Mathematical Modelling in Physics II

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SIMPLIFIED MODELS OF FLUID FLOW

Quantitative behaviour of moving fluids:

- measurements (wind tunnels): time consuming, expensive
- mathematical and computer modelling

PDE's describing the flow:

- quantitative research (uniqueness of a solution)
- numerical simulation (they constitute CFD (=Computation Fluid Dynamics)

The goal of CFD is to simulate the flow with the aid of numerical methods and computers in order to obtain results comparable with measurements.

1.1 Visualization of flow on the basis of computations

Isolines: curves on which a given quantity attains constant values.
Streamlines: such curves that at each point the tangent is parallel to the velocity vector.
Velocity vectors: arrows with direction of the velocity and length proportional to the magnitude of the velocity.

1.2 Basic equations

We shall assume that quantities describing the flow are suffciently smooth.

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \vec{v}) = 0 \tag{1.2.1}$$

in $\mathcal{M} = \{(x,t) : x \in \Omega_t, t \in (0,T)\}$.
Navier-Stokes equations of motion:

$$\frac{\partial\,(\rho v_i)}{\partial t} + \mathrm{div}\ (\rho \vec{v} v_i) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\,(\lambda \mathrm{div}\,\vec{v}) + \\ + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left\{ \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\}, \tag{1.2.2}$$

$i = 1, 2, 3$, where λ, μ are viscosity coefficients. We assume that $\mu > 0$ and usually set $3\lambda + 2\mu = 0$.
Energy equation: is not needed in the case of incompressible fluids (our case).

Simplification:
a) The fluid is incompressible: $\rho = const > 0$.

Assumption: $\vec{v}, p$ and all functions we use are sufficiently smooth.
Then (1.2.1) is equivalent to

$$\operatorname{div} \vec{v} = 0 \quad \text{in } \mathcal{M} \tag{1.2.3}$$

and (1.2.3) is equivalent to

$$\frac{\partial v_i}{\partial t} + \operatorname{div}\,(v_i \vec{v}) = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + 0 + \underbrace{\sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left\{\frac{\mu}{\rho}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right\}}_{(*)}. \tag{1.2.4}$$

We call μ dynamical viscosity and $\nu := \frac{\mu}{\rho}$ kinematical viscosity.

b) Further assumption: Let $\mu = const \Rightarrow \nu = const$.
Then term in $(*)$ takes the form:

$$\nu \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_j^2} + \nu \sum_{j=1}^{3} \frac{\partial^2 v_j}{\partial x_i \partial x_j} = \nu \Delta v_i + \nu \frac{\partial}{\partial x_i} \underbrace{\operatorname{div} \vec{v}}_{=0}.$$

We know that

$$\operatorname{div}\,(v_i \vec{v}) = v_i \operatorname{div} \vec{v} + (\vec{v} \cdot \nabla) v_i = (\vec{v} \cdot \nabla) v_i,$$

and thus equation (1.2.4) can be written as

$$\frac{\partial v_i}{\partial t} + (\vec{v} \cdot \nabla) v_i = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta v_i. \tag{1.2.5}$$

The vector form of equations (1.2.5)

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} = \vec{f} - \frac{1}{\rho} \nabla p + \nu \Delta \vec{v}. \tag{1.2.6}$$

Together with

$$\operatorname{div} \vec{v} = 0 \quad \text{in } \mathcal{M},$$

we have a complete system: four equations for four unknows v_1, v_2, v_3 and p .

1.3 Initial conditions

$$\vec{v}(\vec{x}, 0) = \vec{v}^0(\vec{x}), \quad \vec{x} \in \Omega_0. \tag{1.3.7}$$

1.4 Boundary conditions

Boundary conditions are based on the fact that viscous fluid adheres to walls. Therefore, $\vec{v} = 0$ on fixed impermeable walls.
This condition is generalized so that we set

$$\vec{v} = \vec{g}, \tag{1.4.8}$$

where $\vec{g}$ is a given vector function on $\partial\Omega_t$, $t \in (0, T)$.
Sometimes we need 'softer' boundary conditions for the outlet:

$$-(p - p_{ref})\vec{n} + \frac{\partial \vec{v}}{\partial \vec{n}} = 0 \quad \text{at outlet}, \tag{1.4.9}$$

where $\vec{n}$ is unit outler normal to $\partial\Omega_t$, p_{ref} is a prescribed pressure at outlet (e.g. atmospheric pressure).

1.5 Incompressible Navier-Stokes problem

Let the following data be prescribed: $\vec{f} : \mathcal{M} \to \mathbb{R}, \rho > 0, \rho = \text{const}, \nu > 0, \nu = \text{const}, \vec{v}^0, \vec{g}$. Find $\vec{v}, p$ such that

- $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} = \vec{f} - \frac{\nabla p}{\rho} + \nu \Delta \vec{v} \quad \text{in} \;\; \mathcal{M} = \{(x, t) : x \in \Omega_t, t \in (0, T)\}$,
- $\operatorname{div} \vec{v} = 0 \;\; \text{in} \;\; \mathcal{M}$.
- Conditions (1.3.7) and (1.4.8) are satisfied.

Example 1 *Flow past an airfoil in a wind tunnel.*

1.6 Euler equations of motion

Often $0 < \nu << 1$. For example, $\nu = 1.007 \cdot 10^{-6}$ for water at 20°C and $\nu = 1.5 \cdot 10^{-5}$ for air at 20°C. Therefore, we set $\nu = 0$. Then we get the Euler equations (E.E.)

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} = \vec{f} - \frac{1}{\rho}\nabla p \quad \text{in } \mathcal{M}. \tag{1.6.10}$$

They are considered together with the continuity equation (C.E.):

$$\operatorname{div} \vec{v} = 0. \tag{1.6.11}$$

This system is equipped with the initial condition (1.3.7).
Boundary conditions:
On impermeable fixed wall:

$$\vec{v} \cdot \vec{n} = 0,$$

where $\vec{n}$ is unit outer normal to $\partial\Omega$.
On inlet/outlet:

$$\vec{v} \cdot \vec{n} = \varphi,$$

where φ is a given function on $\partial\Omega$. Thus, we can consider the condition

$$\vec{v} \cdot \vec{n} = \varphi \tag{1.6.12}$$

on the whole boundary.

1.7 Stationary flow

All quantities describing the flow are time independent ($\frac{\partial}{\partial t} = 0$): $\vec{v} = \vec{v}(x), p = p(x), \vec{f} = \vec{f}(x)$ and the domain occupied by the fluid is time independent: $\Omega_t = \Omega$. Model describing stationary inviscid incompressible flow:

$$\text{(E.E.)} \qquad (\vec{v} \cdot \nabla)\,\vec{v} = \vec{f} - \frac{\nabla p}{\rho} \quad \text{in } \Omega, \tag{1.7.13}$$

$$\text{(C.E.)} \qquad \operatorname{div} \vec{v} = 0 \quad \text{in} \quad \Omega. \tag{1.7.14}$$

To this system we add the boundary condition (1.6.12).

1.8 Irrotational flow

$$\operatorname{rot} \vec{v} = 0 \quad \text{in } \Omega. \tag{1.8.15}$$

Physical meaning of $\operatorname{rot} \vec{v} = 0$: very small fluid volumes move so that they are translated, deformed but do not rotate.

Simplification of the E.E. under $\underline{\operatorname{rot} \vec{v} = 0}$:

Lemma 1 *Let $\vec{v} \in C^1(\Omega)$. Then*

$$(\vec{v} \cdot \nabla)\vec{v} = \sum_{j=1}^{3} v_j \frac{\partial \vec{v}}{\partial x_j} = \frac{1}{2}\nabla\, |\vec{v}|^2 - \vec{v} \times \operatorname{rot} \vec{v} \quad \textit{in } \Omega.$$

Proof *(homework) rewrite equation in components.*

Definition 1 *We say that the field $\vec{f} : \Omega \to \mathbb{R}^N$ is potential if there exists $U \in C^1(\Omega)$ such that*

$$\vec{f} = \operatorname{grad} U.$$

Theorem 1 (***Bernoulli's equation***) *Let $\vec{v} \in C^1(\Omega)$, $p \in C^1(\Omega)$ satisfy (1.7.13) and $\operatorname{rot} \vec{v} = 0$ in a domain Ω. Let us assume that such $U \in C^1(\Omega)$ exists that $\vec{f} = \operatorname{grad} U$ in Ω. Then*

$$\frac{p}{\rho} + \frac{1}{2}\,|\vec{v}|^2 - U = \text{const} \quad \textit{in} \quad \Omega. \tag{1.8.16}$$

