí $S^T K X = 0$, $S^T K Y = 0$, kde $S = (x, y, 1)$.

Osy:

$$\vec{v_1} = (1;2) \quad \Longrightarrow \quad \vec{n_1} = (2;-1) \quad \Longrightarrow \quad 2\left(x - \left(-\frac{1}{5}\right)\right) - 1\left(y - \frac{3}{5}\right) = 0$$

$$\underline{\underline{o_1: \quad 2x - y + 1 = 0}}$$

$$\vec{v_2} = (2;-1) \quad \Longrightarrow \quad \vec{n_2} = (1;2) \quad \Longrightarrow \quad 1\left(x - \left(-\frac{1}{5}\right)\right) + 2\left(y - \frac{3}{5}\right) = 0$$

$$\underline{\underline{o_2: \quad x + 2y - 1 = 0}}$$

3.2

$$K = \begin{pmatrix} 5 & 6 & -11 \\ 6 & 0 & -6 \\ -11 & -6 & -19 \end{pmatrix} \qquad \det K = 1296 = 36^2 \neq 0 \Longrightarrow \text{regulární kuželosečka}$$

$$A = \begin{pmatrix} 5 & 6 \\ 6 & 0 \end{pmatrix} \qquad \det A = -36 < 0 \Longrightarrow \text{hyperbola}$$

Asymptoty jsou tečny v nevlastních bodech, musíme tedy najít nevlastní body hyperboly. V homogenních souřadnicích jsouto body $X = (x_0; y_0; 0)$ (přičemž $(x_0; y_0)$ je vektor ukazující do daného nevlastního bodu). Jsou to body hyperboly, tj. splňují následující rovnici.

$$X^T \cdot K \cdot X = 0$$

$$\begin{pmatrix} x_0 & y_0 \end{pmatrix} \cdot A \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$$

$$5x_0^2 + 12x_0y_0 = 0$$

$$x_0\,(5x_0 + 12y_0) = 0$$

$$x_0 = 0 \wedge y_0 \in \mathbb{R} \setminus \{0\} \quad \text{tj. např. bod } M_1 = (0;1;0) \quad \Longrightarrow \quad \vec{s_1} = (0;1)$$

$$x_0 = -\frac{12}{5}y_0 \quad \text{tj. např. bod } M_2 = (-12;5;0) \quad \Longrightarrow \quad \vec{s_2} = (-12;5)$$

Vektory $\vec{s_1}$ a $\vec{s_2}$ jsou vektory směřující do nevlastních bodů, tj. směrové vektory první a druhé asymptoty.

Dva způsoby hledání asymptot:

i) Najdeme střed, známe směrové vektory, asymptoty prochází středem.

$$\left(\begin{array}{cc|c} 5 & 6 & 11 \\ 6 & 0 & 6 \end{array}\right) \quad \Longrightarrow \quad \begin{array}{c} x = 1 \\ y = 1 \end{array} \quad \Longrightarrow \quad S = [1;1]$$

$$\vec{s_1} = (0;1) \quad \Longrightarrow \quad \underline{\underline{u_1: \quad x = 1}}$$

$$\vec{s_2} = (-12;5) \quad \Longrightarrow \quad \vec{n_2} = (5;12) \quad \Longrightarrow \quad 5\,(x-1) + 12\,(y-1) = 0$$

$$\Longrightarrow \quad \underline{\underline{u_2: \quad 5x + 12y - 17 = 0}}$$

ii) Hledáme asymptoty jako tečny v nevlastních bodech.
Rovnice tečny kuželosečky v bodě M:

$$X^T \cdot K \cdot M = 0$$
$$(\text{nebo } M^T \cdot K \cdot X = 0)$$

$$X^T \cdot K \cdot M_1 = 0$$
$$\begin{pmatrix} x & y & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 6 & -11 \\ 6 & 0 & -6 \\ -11 & -6 & -19 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$
$$6x - 6 = 0 \quad / : 6$$
$$\underline{\underline{u_1 : \quad x = 1}}$$

$$M_2^T \cdot K \cdot X = 0$$
$$\begin{pmatrix} -12 & 5 & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 & 6 & -11 \\ 6 & 0 & -6 \\ -11 & -6 & -19 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$
$$-30x - 72y + 102 = 0 \quad / : (-6)$$
$$\underline{\underline{u_2 : \quad 5x + 12y - 17 = 0}}$$