Proof If $\operatorname{rot} \vec{v} = 0$, then $\vec{v}$ and p satisfy:

$$\begin{aligned}
\text{(E.E.)} \Leftrightarrow & \frac{1}{2}\nabla\,|\vec{v}|^2 - \vec{v} \times \operatorname{rot} \vec{v} = \nabla U - \frac{\nabla p}{\rho} \\
\Leftrightarrow & \nabla\left(\frac{p}{\rho} + \frac{1}{2}\,|\vec{v}|^2 - U\right) = 0 \quad \text{in } \Omega \\
\Leftrightarrow & \frac{p}{\rho} + \frac{1}{2}\,|\vec{v}|^2 - U = \text{const} \quad \text{in } \Omega.
\end{aligned}$$

Remark 1.1 1. *If the axis x_3 is perpendicular to the Earth, then the gravity force is $\vec{f} = (0, 0, -g)$ and the potential has the form $U = -gx_3$, where g is a gravity constant.*

2. *If $U \equiv 0$, then $\vec{f} = 0$. In this case it follows from (B.E.) that for $|\vec{v}|$ large the pressure p is small and for $|\vec{v}|$ small p is large.*
3. *From (B.E.) we can expres p as a function of $|\vec{v}|^2$ and U.*
4. *The constant in (B.E.) can be determined on the basis of given $|\vec{v}|^2$, p and U at a fixed point.*

1.9 Mathematical formulation of stationary inviscid irrotational flow

Now we consider stationary inviscid incompressible and irrotational flow (I.F.). We consider the problem to find $\vec{v}$ and p such that $\vec{v} \in C^1(\Omega) \cap C(\overline{\Omega})$, $p \in C^1(\Omega)$ and

$$\begin{aligned} \operatorname{div} \vec{v} &= 0 \quad \text{in } \Omega, \\ \operatorname{rot} \vec{v} &= 0 \quad \text{in } \Omega, \\ p &= \rho \left(\text{const} - \frac{1}{2} |\vec{v}|^2 + U \right) \quad \text{in } \Omega, \\ \vec{v} \cdot \vec{n} &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

Definition 2 *We say that $\Phi : \Omega \to \mathbb{R}$ is a velocity potential, if $\Phi \in C^2(\Omega)$ and $\vec{v} = \operatorname{grad} \Phi$ in Ω.*

Lemma 2 *Let Φ be a potential to $\vec{v}$ in Ω. Then* $\operatorname{rot} \vec{v} = 0$.

Proof *Components of* $\operatorname{rot} \vec{v}$

$$\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = \frac{\partial^2 \Phi}{\partial x_j \partial x_i} - \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = 0,$$

in Ω, since $\Phi \in C^2(\Omega)$.

1.9.1 *Transformation of the problem for $\vec{v}$ with the aid of the velocity potential Φ*

Let us assume that the velocity potential Φ to $\vec{v}$ exists. (Then $\operatorname{rot} \vec{v} = 0$).
From (C.E.) we have

$$0 = \operatorname{div} \vec{v} = \operatorname{div} (\nabla \Phi) = \Delta \Phi \quad \text{in } \Omega.$$

From the boundary condition we have

$$\varphi = \vec{v} \cdot \vec{n} = \nabla \Phi \cdot \vec{n} \equiv \frac{\partial \Phi}{\partial \vec{n}} \quad \text{on } \partial\Omega.$$

We are interested in the question, whether the following implication holds:
$\vec{v} \in C^1(\Omega)$, $\operatorname{rot} \vec{v} = 0$ in $\Omega \Longrightarrow$ there exists $\Phi \in C^2(\Omega)$ such that $\nabla\Phi = \vec{v}$ in Ω.

Definition 3 1. We say that φ is a *curve* (in Ω), if $\varphi : [a,b] \to \Omega$ ($[a,b]$ is a closed interval) and φ is continuous.

2. A curve $\varphi : [a,b] \to \Omega$ is *smooth*, if $\varphi \in C^1([a,b])$ and $\varphi'(\tau) \neq 0$ for all $\tau \in [a,b]$.
3. A curve $\varphi : [a,b] \to \Omega$ is *piecewise smooth*, if there exists a partition $a = a_0 < a_1 < \ldots < a_n = b$ such that $\varphi|_{[a_i,a_{i+1}]}$ is a smooth curve for all $i = 1, 2, \ldots, n-1$.
4. A curve $\varphi : [a,b] \to \Omega$ is *piecewise linear*, if there exists a partition $a = a_0 < a_1 < \ldots < a_n = b$, if the mapping $\tau \in [a_i, a_{i+1}] \to \varphi(\tau)$ is linear for all $i = 1, 2, \ldots, n-1$.

A curve $\varphi : [a,b] \to \Omega$ is *closed*, if $\varphi(a) = \varphi(b)$.
i.p. φ = *initial point* of $\varphi = \varphi(a)$, t.p. φ = *terminal point* of $\varphi = \varphi(b)$.
Geometric image of φ

$$\langle\varphi\rangle = \{\varphi(\tau) : \tau \in [a,b]\}.$$

Unit tangent to φ at the point $\varphi(\tau)$: $\vec{t}(\tau) = \frac{\varphi'(\tau)}{|\varphi'(\tau)|}$.
Element of φ: $\mathrm{d}s = |\varphi'(\tau)|\,\mathrm{d}\tau$, so $\vec{t}\mathrm{d}s = \varphi'(\tau)\mathrm{d}\tau$.

Definition 4 We say that a domain $\Omega \subset \mathbb{R}^3$ is *simply connected*, if for each smooth closed curve $\varphi : [a,b] \to \Omega$ there exist a mapping $H : [a,b] \times [0,1] \to \Omega$, $H \in C^1([a,b] \times [0,1])$ and a point $g \in \Omega$ such that

$$\begin{aligned} &H(a,s) = H(b,s) \quad \forall s \in [0,1],\\ &H(\tau,0) = \varphi(\tau) \quad \forall \tau \in [a,b],\\ &H(\tau,1) = g \quad \forall \tau \in [a,b]. \end{aligned}$$

Simply, each smooth closed curve in Ω can be smoothly transformed in Ω to a point.

Definition 5 We say that a domain $\Omega \subset \mathbb{R}^2$ is simply connected, if $\mathbb{R}^2 - \Omega$ does not contain any bounded component.

Example 2 $\mathbb{R}^2$ *is simply connected. Circle is simply connected. In general, any convex set is simply connected.*

Definition 6 Let $\Omega \subset \mathbb{R}^N$, $\vec{v} : \Omega \to \mathbb{R}^N$, $\vec{v} \in C(\Omega)$ and let $\varphi : [a,b] \to \Omega$ be a piecewise smooth curve. Then we define the circulation of $\vec{v}$ along φ

$$\gamma = \int_{\varphi} \vec{v} \cdot \vec{t}\ \mathrm{d}s = \int_a^b \vec{v}(\varphi(\tau)) \cdot \varphi'(\tau)\ \mathrm{d}\tau. \tag{1.9.17}$$

Theorem 2 *Let* $\vec{v} : \Omega \to \mathbb{R}^N$, $\vec{v} \in C^1(\Omega)(N = 2$ or $3)$. *Then there exists a potential* Φ *in* Ω *to* $\vec{v}$ *if and only if*

$$\int_\varphi \vec{v} \cdot \vec{t} \,\mathrm{d}s = 0 \tag{1.9.18}$$

for arbitrary piecewise linear closed curve φ *in* Ω.

Proof Let, e.g. $N = 3$.
a) Let (1.9.18) be valid. We prove that there exists $\Phi \in C^2(\Omega)$ such that $\nabla\Phi = \vec{v}$ in Ω. Let $x_0 \in \Omega$ be arbitrary fixed, $\Phi(x_0) \in \mathbb{R}$ arbitrary fixed. Then we set

$$\Phi(x) = \Phi(x_0) + \int_{\varphi(x_0,x)} \vec{v} \cdot \vec{t} \,\mathrm{d}S, \quad x \in \Omega,$$

where $\varphi(x_0, x)$ is a piecewise linear curve in Ω with i.p.$\varphi(x_0, x) = x_0$ and t.p.$\varphi(x_0, x) = x$.
(i) Now we prove that the above integral is independent of the choice of $\varphi(x_0, x)$. Let us consider two piecewise linear curves $\varphi_i = \varphi_i(x_0, x)$, $i = 1, 2$, in Ω. Then $\varphi_1 - \varphi_2$ is a closed piecewise linear curve in Ω and, in view of (1.9.18),

$$\int_{\varphi_1} \vec{v} \cdot \vec{t} \,\mathrm{d}S - \int_{\varphi_2} \vec{v} \cdot \vec{t} \,\mathrm{d}S = \int_{\varphi_1 - \varphi_2} \vec{v} \cdot \vec{t} \,\mathrm{d}S = 0 \quad (1.9.18).$$