3.3

$$K = \begin{pmatrix} 12 & -6 & -13 \\ -6 & 3 & 4 \\ -13 & 4 & -\sqrt{5} \end{pmatrix} \qquad \det K = -75 \neq 0 \implies \text{regulární kuželosečka}$$
$$A = \begin{pmatrix} 12 & -6 \\ -6 & 3 \end{pmatrix} \qquad \det A = 0 \implies \text{parabola}$$

Směry osy paraboly a její vrcholové tečny jsou určeny vlastními vektory.
Vlastní čísla:

$$|A - \lambda E| = 0$$
$$\begin{vmatrix} 12-\lambda & -6 \\ -6 & 3-\lambda \end{vmatrix} = (12-\lambda)(3-\lambda) - 36 = \lambda^2 - 15\lambda$$
$$= \lambda(\lambda - 15) = 0$$

$$\lambda_1 = 15$$
$$\lambda_2 = 0$$

Nulové vlastní číslo určuje směr osy paraboly.
Vlastní vektory:

$$\lambda_1 = 15 : \quad \begin{pmatrix} -3 & -6 \\ -6 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \vec{v_1} = (2; -1) \quad \text{(směr vrcholové tečny)}$$
$$\lambda_2 = 0 : \quad \begin{pmatrix} 12 & -6 \\ -6 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad \vec{v_2} = (1; 2) \quad \text{(směr osy)}$$

Dva způsoby hledání osy a vrcholu:

i) Přes rovnici

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \cdot K \cdot \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = 0,$$

kde $(u_1; u_2)$ je směr vrcholové tečny.

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \cdot \begin{pmatrix} 12 & -6 & -13 \\ -6 & 3 & 4 \\ -13 & 4 & -\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$\underline{\underline{o: \quad 2x - y - 2 = 0}}$$

Vrchol je průsečík osy s parabolou.

$$y = 2x - 2$$

$$\begin{aligned} 12x^2 - 12x\,(2x-2) + 3\,(2x-2)^2 - 26x + 8\,(2x-2) - \sqrt{5} &= 0 \\ x^2\,(12 - 24 + 12) + x\,(24 - 24 - 26 + 16) + \left(12 - 16 - \sqrt{5}\right) &= 0 \\ -10x - 4 - \sqrt{5} &= 0 \\ x &= -\frac{4+\sqrt{5}}{10} \\ y &= -\frac{4+\sqrt{5}}{5} - 2 \end{aligned}$$

$$\boxed{V = \left[-\frac{4+\sqrt{5}}{10}; -\frac{14+\sqrt{5}}{5}\right]}$$

ii) Najdeme vrcholovou tečnu t_V jako přímku směru $\vec{v_1}$, která má s parabolou jediný společný bod — vrchol. Osa pak prochází vrcholem ve směru $\vec{v_2}$.

$$\vec{v_1} = (2; -1) \qquad \Longrightarrow \qquad \vec{n_1} = (1; 2) \qquad \Longrightarrow \qquad t_V: \quad x + 2y + c = 0$$