(ii) Further, we shall show that $\nabla\Phi(x) = \vec{v}$ for each $x \in \Omega$. Let us prove, e.g. $\frac{\partial\Phi}{\partial x_1}(x) = v_1(x)$ for all $x \in \Omega$. Let $x = (x_1, x_2, x_3)$. Then $x(h) = (x_1 + h, x_2, x_3) \in \Omega$, if h is sufficiently small. We have

$$\frac{\partial\Phi}{\partial x_1}(x) = \lim_{h\to 0} \frac{\Phi(x_1 + h, x_2, x_3) - \Phi(x_1, x_2, x_3)}{h}, \tag{1.9.19}$$

where $\Phi(x) = \Phi(x_0) + \int_{\varphi(x_0,x))} \vec{v} \cdot \vec{t} \,\mathrm{d}S$.
It is clear that

$$\Phi(x(h)) = \Phi(x) + \int_{x_1}^{x_1+h} v_1(\xi, x_2, x_3)\mathrm{d}\xi.$$

If we make the substitution $h = s - x_1$, (1.9.19) takes the form

$$\begin{aligned}\frac{\partial\Phi(x)}{\partial x_1} = \lim_{s\to x_1} \frac{\int_{x_1}^{s} v_1(\xi, x_2, x_3)\mathrm{d}\xi - \int_{x_1}^{x_1} v_1(\xi, x_2, x_3)\mathrm{d}\xi}{s - x_1} &= \lim_{s\to x_1} \frac{F(s) - F(x_1)}{s - x_1}, \\ &= F'(x_1),\end{aligned}$$

where $F(s) = \int_{x_1}^{s} v_1(\xi, x_2, x_3)\mathrm{d}\xi$. This and the relation $F'(x_1) = v_1(x_1, x_2, x_3) = v_1(x)$ imply that $\frac{\partial\Phi(x)}{\partial x_1} = v_1(x)$ for all $x \in \Omega$.

Of course, if $\vec{v} \in C^1(\Omega)$ and $\nabla\Phi = \vec{v}$, then $\Phi \in C^2(\Omega)$.

b) Let $\Phi \in C^2(\Omega)$ and $\nabla\Phi = \vec{v}$ in Ω. We prove that $\int_\varphi \vec{v} \cdot \vec{t} \, \mathrm{d}S = 0$ for any piecewise linear closed curve φ in Ω. For such φ we have

$$\int_\varphi \vec{v} \cdot \vec{t} \, \mathrm{d}S = \int_a^b \vec{v}(\varphi(\tau)) \cdot \varphi'(\tau) \, \mathrm{d}\tau = \int_a^b (\nabla\Phi)(\varphi(\tau)) \cdot \varphi'(\tau) \, \mathrm{d}\tau =$$
$$= \int_a^b \frac{\mathrm{d}}{\mathrm{d}\tau}(\Phi(\varphi(\tau))) \, \mathrm{d}\tau = \Phi(\varphi(b)) - \Phi(\varphi(a)) = 0.$$

This completes the proof.

Theorem 3 *Let $\Omega \subset \mathbb{R}^3$ be a simply connected domain, $\vec{v} \in C^1(\Omega)$, $\operatorname{rot} \vec{v} = 0$ in Ω. Then there exists a potential Φ to $\vec{v}$ in Ω.*

Proof *See, e.g., Feistauer: Mathematicals Method in Fluid Dynamics, Longman 1993.*

Similar theorem for $\Omega \subset \mathbb{R}^2$ will be proven in the next section.

Remark 1.2 *If Ω is not simply connected, then it may happen that Φ does not exist.*

2

2D MODEL OF FLOW

We speak about 2D flow, if Cartesian coordinates x_1, x_2, x_3 can be chosen in such a way that the domain occupied by fluid can be expressed in the form $\Omega_3 = \Omega \times (0, L)$, $\Omega \subset \mathbb{R}^2$, $L > 0$, the functions $\vec{v}$ and p depend on x_1, x_2 only $\left(\frac{\partial}{\partial x_3} \equiv 0\right)$ and $v_3 \equiv 0$. Thus,

$$\vec{v} = \vec{v}(x) = \vec{v}(x_1, x_2) = (v_1(x_1, x_2), v_2(x_1, x_2)),$$
$$p = p(x) = p(x_1, x_2) \quad \text{for all } x = (x_1, x_2) \in \Omega.$$

Notice that

$$\operatorname{div} \vec{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$$

and

$$\operatorname{rot} \vec{v} = \left(0, 0, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right).$$

Therefore, a 2D stationary irrotational incompressible flow is described by the equations

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \quad \text{in} \quad \Omega, \tag{2.0.1}$$
$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0 \quad \text{in} \quad \Omega. \tag{2.0.2}$$

and Bernoulli's equation.

2.1 Velocity potential in 2D

Theorem 4 *Let $\vec{v} \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^2$ be a simply connected domain and*

$$\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} = 0 \quad \textit{in} \quad \Omega. \tag{2.1.3}$$

Then there exists a potential Φ to $\vec{v}$ in Ω (i.e. $\Phi \in C^2(\Omega)$, $\nabla\Phi = \vec{v}$ in Ω).

Proof It is enough to show that

$$\int_{\varphi} \vec{v} \cdot \vec{t} \, \mathrm{d}S = 0$$

for all piecewise linear closed curves φ in Ω. It is clear that

$$\int_{\varphi} = \sum_{i=1}^{n} \int_{\varphi_i},$$

where φ_i are piecewise linear closed simple curves for all $i = 1, 2, \ldots, n$. (Note that a curve ϑ is a *simple closed curve* if $\vartheta : [a, b] \to \mathbb{R}^2$ and $\vartheta(t) \neq \vartheta(t')$ for all $t, t' \in [a, b], t \neq t'$, $|t - t'| < b - a$.) Thus, $\mathbb{R}^2 - \langle\varphi_i\rangle$ has exactly two components. One is bounded (denoted as Int φ_i = interior of φ_i)) and second one is unbounded (Ext φ_i = exterior of φ_i). Since Ω is simply connected, we have $\overline{\text{Int}}\ \varphi_i \subset \Omega$. From the condition

$$\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} = 0$$

in Ω, by Green's theorem we have

$$0 = \int_{Int\ \varphi_i} \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) \mathrm{d}x = \int_{\partial(Int\ \varphi_i)} (v_1 n_2 - v_2 n_1) \mathrm{d}S,$$

where $\vec{n} = (n_1, n_2)$ is the unit outer normal to $\partial(\text{Int } \varphi_i)$. Then

$$\int_{\partial(Int\ \varphi_i)} (v_1 n_2 - v_2 n_1) \mathrm{d}S = \pm \int_{\varphi_i} (v_1 t_1 + v_2 t_2) \mathrm{d}S = \pm \int_{\varphi_i} \vec{v} \cdot \vec{t}\, \mathrm{d}S,$$

where $\vec{t} = (t_1, t_2) = (n_2, -n_1)$ is unit tangent. This implies that

$$\int_{\varphi} \vec{v} \cdot \vec{t}\, \mathrm{d}S = \sum_{i=1}^{r} \int_{\varphi_i} \vec{v} \cdot \vec{t}\, \mathrm{d}S = 0,$$

what we wanted to prove.

Example 3 *Flow in a 2D channel: Find $\vec{v} \in C^1(\Omega) \cap C(\overline{\Omega})$:*

- $\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0$ *in* Ω,
- $\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} = 0$ *in* Ω,
- $\vec{v} \cdot \vec{n}|_{\partial\Omega} = g$.

Then there exist a potential Φ to $\vec{v}$: $\nabla\Phi = \vec{v}$.
The problem for $\vec{v}$ is equivalent to find $\Phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

- $\Delta\Phi = 0$ in Ω,
- $\frac{\partial \Phi}{\partial \vec{n}}|_{\partial\Omega} = g$.

Remark 2.1 *The necessary condition on g: Let $\vec{v}$ satisfy the continuity equation. Then*

$$0 = \int_\Omega \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) \mathrm{d}x \overset{Green}{=} \int_{\partial\Omega} (v_1 n_1 + v_2 n_2)\ \mathrm{d}S$$

$$= \int_{\partial\Omega} \vec{v} \cdot \vec{n}\ \mathrm{d}S = \int_{\partial\Omega} g\ \mathrm{d}S.$$

The equation

$$\int_{\partial\Omega} g\ \mathrm{d}S = 0 \tag{2.1.4}$$

means that flux through $\partial\Omega$ is zero.

2.2 Modelling of flow past a profile (airfoil)

This is important in the desing of airplane wings. *Profile* or *airfoil* means a plane cut through a wing.

Mathematical interpretation: *Profile* C_0 is the geometric image of a simple closed negatively oriented curve φ in $\mathbb{R}^2$, smooth with exception of at most one point (lying at the backward side with respect to the flow direction).

Mathematical formulation of the flow past an airfoil

Let $\vec{v}_\infty$ be given, $\Omega = \mathrm{Ext}\varphi$, $C_0 = \langle\varphi\rangle = \partial\Omega$. Find $\vec{v} \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

- $\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0$ in Ω,
- $\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} = 0$ in Ω,
- $\vec{v} \cdot \vec{n}|_{C_0} = 0$ (or $\vec{v} \cdot \vec{n} = 0$ on C_0),
- $\lim_{|x|\to\infty} \vec{v}(x) = \vec{v}_\infty$.

Question: Does the velocity potential exist in Ω?
In general it does not exist because the domain Ω is not simply connected!

2.3 Velocity potential

We cut $\mathbb{R}^2$ by a half line Σ starting from C_0 and going to ∞. Then the domain $\tilde{\Omega} = \Omega - \Sigma$ is simply connected, because $\mathbb{R}^2 - \tilde{\Omega}$ does not contain any bounded component. Then there exists Φ in $\tilde{\Omega}$ such that $\Phi \in C^2(\tilde{\Omega}) \cap C^1(\tilde{\Omega} \cup C_0)$ and $\nabla\Phi = \vec{v}$ in $\tilde{\Omega}$.

Let $x_0 \in \tilde{\Omega}$ and $\Phi(x_0) \in \mathbb{R}$ be fixed. Then for arbitrary $x \in \tilde{\Omega}$ we can write

$$\Phi(x) = \Phi(x_0) + \int_{\varphi(x_0,x)} \vec{v} \cdot \vec{t}\ \mathrm{d}S, \tag{2.3.5}$$

where $\varphi(x_0, x)$ is a piecewise linear curve in $\tilde{\Omega}$, $i.p.\varphi(x_0, x) = x_0$, $t.p.\varphi(x_0, x) = x$. If $x \in \Sigma$, let $U(x)$ be a sufficiently small neighbourhood of x. Then $U(x) - \Sigma$ has

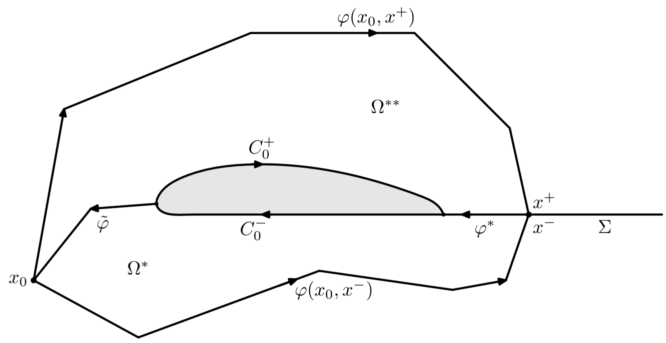

FIG. 2.3.1. Curves φ^- and φ^+

exactly two components $U^-(x)$ and $U^+(x)$ and, thus, $U(x) - \Sigma = U^+(x) \cup U^-(x)$.
Question: Does exist $\lim_{y \to x \in \Sigma, y \in U^\pm(x)} \Phi(y)$?
We have

- $\Phi|_{U^\pm(x)}$ is defined,
- $\nabla\Phi|_{U^\pm(x)} = \vec{v}$, $\vec{v} \in C^1(\Omega)$.

Since $\vec{v}$ is bounded in $\overline{U^\pm(x)}$, then $\nabla\Phi|_{U^\pm(x)}$ is also bounded and thus, $\Phi|_{U^\pm(x)}$ is Lipschitz-continuous. This implies that Φ can be extended from $U^\pm(x)$ continously to $x \in \Sigma$. By a similar arguments we can show that there exists $\lim_{y \to x} \nabla\Phi(x)$.

Since $\nabla\Phi = \vec{v}$ in $\tilde{\Omega}$ and $\vec{v} \in C^1(\Omega)$ we have

- $\Phi \in C^2(\tilde{\Omega}) \cap C^1(\tilde{\Omega} \cup C_0)$,
- $\lim_{y \to x, y \in U^\pm(x)} \Phi(y) \equiv \Phi(x^\pm)$ for $x \in \Sigma$,
- $\lim_{y \to x, y \in U^\pm(x)} \nabla\Phi(y) \equiv \nabla\Phi(x^\pm) = \vec{v}(x)$ for $x \in \Sigma$,
- $\Phi(x^+) - \Phi(x^-) = \gamma$, where γ is the velocity circulation along C_0.

Proof of the last relation: Let us consider closed curves

$$\varphi^- \equiv \varphi(x_0, x^-) \oplus \varphi^\star \oplus C_0^- \oplus \tilde{\varphi} \equiv \text{ boundary of } \Omega^\star,$$
$$\varphi^+ \equiv \varphi(x_0, x^+) \oplus \varphi^\star \ominus C_0^+ \oplus \tilde{\varphi} \equiv \text{ boundary of } \Omega^{\star\star}.$$

shown in Figure 2.3.1.

We have $\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} = 0$ in $\Omega^\star \cup \Omega^{\star\star}$. Then, using Green's theorem, we find that

$$\begin{aligned} 0 &= \int_{\Omega^\star \cup \Omega^{\star\star}} \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) \, \mathrm{d}x = \int_{\varphi^+} \vec{v} \cdot \vec{t} \, \mathrm{d}S - \int_{\varphi^-} \vec{v} \cdot \vec{t} \, \mathrm{d}S \\ &= \int_{\tilde{\varphi}} - \int_{C_0^+} + \int_{\varphi^\star} - \int_{\tilde{\varphi}} - \int_{C_0^-} - \int_{\varphi^\star} \\ &= (\Phi(x+) - \Phi(x_0)) - (\Phi(x-) - \Phi(x_0)). \end{aligned}$$

This implies that

$$0 = \Phi(x+) - \Phi(x-) - \int_{C_0} \vec{v} \cdot \vec{t} \, \mathrm{d}S.$$

Hence,

$$\gamma = \int_{C_0} \vec{v} \cdot \vec{t} \, \mathrm{d}S = \Phi(x+) - \Phi(x-)$$

.

If $\vec{v} \in C(\overline{\Omega})$ ($\vec{v}$ is continuous across Σ) and $x \in \Sigma$, then

- $\vec{v}(x+) = \vec{v}(x-)$,
- $\vec{v}(x+) \cdot \vec{n} = \vec{v}(x-) \cdot \vec{n}$ with a unit normal $\vec{n}$ to Σ,
- $\vec{v}(x+) \cdot \vec{t} = \vec{v}(x-) \cdot \vec{t}$ with a unit tangent $\vec{t}$ to Σ

We note that $\vec{v}(x\pm) \cdot \vec{n} = \frac{\partial \Phi}{\partial \vec{n}}(x\pm)$ and $\vec{v}(x\pm) \cdot \vec{t} = \frac{\partial \Phi}{\partial \vec{t}}(x\pm)$.

2.4 Problem for the velocity potential

Let a profile $C_0, v_\infty \in \mathbb{R}^2, \gamma \in \mathbb{R}$ be given. Find $\Phi \in C^2(\tilde{\Omega}) \cap C^1(\tilde{\Omega} \cup C_0)$ such that

- $\Phi, \frac{\partial \Phi}{\partial x_i}, \frac{\partial^2 \Phi}{\partial x_i \partial x_j}$ have one-sides limits on Σ,
- $\Delta \Phi = 0$ in $\tilde{\Omega}$,
- $\frac{\partial \Phi}{\partial \vec{n}} = 0$ on C_0,
- $\frac{\partial \Phi}{\partial \vec{n}}(x+) = \frac{\partial \Phi}{\partial \vec{n}}(x-)$ for all $x \in \Sigma$,
- $\Phi(x+) - \Phi(x-) = \gamma$ for all $x \in \Sigma$,
- $\lim_{|x| \to \infty} \nabla \Phi(x) = \vec{v}_\infty$.