Hledáme jednobodový průnik tečny a paraboly:

$$x = -2y - c$$

$$\begin{aligned} 12\,(-2y-c)^2 - 12\,(-2y-c)\,y + 3y^2 - 26\,(-2y-c) + 8y - \sqrt{5} &= 0 \\ (48 + 24 + 3)\,y^2 + (48c + 12c + 52 + 8)\,y + \left(12c^2 + 26c - \sqrt{5}\right) &= 0 \\ 75y^2 + (60 + 60c)\,y + \left(12c^2 + 26c - \sqrt{5}\right) &= 0 \end{aligned}$$

$$\begin{aligned} D &= (60\,(1+c))^2 - 4 \cdot 75 \cdot \left(12c^2 + 26c - \sqrt{5}\right) \\ &= -600c + 3600 + 300\sqrt{5} \\ &= 300\left(-2c + 12 + \sqrt{5}\right) = 0 \iff c = \frac{12+\sqrt{5}}{2} \end{aligned}$$

$$y = \frac{-60\left(1+\frac{12+\sqrt{5}}{2}\right)}{2\cdot 75} = -\frac{14+\sqrt{5}}{5}$$
$$x = -2y - c = -\frac{-4+\sqrt{5}}{10}$$

$$\boxed{V = \left[-\frac{4+\sqrt{5}}{10}; -\frac{14+\sqrt{5}}{5}\right]}$$

Vrcholová tečna t_V:

$$x + 2y + \frac{12+\sqrt{5}}{2} = 0$$
$$t_V: \quad 2x + 4y + 12 + \sqrt{5} = 0$$

Osa paraboly o:

$$\vec{n_o} = (2;-1) \quad \wedge \quad V \in o$$

$$2x - y + c = 0$$
$$2\left(-\frac{4+\sqrt{5}}{10}\right) - \left(-\frac{14+\sqrt{5}}{5}\right) + c = 0 \quad / \cdot 5$$
$$-4 - \sqrt{5} + 14 + \sqrt{5} + 5c = 0$$
$$5c = -10$$
$$c = -2$$

$$\underline{\underline{o: \quad 2x - y - 2 = 0}}$$

3.4 a)

$$K = \begin{pmatrix} 13 & 5 & 21 \\ 5 & 13 & -3 \\ 21 & -3 & -27 \end{pmatrix} \qquad \det K = -10\,368 \neq 0 \implies \text{regulární kuželosečka}$$
$$A = \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} \qquad \det A = 144 > 0 \implies \text{elipsa}$$

Vlastní čísla:

$$|A - \lambda E| = 0$$
$$\begin{vmatrix} 13-\lambda & 5 \\ 5 & 13-\lambda \end{vmatrix} = (13-\lambda)^2 - 5^2$$
$$= (13-\lambda-5)(13-\lambda+5)$$
$$= (8-\lambda)(18-\lambda) = 0$$

$$\lambda_1 = 8$$
$$\lambda_2 = 18$$

Vlastní směry:

$$\lambda_1 = 8: \qquad \begin{pmatrix} 13-8 & 5 \\ 5 & 13-8 \end{pmatrix} \sim \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \qquad \vec{v_1} = (1; -1)$$

$$\lambda_2 = 18: \qquad \begin{pmatrix} 13-18 & 5 \\ 5 & 13-18 \end{pmatrix} \sim \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \qquad \vec{v_2} = (1; 1)$$

Střed elipsy:

$$\left(\begin{array}{cc|c} 13 & 5 & -21 \\ 5 & 13 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 13 & 5 & -21 \\ 0 & 144 & 144 \end{array}\right) \implies \begin{array}{c} x = -2 \\ y = 1 \end{array} \implies S = [-2; 1]$$

Rovnice elipsy:

$$\begin{aligned} \lambda_1 x^2 + \lambda_2 y^2 + \frac{\det K}{\det A} &= 0 \\ 8x^2 + 18y^2 - 72 &= 0 \\ \frac{x^2}{\frac{72}{8}} + \frac{y^2}{\frac{72}{18}} &= 1 \\ \frac{x^2}{3^2} + \frac{y^2}{2^2} &= 1 \end{aligned}$$