Remark 2.2 *The fact that the domains Ω and $\tilde{\Omega}$ are unbounded causes problems in the numerical solutions. Therefore, in the numerical simulation, Ω or $\tilde{\Omega}$ are replaced by a bounded sufficiently large domain with an artificial outer boundary Γ_∞.*

2.5 Force acting on the profile

Let $\vec{n}$ be the unit outer normal to $\partial\Omega = C_0$ (pointing into the profile). If we neglect the gravity force, then the force acting on C_0 is

$$\vec{F} = \int_{C_0} p\vec{n}\,\mathrm{d}S. \tag{2.5.6}$$

We shall use Bernoulli's equation

$$p = \rho(\mathrm{const} - \frac{1}{2}|v|^2).$$

Theorem 2.3 *(Chaplygin) Let us consider inviscid, incompressible and irrotational stacionary flow past a profile C_0 (= closed simple negatively oriented curve), $\Omega = \operatorname{Ext} C_0$, $\vec{v} \in C^1(\Omega) \cap C(\Omega \cup C_0)$ (= velocity of the fluid with the far field velocity $\vec{v}_\infty$), velocity circulation γ along C_0 and we denote by ρ (= const>0) the density of the fluid. Then the force acting on C_0 has the form*

$$\vec{F} = \gamma\rho\,|\vec{v}_\infty|\,(-\sin\Theta_\infty, \cos\Theta_\infty), \tag{2.5.7}$$

where Θ_∞ is the angle of attack ($\vec{v}_\infty = |\vec{v}_\infty|\,(\cos\Theta_\infty, \sin\Theta_\infty)$).

Remark 2.4 *$\vec{F}\perp\vec{v}_\infty$. Let $\Theta_\infty \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The airplane flies, if $F_2 > 0 \Leftrightarrow \gamma > 0$.*

Proof will be carried out with the aid of the complex function theory. Let

$$z = x_1 + ix_2, w(z) = v_1(x_1, x_2) - iv_2(x_1, x_2).$$

w is called the *complex velocity*. Since

$$v_1, v_2 \in C^1(\Omega), \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0,$$
$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0 \text{ in } \Omega,$$

w satisfies the Cauchy - Riemann conditions and w is a holomorphic function in Ω. If $v_1, v_2 \in C(\overline{\Omega})$, then $w \in C(\overline{\Omega})$, $\lim_{z\to\infty} w(z) = \overline{V}_\infty = v_{\infty 1} - iv_{\infty 2}$, where C_0 is the geometric image of a simple closed negatively oriented curve $\varphi : [a, b] \to \Omega(\subset \mathbb{R}^2), \varphi$ smoot except at most one point, $\varphi(a) = \varphi(b)$. We have $\varphi = (\varphi_1, \varphi_2), \vec{t} = \varphi'(\tau) = (\varphi_1'(\tau), \varphi_2'(\tau)), \tau \in [a, b]$, $\varphi'(\tau)$ is a tangent vector to C_0. Let $|\varphi'(\tau)| = 1$ for all $\tau \in [a, b]$, $\varphi = \varphi_1 + i\varphi_2$, $t = \varphi_1' + i\varphi_2'$. If $\vec{n}$ is the unit outer normal to $C_0 = \partial\Omega$, $\vec{n} = (n_1, n_2)$, then we set $n = n_1 + in_2 = -i\vec{t} = -i\varphi'(\tau) = -i\varphi_1(\tau) + \varphi_2(\tau)$.

In view of the Bernoulli equation we have

$$p = \rho(c - \frac{1}{2}|\vec{v}|^2) = \rho(c - \frac{1}{2}|\vec{w}|^2),$$

where c is a constant. Then

$$\mathcal{F} := F_1 + iF_2 = \int_{C_0} pn \,\mathrm{d}S = \int_a^b \rho(c - \frac{1}{2}|\vec{w}(\varphi(\tau))|^2)(-i\varphi'(\tau)) \,\mathrm{d}\tau.$$

Now we compute

$$\begin{aligned} w(\varphi(\tau))\varphi'(\tau) &= (v_1 - iv_2)(\varphi_1' + i\varphi_2') \\ &= v_1\varphi_1' + v_2\varphi_2' + i(v_1\varphi_2 - v_2\varphi_1), \end{aligned}$$

where the first term $v_1\varphi_1' + v_2\varphi_2' = \vec{v}\cdot\vec{t}$ and the second term $v_1\varphi_2 - v_2\varphi_1 = \vec{v}\cdot\vec{n} = 0$. Hence, $\mathrm{Im}(w(\varphi(\tau))\varphi'(\tau)) = 0$ for all $\tau \in [a, b]$ and $w(\varphi(\tau))\varphi'(\tau) = \vec{v}\cdot\vec{t}$ is a real-valued function.

Circulation along C_0 is

$$\begin{aligned} \gamma = \int_{C_0} \vec{v}\cdot\vec{t}\,\mathrm{d}S &= \int_a^b (v_1(\varphi(\tau))\varphi_1'(\tau) + v_2(\varphi(\tau))\varphi_2'(\tau)) \,\mathrm{d}\tau \\ &= \int_a^b w(\varphi(\tau))\varphi'(\tau) \,\mathrm{d}\tau = \int_\varphi w(z) \,\mathrm{d}z. \end{aligned}$$

Theorem 2.5 ***(Cauchy-Goursat)*** *Let $\Omega = \mathrm{Ext}\, C_0$. Let $C_0 \subset \mathrm{Int}\, K_R$, where K_R is a simple closed smooth negatively oriented curve, $\langle K_R\rangle \subset \mathrm{Ext}\, C_0 = \Omega$. Let $\vec{w}$ be a holomorphic function in $\tilde{\Omega} = \mathrm{Int}\, K_R \cap \mathrm{Ext}\, C_0$, $w \in C(\overline{\tilde{\Omega}})$. Then*

$$\int_\varphi w(z) \,\mathrm{d}z = \int_{K_R} w(z) \,\mathrm{d}z.$$

Let K_R be a curve with geometric image, which is the circle with centre at the origin and diameter R (sufficiently large). Let K_r be the circle with diameter $r < R$, $C_0 \subset \mathrm{Int}\, K_r$. Since w is holomorphic in Ext K_r and $\lim_{|z|\to\infty} w(z) = \overline{V}_\infty$, the function w is holomorphic in a neighbourhood of ∞. From this we see that w can be written in the form

$$w(z) = \overline{V}_\infty + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots, \quad z \in \mathrm{Ext}\, K_r.$$

Using the Cauchy - Goursat theorem, we see that

$$\gamma = \int_\varphi w(z) \,\mathrm{d}z = \int_{K_R} w(z) \,\mathrm{d}z = \int_{K_R} \frac{a_1}{z} \,\mathrm{d}z = -2\pi i a_1.$$

Hence, we can write

$$w(z) = \overline{V}_\infty - \frac{\gamma}{2\pi i z} + \frac{a_2}{z^2} + \dots, \quad z \in \mathrm{Ext} K_r.$$

Further, we have

$$|w(\varphi(\tau))|^2 \varphi'(\tau) = \overline{w(\varphi(\tau))} w(\varphi(\tau))\varphi'(\tau) = \overline{(w(\varphi(\tau)))^2 \varphi'(\tau)},$$

because we know that $w(\varphi(\tau))\varphi'(\tau)$ is a real-valued function. Since

$$\int_a^b \varphi' \,\mathrm{d}\tau = 0,$$

we find that

$$\mathcal{F} = \frac{1}{2} i\rho \int_a^b \overline{w^2(\varphi(\tau))\varphi'(\tau)} \,\mathrm{d}\tau = \frac{1}{2} i\rho \overline{\int_a^b w^2(\varphi(\tau))\varphi'(\tau) \,\mathrm{d}\tau}$$
$$= \frac{1}{2} i\rho \overline{\int_\varphi w^2(z) \,\mathrm{d}z}.$$

It follows from the above considerations that
- $w^2(z)$ is holomorphic function in $\Omega = \mathrm{Ext} C_0$,
- $w^2 \in C(\overline{\Omega})$,
- $\lim_{|z|\to\infty} = \overline{V}_\infty^2$.

Then the Cauchy-Goursat theorem implies that

$$\int_\varphi w^2(z) \,\mathrm{d}z = \int_{K_R} w^2(z) \,\mathrm{d}z.$$

In $\mathrm{Ext} K_r$ we have

$$w^2(z) = (\overline{V}_\infty)^2 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots.$$

On the other hand,

$$w^2(z) = (\overline{V}_\infty - \frac{\gamma}{2\pi i z} + \frac{a_2}{z} + \dots)(\overline{V}_\infty - \frac{\gamma}{2\pi i z} + \frac{a_2}{z} + \dots)$$
$$= (\overline{V}_\infty)^2 - \frac{\gamma \overline{V}_\infty}{\pi i z} + \frac{b_2}{z^2} + \dots.$$

Therefore,

$$\int_{K_R} w^2(z) \,\mathrm{d}z = -\frac{\overline{V}_\infty}{\pi i} \gamma(-2\pi i) = 2\gamma \overline{V}_\infty. \tag{2.5.8}$$

Finally we get

$$\mathcal{F} = \frac{1}{2} i\rho \overline{\int_{K_R} w^2(z) \,\mathrm{d}z} = 2\gamma \overline{V}_\infty \frac{1}{2} i\rho = \rho\gamma(v_{\infty 1} + i v_{\infty 2}) i$$
$$= \rho\gamma(i v_{\infty 1} - v_{\infty 2}) = \rho\gamma \,|\vec{v}_\infty|\, (-\sin\theta_\infty + i\cos\theta_\infty),$$

where we used $\vec{v}_\infty = (v_{\infty 1}, v_{\infty 2}) = |\vec{v}_\infty| (\cos\theta_\infty, \sin\theta_\infty)$. This means that

$$\vec{F} = \rho\gamma\, |\vec{v}_\infty|\, (-\sin\theta_\infty, \cos\theta_\infty),$$

what we wanted to prove.

2.6 The choice of the velocity circulation γ

Let the profile have a sharp trailing edge at the point z_0. If γ is not chosen in a suitable way, then z_0 behaves as a singularity, i.e. $|\vec{v}(x)|_{x\to z_0} \to +\infty$ and from the Bernoulli theorem $p(x)_{x\to z_0} \to -\infty$! Nature causes that the stagnation point $z_1 \in C_0$, where $\vec{v}(z_1) = 0$ is moved to the point z_0 and the velocity becomes bounded in a neighbourhood of z_0. This implies that γ must be chosen so that $|\vec{v}|$ is bounded in a neighbourhood of z_0. This condition is called the *Kutta-Joukowski condition* for profiles with a sharp trailing edge.

2.7 Profile with a rounded trailing edge

Kutta-Joukowski condition is formulated now as $\vec{v}(z_0) = 0 \Leftrightarrow (\vec{v}\cdot\vec{t})(z_0) = 0 \Leftrightarrow$ $\underline{\frac{\partial \Phi}{\partial t}(z_0) = 0}$.

2.8 Problem for the velocity potential describing the flow past an airfoil

Let C_0 be a smooth airfoil, let a point $z_0 \in C_0$ be given and let $\vec{v}_\infty \in \mathbb{R}^2$ be given (Let Σ be a suitable artificial cut in $\Omega = \mathrm{Ext}C_0$.)

We want to find a function Φ and a constant $\gamma \in \mathbb{R}$ satisfying the following conditions

1. $\Phi \in C^2((\Omega \cup C_0) - \Sigma)$,
2. Φ has one-sides limits $\Phi(x\pm)$ at each point $x \in \Sigma$ and also first and second derivatives of Φ have one-sided limits on Σ,
3. $\Delta\Phi = 0$ in $\Omega - \Sigma$ ($\Leftrightarrow$ continuity equation),
4. $\frac{\partial \Phi}{\partial n}|_{C_0} = 0$ (normal component of $\vec{v}$ is zero on the profile),
5. $\Phi(x+) - \Phi(x-) = \gamma$ (condition for the velocity circulation),
6. $\frac{\partial \Phi}{\partial \vec{n}}(x+) = \frac{\partial \Phi}{\partial \vec{n}}(x-)$ (normal component of $\vec{v}$ is continuous across Σ),
7. $\lim_{|x|\to\infty} \nabla\Phi(x) = \vec{v}_\infty$ ($\lim_{|x|\to\infty} \vec{v}(x) = \vec{v}_\infty$),
8. Kutta-Joukowski condition $\frac{\partial \Phi}{\partial \vec{t}}(z_0) = 0$ ($\frac{\partial}{\partial \vec{t}}$ = derivative in the tangential direction to C_0).

Note that if 5 and 6 hold, then $\vec{v}$ is continuous across Σ.

Remark 2.6 *If $\gamma \in \mathbb{R}$ is prescribed, then it is possible to solve the problem for Φ satisfying $1, 2, \ldots, 7$. (It is posible to introduce the so called weak formulation and then apply the FEM-finite element method).*

Remark 2.7 *In the case of the rounded trailing edge, the trailing stagnation point $z_0 \in C_0$ is chosen as the point lying on the backward part of the airfoil (with respect to the flow direction), with the largest curvature. If the backward part of the airfoil is formed by a part of a circle, then z_0 is chosen as the midpoint of this circular arc.*

Let us assume that we are able to solve problem $1. - 7.$ with an arbitrary given $\gamma \in \mathbb{R}$.

Question: How to obtain the solution (Φ, γ) of the problem $1. - 8.$?

2.9 The solution of the problem

Let Φ_0 (or Φ_1) be the velocity potential and solution of problem $1. - 7.$ with $\gamma := \gamma_0 = 0$ (or $\gamma := \gamma_1 = 1$). Let us set $\Phi_\theta := \Phi_0 + \theta(\Phi_1 - \Phi_0)$, $\theta \in \mathbb{R}$.

Lemma 3 *Φ_θ is a solution of problem $1.-7.$ with prescribed $\gamma = \gamma_0 + \theta(\gamma_1 - \gamma_0)$.*

Proof *Problem $1. - 7.$ is linear (Homework).*

Goal: Determine $\theta \in \mathbb{R} : \frac{\partial \Phi_\theta}{\partial \vec{t}}(z_0) = 0$

Solution:

$$0 = \frac{\partial \Phi_\theta}{\partial \vec{t}}(z_0) = \frac{\partial \Phi_0}{\partial \vec{t}}(z_0) + \theta \left(\frac{\partial \Phi_1}{\partial \vec{t}}(z_0) - \frac{\partial \Phi_0}{\partial \vec{t}}(z_0) \right).$$

If $\frac{\partial \Phi_1}{\partial \vec{t}}(z_0) - \frac{\partial \Phi_0}{\partial \vec{t}}(z_0) \neq 0$, then

$$\theta = -\frac{\frac{\partial \Phi_0}{\partial \vec{t}}(z_0)}{\frac{\partial \Phi_1}{\partial \vec{t}}(z_0) - \frac{\partial \Phi_0}{\partial \vec{t}}(z_0)}.$$

3

POROUS MEDIA FLOW

Example 4 *Infiltration*

1. *Infiltration through a dam. Water can flow through the concrete which forms the dam.*
2. *Infiltration of liquids from reservois (danger of the pollution of groundwater).*
3. *Chemical mining of uranium.*
4. *Mining of oil.*

3.1 Modelling of flow in porous media

Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be a domain in which we consider the flow,

$$\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_s,$$

where Ω_f is a domain occupied by a fluid and Ω_s is a domain formed by a solid material (porous media). Note that Ω_f and Ω_s have an unknown microstructure (difficult to describe). For $x = (x_1, \dots, x_N) \in \Omega$ and $\epsilon > 0$ we set

$$V_\epsilon(x) = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \ldots \times (x_N - \epsilon, x_N + \epsilon).$$

Let us assume that the material has a periodic structure. Let $V_{\epsilon,f}(x) = V_\epsilon(x) \cap \Omega_f$. Let $\vec{v}^\star = \vec{v}^\star(x,t)$ be fluid velocity at $x \in \Omega_f$ and time t. Now we define the averaged velocity $\vec{v}(x,t)$ at $x \in \Omega$ and time t

$$\vec{v}(x,t) = \frac{1}{|V_\epsilon(x)|} \int_{V_{\epsilon,f}(x)} \vec{v}^\star(y,t) \ \mathrm{d}y. \tag{3.1.1}$$

3.2 Derivation of equations for velocity

Let $q = q(x,t)$ denote sources of the liquid in Ω. Let the fluid be incompressible. Then density $\rho = \text{const} > 0$. Moreover, we shall consider stationary flow, i.e. $\frac{\partial}{\partial t} \equiv 0$. By the symbol $\sigma (\subset \bar{\sigma} \subset \Omega)$ we denote a control volume.

Physical postulate: Total mass flux through the boundary $\partial\sigma$ is equal to the production of fluid in σ due to sources.
Mathematical formulation:

$$\int_{\partial\sigma} (\rho\vec{v})(x,t) \cdot \vec{n}(x) \ \mathrm{d}s = \int_\sigma \rho(x,t) q(x,t) \ \mathrm{d}x \quad \forall\sigma \quad \text{control volume} \tag{3.2.2}$$

Let $\vec{v} \in C^1(\Omega)^N$, $q \in C(\Omega)$. By Green's theorem we get

$$\int_\sigma \rho \mathrm{div}\,\vec{v}\ \mathrm{d}x = \int_\sigma \rho q\ \mathrm{d}x \quad \text{in} \quad \forall \sigma \subset \Omega. \tag{3.2.3}$$

This is equivalent with the differential equation

$$\mathrm{div}\,\vec{v} = q \quad \text{in} \quad \Omega. \tag{3.2.4}$$

If $q = 0$ in Ω, then we get a standard continuity equation for incompressible flow. We have N unknown functions and 1 equation.