Prvky elipsy:

$$\begin{aligned} a &= 3 \\ b &= 2 \\ e &= \sqrt{9-4} = \sqrt{5} \end{aligned}$$

$$A, B: \quad S \pm a \cdot \frac{\vec{v_1}}{||\vec{v_1}||} = [-2; 1] \pm \left(\frac{3}{\sqrt{2}}; -\frac{3}{\sqrt{2}}\right)$$

$$A = \left[-2 - \frac{3\sqrt{2}}{2}; 1 + \frac{3\sqrt{2}}{2}\right]$$

$$B = \left[-2 + \frac{3\sqrt{2}}{2}; 1 - \frac{3\sqrt{2}}{2}\right]$$

$$C, D: \quad S \pm b \cdot \frac{\vec{v_2}}{||\vec{v_2}||} = [-2; 1] \pm \left(\frac{2}{\sqrt{2}}; \frac{2}{\sqrt{2}}\right)$$

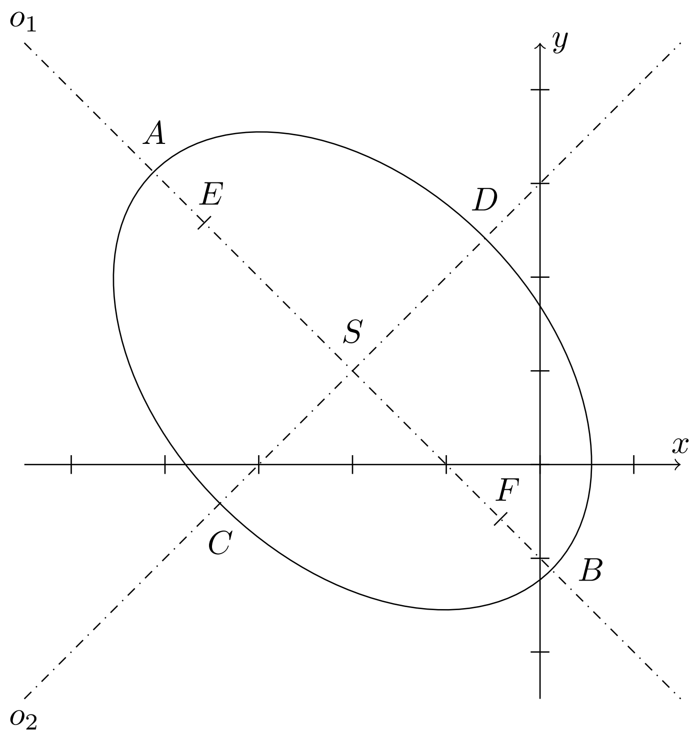
$$C = \left[-2 - \frac{2}{\sqrt{2}}; 1 - \frac{2}{\sqrt{2}}\right]$$

$$D = \left[-2 + \frac{2}{\sqrt{2}}; 1 + \frac{2}{\sqrt{2}}\right]$$

$$E, F: \quad S \pm e \cdot \frac{\vec{v_1}}{||\vec{v_1}||} = [-2; 1] \pm \left(\frac{\sqrt{5}}{\sqrt{2}}; -\frac{\sqrt{5}}{\sqrt{2}}\right)$$

$$E = \left[-2 - \sqrt{\frac{5}{2}}; 1 + \sqrt{\frac{5}{2}}\right]$$

$$F = \left[-2 + \sqrt{\frac{5}{2}}; 1 - \sqrt{\frac{5}{2}}\right]$$



b)

$$K = \begin{pmatrix} 11 & 12 & 23 \\ 12 & 4 & 16 \\ 23 & 16 & 19 \end{pmatrix} \qquad \det K = 2000 \neq 0 \implies \text{regulární kuželosečka}$$

$$A = \begin{pmatrix} 11 & 12 \\ 12 & 4 \end{pmatrix} \qquad \det A = -100 < 0 \implies \text{hyperbola}$$

Vlastní čísla:

$$\begin{aligned} |A - \lambda E| &= 0 \\ \begin{vmatrix} 11-\lambda & 12 \\ 12 & 4-\lambda \end{vmatrix} &= (11-\lambda)(4-\lambda) - 12^2 \\ &= \lambda^2 - 15\lambda - 100 \\ &= (\lambda - 20)(\lambda + 5) = 0 \\ \\ &\lambda_1 = 20 \\ &\lambda_2 = -5 \end{aligned}$$

Vlastní směry:

$$\lambda_1 = 20: \qquad \begin{pmatrix} 11-20 & 12 \\ 12 & 4-20 \end{pmatrix} \sim \begin{pmatrix} -9 & 12 \\ 12 & -16 \end{pmatrix} \qquad \vec{v_1} = (4; 3)$$

$$\lambda_2 = -5: \qquad \begin{pmatrix} 11+5 & 12 \\ 12 & 4+5 \end{pmatrix} \sim \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} \qquad \vec{v_2} = (3; -4)$$

Střed hyperboly:

$$\left(\begin{array}{cc|c} 11 & 12 & -23 \\ 12 & 4 & -16 \end{array}\right) \sim \left(\begin{array}{cc|c} 11 & 12 & -23 \\ -25 & 0 & 25 \end{array}\right) \implies \begin{array}{c} x = -1 \\ y = -1 \end{array} \implies S = [-1; -1]$$

Rovnice hyperboly:

$$\begin{aligned}\lambda_1 x^2 + \lambda_2 y^2 + \frac{\det K}{\det A} &= 0\\ 20x^2 - 5y^2 - 20 &= 0\\ \frac{x^2}{1^2} - \frac{y^2}{2^2} &= 1\end{aligned}$$

Prvky hyperboly:

$$\begin{aligned}a &= 1\\ b &= 2\\ e &= \sqrt{1+4} = \sqrt{5}\end{aligned}$$

$$A, B: \quad S \pm a \cdot \frac{\vec{v_1}}{||\vec{v_1}||} = [-1;-1] \pm \left(\frac{4}{5}; \frac{3}{5}\right)$$

$$A = \left[-\frac{1}{5}; -\frac{2}{5}\right]$$

$$B = \left[-\frac{9}{5}; -\frac{8}{5}\right]$$

$$E, F: \quad S \pm a \cdot \frac{\vec{v_1}}{||\vec{v_1}||} = [-1;-1] \pm \left(\frac{4\sqrt{5}}{5}; \frac{3\sqrt{5}}{5}\right)$$

$$E = \left[-1 + \frac{4\sqrt{5}}{5}; -1 + \frac{3\sqrt{5}}{5}\right]$$

$$F = \left[-1 - \frac{4\sqrt{5}}{5}; -1 - \frac{3\sqrt{5}}{5}\right]$$

Asymptoty:

Nevlastní body (směry):

$$\begin{aligned}\begin{pmatrix} x_0 & y_0 \end{pmatrix} \cdot A \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= 0\\ \begin{pmatrix} x_0 & y_0 \end{pmatrix} \cdot \begin{pmatrix} 11 & 12 \\ 12 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= 0\\ 11x_0^2 + 4y_0^2 + 24x_0y_0 &= 0\end{aligned}$$