3.3 Closing relations

Let us have a simple case when the domain Ω is a narrow layer orthogonal to the Earth of the length l. Then the fluid moves in the direction x_3 orthogonal to the Earth. This motion is caused by the pressure drop $\frac{p_2 - p_1}{l}$ and gravity force. Thus,

$$v_3 \propto \frac{p_2 - p_1}{l} + \rho g,$$

where g is the gravity constant. It is clear that $v_3 < 0$, if $\frac{p_2-p_1}{l} + \rho g > 0$. This means that $v_3 \propto -C(\frac{p_2-p_1}{l} + \rho g)$, where $C > 0$ is a constant. Let $l \to 0+$. Then we get the relation

$$v_3 = -C(\frac{\partial p}{\partial x_3} + \rho g).$$

Its generalization reads

$$\vec{v} = -C(\mathrm{grad}\, p + \rho \vec{g}) \quad \text{in } \Omega, \tag{3.3.5}$$

called the Darcy law. Here $\vec{g} = (0, 0, g), C > 0$ is a constant depending on material properties, namely the permeability k and the liquid viscosity $\mu =:$ $C = \frac{k}{\mu}$. In general, porous medium has different properties in different direction (= *anisotropy).* Then we can write

$$\vec{v} = -\mathbb{K}(\nabla p + \rho \vec{g}), \tag{3.3.6}$$

where $\mathbb{K}$ is a symmetric positive definite matrix ($\mathbb{K} = \mathbb{K}^T$ and $\xi \cdot \mathbb{K} \cdot \xi^T > 0 \quad \forall \xi \in \mathbb{R}^N, \xi \neq 0$). If we substitute (3.3.6) into equation (3.2.4), we get

$$-\mathrm{div}\,(\mathbb{K}\nabla p) = q \quad \text{in} \quad \Omega. \tag{3.3.7}$$

This is a 2nd order PDE of elliptic type.

3.4 Boundary conditions

Let $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ .We consider two types of boundary conditions:

1. $p|_{\Gamma_D} = p_D$ (Dirichlet BC prescribing the pressure),

2. $\vec{v} \cdot \vec{n}|_{\Gamma_N} = \tilde{\varphi}_N$ (prescribed flux).

When we use equation (3.3.6), then we get

$$-\mathbb{K}(\nabla p + \rho\vec{g}) \cdot \vec{n} = \tilde{\varphi}_N \quad \text{on } \Gamma_N, \tag{3.4.8}$$

i.e.

$$-(\mathbb{K}\nabla p) \cdot \vec{n} = \tilde{\varphi}_N + \rho\mathbb{K}\vec{g} \cdot \vec{n} =: \varphi_N \quad \text{on} \quad \Gamma_N \tag{3.4.9}$$

(Neumann boundary condition).

Example 5 *What happens if we use the Darcy law in the form (3.3.5). Then If $\mathbb{K} = C\mathbb{I}$ and we get the boundary condition*

$$-\nabla p \cdot \vec{n} = \varphi_N / C$$

or

$$-\frac{\partial p}{\partial \vec{n}} = \varphi_N / C$$

(standard Neumann BC).

3.5 Classical formulation of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Let $\mathbb{K}$ be a symmetric positive definite constant matrix (depending on the material properties (i.e., permeability and fluid viscosity). Let $\partial\Omega = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset$. Let

- $q : \Omega \to \mathbb{R}$,
- $p_D : \Gamma_D \to \mathbb{R}$,
- $\varphi_N : \Gamma_N \to \mathbb{R}$

be given functions. We want to find $p \in C^2(\overline{\Omega})$ such that

- $-\text{div}\,(\mathbb{K}\nabla p) = q$ in Ω,

- $p|_{\Gamma_D} = p_D$,

- $-(\mathbb{K}\nabla p) \cdot \vec{n}|_{\Gamma_N} = \varphi_N$.

If we obtain the pressure p as a solution of the above problem, then we compute the velocity

- $\vec{v} = -\frac{k}{\mu}(\nabla p + \rho\vec{g})$ in Ω (Darcy Law)

or

- $\vec{v} = -\mathbb{K}(\nabla p + \rho\vec{g})$ (generalized Darcy Law).

Remark 3.1 *We derived the Darcy Law with the aid of a very simple engineering approach. Mathematically precise derivation of the Darcy Law can be carried out by the homogenization (i.e. averaging) of the Stokes problem*

$$\operatorname{div} \vec{v} = 0,$$
$$-\nu \Delta \vec{v} + \nabla p = \vec{g}.$$

(See, e.g. Sanchez-Palenzia: Non-homogenous media and vibration theory, Springer 1980).

3.6 Flow in a domain consisting of different porous materials

Let $\Omega \subset \mathbb{R}^N$ be bounded, $\overline{\Omega} = \overline{\Omega}_1 \cup \ldots \cup \overline{\Omega}_r$, where Ω_i represent subdomains formed by different materials. Let $\mathbb{K}^i, i = 1, \ldots, r$, be positive definite symmetric constant matrices. Let

1.
$$-\operatorname{div}(\mathbb{K}^{(i)} \nabla p^{(i)}) = q \quad \text{in } \Omega_i,\ i = 1, \ldots, r, \tag{3.6.10}$$
2. $p^{(i)} = p|_{\Omega_i}$, $p|_{\Omega_i}$ is the restriction of the pressure on the subdomain Ω_i,
3. $\mathbb{K} : \bigcup_{i=1}^r \Omega_i \to \mathbb{R}^{N \times N}$, $\mathbb{K}|_{\Omega_i} = \mathbb{K}^{(i)},\ i = 1, \ldots, r,$
4. $p : \bigcup_{i=1}^r \Omega_i \to \mathbb{R}$.

Instead of equations (3.6.10), $i = 1, \ldots, r$, we simply write

$$-\operatorname{div}(\mathbb{K} \nabla p) = q \quad \text{in } \Omega.$$

This is a PDE with discontinuous coefficients, interpreted in the sense of equations (3.6.10).
Boundary conditions:

1. $p|_{\Gamma_D} = p_D$ (i.e., $p^{(i)}|_{\Gamma_D \cap \partial\Omega_i} = p_D|_{\Gamma_D \cap \partial\Omega_i}$),
2. $-(\mathbb{K}^{(i)} \nabla p^{(i)}) \cdot \vec{n} = \varphi_N$ on $\Gamma_N \cap \partial\Omega_i$, $i = 1, \ldots, r$.

3.7 Transmission conditions

Let $\partial\Omega_i \cap \partial\Omega_j = \Gamma_{ij} = \Gamma_{ji}$, for $i, j = 1, \ldots, r$. We consider such Γ_{ij} that $\text{meas}_{N-1}(\Gamma_{ij}) > 0$. Denote by $\vec{n}_{ij}$ the unit normal to Γ_{ij} pointing from Γ_i to Γ_j ($\vec{n}_{ij} = -\vec{n}_{ji}$) and prescribe here the conditions

- continuity of the pressure across Γ_{ij}: $p^{(i)}|_{\Gamma_{ij}} = p^{(j)}|_{\Gamma_{ij}}$,
- continuity of the flux across Γ_{ij}: $\vec{v}^{(i)} \cdot \vec{n}_{ij}|_{\Gamma_{ij}} = -\vec{v}^{(j)} \cdot \vec{n}_{ji}|_{\Gamma_{ij}}$.

This is equivalent with the relation

$$\begin{aligned}
&(\mathbb{K}^{(i)} \nabla p^{(i)}) \cdot \vec{n}_{ij} + (\mathbb{K}^{(j)} \nabla p^{(j)}) \cdot \vec{n}_{ji} \\
&= \sigma_{ij} := -\rho \left((\mathbb{K}^{(i)} \vec{g}) \cdot \vec{n}_{ij} + (\mathbb{K}^{(j)} \vec{g}) \cdot \vec{n}_{ji} \right) \\
&\quad \text{on } \Gamma_{ij},\ i, j = 1, \ldots, r, i < j.
\end{aligned} \tag{3.7.11}$$

3.8 Classical formulation

Given $\mathbb{K}^{(i)} \in [C^1(\overline{\Omega}_i)]^{N\times N}$, $\mathbb{K} : \bigcup_{i=1}^r \Omega_i \to \mathbb{R}^{N\times N}$, $\mathbb{K}|_{\Omega_i} = \mathbb{K}^{(i)}$, p_D, φ_N, q as before. We want to find a function $p : p|_{\Omega_i} = p^{(i)}$ such that

1. $p \in C(\overline{\Omega})$,
2. $p^{(i)} \in C^2(\overline{\Omega}_i)$, $i = 1, \ldots, r$,
3. $-\mathrm{div}\,(\mathbb{K}\nabla p) = q$ in $\bigcup_{i=1}^r \Omega_i$,
4. $(\mathbb{K}_i \nabla p^{(i)}) \cdot \vec{n}_{ij} + (\mathbb{K}_j \nabla p^{(j)}) \cdot \vec{n}_{ji} = \sigma_{ij}$ on Γ_{ij}, $\mathrm{meas}(\Gamma_{ij}) > 0$, $i, j = 1, \ldots, r$, $i < j$,
5. $p|_{\Gamma_D} = p_D$,
6. $-(\mathbb{K}^{(i)} \nabla p^{(i)}) \cdot \vec{n} = \varphi_N$ on $\Gamma_N \cap \partial\Omega_i$, $i = 1, \ldots, r$.