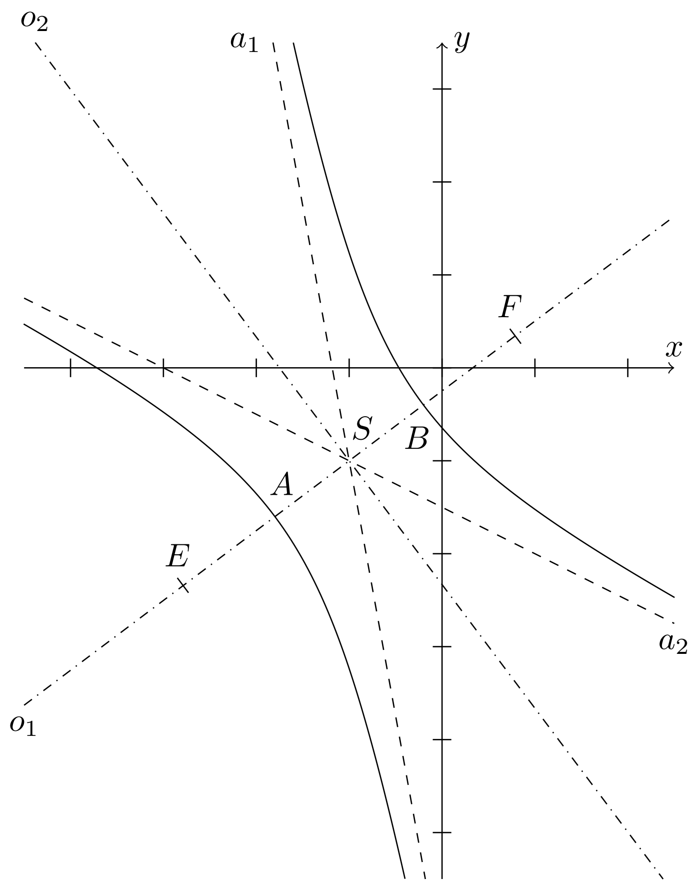
Zvolme $x_0 = 1$, pak:

$$y_0 = \frac{-24 \pm \sqrt{24^2 - 4 \cdot 4 \cdot 11}}{2 \cdot 4} = \begin{cases} -\frac{1}{2} => \vec{s_1} = (2; -1) \\ -\frac{11}{2} => \vec{s_2} = (2; -11) \end{cases}$$

Asymptoty prochází středem:

$$\begin{aligned}S &= [-1;-1]\\ \vec{n_1} &= (1;2)\\ \vec{n_2} &= (11;2)\end{aligned}$$

$$\begin{aligned} a_1: &\quad (x+1) + 2(y+1) = 0 &\implies&\quad a_1: \quad x + 2y + 3 = 0\\ a_2: &\quad 11(x+1) + 2(y+1) = 0 &\implies&\quad a_2: \quad 11x + 2y + 13 = 0\end{aligned}$$



c)

$$K = \begin{pmatrix} 9 & 3 & 3 \\ 3 & 1 & 11 \\ 3 & 11 & -20 \end{pmatrix} \qquad \det K = -900 \neq 0 \implies \text{regulární kuželosečka}$$

$$A = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \qquad \det A = 0 \implies \text{parabola}$$

Vlastní čísla:

$$|A - \lambda E| = 0$$

$$\begin{vmatrix} 9-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (9-\lambda)(1-\lambda) - 9$$
$$= \lambda - 10\lambda$$
$$= \lambda(\lambda - 10) = 0$$

$$\lambda_1 = 10$$
$$\lambda_2 = 0$$

Vlastní směry:

$$\lambda_1 = 10: \quad \begin{pmatrix} 9-10 & 3 \\ 3 & 1-10 \end{pmatrix} \sim \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \quad \vec{v_1} = (3;1)$$

$$\lambda_2 = 0: \quad \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \qquad \vec{v_2} = (1;-3) \quad \text{(směr osy)}$$

Vrchol:

Vrcholová tečna:

$$\vec{s_{t_V}} = \vec{v_1} = (3; 1) \qquad \vec{n_{t_V}} = (1; -3) \qquad t_V: \quad x - 3y + c = 0$$

Hledáme jednobodový průnik tečny a paraboly:

$$x = 3y - c$$

$$\begin{aligned} 9\,(3y-c)^2 + y^2 + 6\,(3y-c)\,y + 6\,(3y-c) + 22y - 20 &= 0 \\ 100y^2 + (-60c+40)\,y + \left(9c^2 - 6c - 20\right) &= 0 \end{aligned}$$

$$\begin{aligned} D &= 100\,(4-6c)^2 - 4 \cdot 100 \cdot \left(9c^2 - 6c - 20\right) \\ &= 4 \cdot 100 \cdot (-6c + 24) \\ &= 40^2\,(4 - c) \end{aligned}$$

$$\begin{aligned} D = 0 &\iff c = 4 \\ v: &\quad x - 3y + 4 = 0 \end{aligned}$$

$$\begin{aligned} y &= \frac{60 \cdot 4 - 40}{2 \cdot 100} = 1 \\ x &= 3 \cdot 1 - 4 = -1 \end{aligned}$$

$$V = [-1; 1]$$

Orientace paraboly:

$$D > 0 \text{ (2 kořeny)} \iff c < 4$$

parabola leží „pod" tečnou t_V

Rovnice paraboly:

$$\lambda_1 y^2 = 2p'x$$

$$p' = \sqrt{\frac{-\det K}{\lambda_1}} = \sqrt{\frac{-900}{10}} = 3\sqrt{10}$$

$$\begin{aligned} 10y^2 &= 2 \cdot 3\sqrt{10}x \\ y^2 &= 2 \cdot \frac{3}{\sqrt{10}}x \end{aligned}$$

$$p = \frac{3}{\sqrt{10}}$$

Osa paraboly:

$$\begin{aligned} \vec{n_o} = (3; 1) &\quad \wedge \quad V \in o \\ o: &\quad 3x + y + 2 = 0 \end{aligned}$$

Ohnisko paraboly:

$$\begin{aligned} F &= V + \frac{p}{2} \cdot \frac{\vec{v_2}}{||\vec{v_2}||} \\ &= [-1;1] + \frac{3}{2\sqrt{10}} \cdot \frac{(1;-3)}{\sqrt{1+9}} \\ &= \left[-\frac{17}{20}; \frac{11}{20}\right] \end{aligned}$$

Řídící přímka:

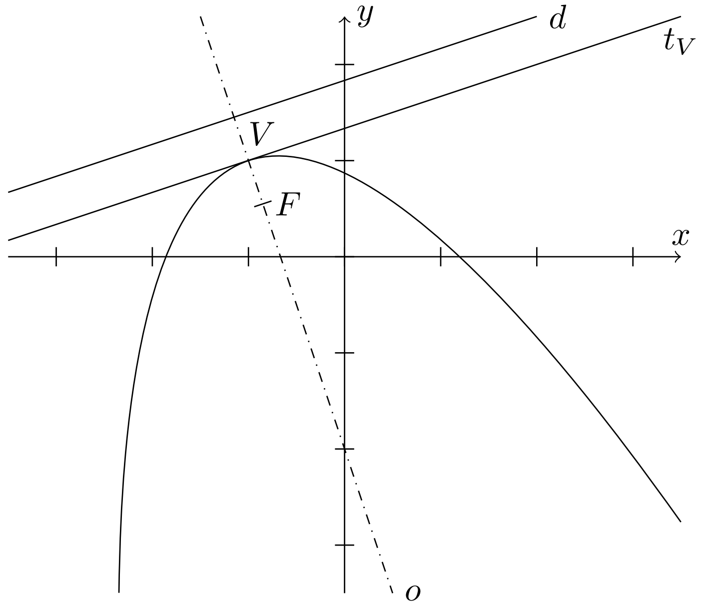
$$\begin{aligned} d &\parallel t_V \\ d: \quad & x - 3y + c = 0 \end{aligned}$$

Prochází bodem D:

$$\begin{aligned} D &= V - \frac{p}{2} \cdot \frac{\vec{v_2}}{||\vec{v_2}||} \\ &= \left[-\frac{23}{20}; \frac{29}{20}\right] \end{aligned}$$

$$\begin{aligned} -\frac{23}{20} - 3 \cdot \frac{29}{20} + c &= 0 \\ c &= \frac{110}{20} = \frac{11}{2} \end{aligned}$$

$$d: \quad x - 3y + \frac{11}{2}$$



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