3.9 Numerical method

An elegant numerical method, the FEM (= the finite element method), is based on the so-called weak formulation, which is represented by an integral identity valid for test functions from the space

$$V = \{v \in C^1(\overline{\Omega}) : v|_{\Gamma_D} = 0\}.$$

We derive this identity in such a way that we consider the equation $-\mathrm{div}\,(\mathbb{K}\nabla p) = q$ in $\bigcup_{i=1}^r \Omega_i$, multiply by any function $v \in V$, integrate over Ω_i, sum over all $i = 1, \ldots, r$ and apply Green's theorem. For the right-hand side of the resulting equation we have

$$\sum_{i=1}^r \int_{\Omega_i} qv \,\mathrm{d}x = \int_\Omega qv \,\mathrm{d}x. \tag{3.9.12}$$

For the left-hand side of this equation we have

$$-\sum_{i=1}^r \int_{\Omega_i} \mathrm{div}\,(\mathbb{K}\nabla p) v \,\mathrm{d}x = -\sum_{i=1}^r \int_{\Omega_i} \mathrm{div}\,(\mathbb{K}_i \nabla p^{(i)}) v \,\mathrm{d}x \overset{Green}{=}$$
$$= -\sum_{i=1}^r \int_{\partial\Omega_i} (\mathbb{K}_i \nabla p^{(i)}) \cdot \vec{n} v \,\mathrm{d}S + \sum_{i=1}^r \int_{\Omega_i} (\mathbb{K}_i \nabla p^{(i)}) \cdot \nabla v \,\mathrm{d}x$$

Note that

$$\partial\Omega_i = \bigcup_{j=1}^r \Gamma_{ij} \cup (\partial\Omega_i \cap \Gamma_D) \cup (\partial\Omega_i \cap \Gamma_N).$$

Then

$$
\begin{aligned}
-\sum_{i=1}^{r}\int_{\partial\Omega_i}(\mathbb{K}_i\nabla p_i)\cdot\vec{n}v\ \mathrm{d}S = -\sum_{i=1}^{r}\int_{\partial\Omega_i\cap\Gamma_N}(\mathbb{K}_i\nabla p_i)\cdot\vec{n}v\ \mathrm{d}S -\\
-\sum_{i=1}^{r}\int_{\partial\Omega_i\cap\Gamma_D}(\mathbb{K}_i\nabla p_i)\cdot\vec{n}v\ \mathrm{d}S - \sum_{i,j=1}^{r}\int_{\Gamma_{ij}}(\mathbb{K}_i\nabla p_i)\cdot\vec{n_{ij}}v\ \mathrm{d}S
\end{aligned}
$$

From the boundary conditions we have $-(\mathbb{K}^{(i)}\nabla p^{(i)})\cdot\vec{n} = \varphi_N$ on $\Gamma_N\cap\partial\Omega_i$, $i = 1,\dots,r$, which implies that the first term takes the form

$$
-\sum_{i=1}^{r}\int_{\partial\Omega_i\cap\Gamma_N}(\mathbb{K}_i\nabla p_i)\cdot\vec{n}v\ \mathrm{d}S = \sum_{i=1}^{r}\int_{\partial\Omega_i\cap\Gamma_N}\varphi_N v\ \mathrm{d}S. \tag{3.9.13}
$$

The second term is zero because $v|_{\Gamma_D} = 0$ for $v\in V$. The third term is zero due to the transmission conditions:

$$
\begin{aligned}
-\sum_{i,j=1}^{r}\int_{\Gamma_{ij}}(\mathbb{K}_i\nabla p_i)\cdot\vec{n_{ij}}v\ \mathrm{d}S = -\sum_{i,j=1;i<j}^{r}\int_{\Gamma_{ij}}[(\mathbb{K}_i\nabla p_i)\cdot\vec{n}_{ij} + (\mathbb{K}_j\nabla p_j)\cdot\vec{n}_{ji}]v\ \mathrm{d}S\\
= -\sum_{i,j=1;i<j}^{r}\int_{\Gamma_{ij}}\sigma_{ij}v\,\mathrm{d}S.
\end{aligned}
$$

Moreover, we can write $\sum_{i=1}^{r}\int_{\Omega_i}(\mathbb{K}_i\nabla p_i)\cdot\nabla v\ \mathrm{d}x = \int_\Omega(\mathbb{K}\nabla p)\cdot\nabla v\ \mathrm{d}x$ and, thus, we obtain the integral identity

$$
\begin{aligned}
\int_\Omega(\mathbb{K}\nabla p)\cdot\nabla v\ \mathrm{d}x = \int_\Omega qv\ \mathrm{d}x - \sum_{i=1}^{r}\int_{\partial\Omega_i\cap\Gamma_N}\varphi_N v\ \mathrm{d}S \qquad (3.9.14)\\
-\sum_{i,j=1;i<j}^{r}\int_{\Gamma_{ij}}\sigma_{ij}v\,\mathrm{d}S \quad \forall v\in\mathbb{V},
\end{aligned}
$$

which is the basis for the application of the FEM.

3.10 FEM

The simpliest possibility is to use the piecewise linear conforming finite elements. Let Ω, Ω_i be polygonal $2D$ domains. Let $\mathcal{T}_h^i$ be a partition of Ω_i into a finite number of triangles. Then $\mathcal{T}_h = \cup_{i=1}^{r}\mathcal{T}_h^i =$ is a triangulation of Ω. For $K, K'\in\mathcal{T}_h$, $K\neq K'$, we assume that $K\cap K'$ is an empty set or a common vertex of K, K' or a common side of K, K'.

Let $\Gamma_N\cap\Gamma_D$ be formed by vertices of some $K\in\mathcal{T}_h$. We denote $X_h = \{\varphi_h\in C(\overline{\Omega}) : \varphi_h|_K$ is linear function $\forall K\in\mathcal{T}_h\}$ and $V_h = \{\varphi_h\in X_h : v_h|_{\Gamma_D} = 0\}$.

In the 'weak' formulation (3.9.14) we approximate

$$
V\approx V_h,\ p\approx p_h\in X_h,\ v\approx v_h\in V_h
$$

and get the discrete problem.

3.11 Discrete problem

Find $p_h \in X_h$ such that

$$\int_\Omega (\mathbb{K}\nabla p_h) \cdot \nabla v_h \,\mathrm{d}x = \int_\Omega q v_h \,\mathrm{d}x - \sum_{i=1}^{r} \int_{\partial\Omega_i \cap \Gamma_N} \varphi_N v_h \,\mathrm{d}S \quad (3.11.15)$$
$$- \sum_{i,j=1;i<j}^{r} \int_{\Gamma_{ij}} \sigma_{ij} v_h \,\mathrm{d}S \quad \forall v_h \in V_h$$

with Dirichlet boundary condition $p_h(P) = p_D(P)$ for all vertices $P \in \Gamma_D$. Problem (3.11.15) is equivalent to a system of linear equations for unknown values $p_h(P)$, $P \in \sigma_h$, where σ_h is the set of all vertices of all $K \in \mathcal{T}_h$, $P \notin \Gamma_D$.

4

PROPAGATION OF ALLOYS IN MOVING FLUID

Applications:

- **propagation of exhalations in air, in water of rivers, lakes, seas,** ...
- **propagation of some material in a blood,**
- **propagation of impurities in underground water,**
- **oil production.**

Importance: In ecology, environmental protection, biology, medicine, meteorology, oceanology, ...
Description: Eulerian description of moving fluid

- $x = (x_1, \ldots, x_N)$ point in the space $\mathbb{R}^N$, $N = 2, 3$, x_i cartesian coordinates in $\mathbb{R}^N$,
- $\vec{v} = \vec{v}(x, t)$ velocity vector,
- $\rho = \rho(x, t)$ density of the fluid,
- $c = c(x, t)$ concentration of alloys (refered to the density of the fluid)

Mass of alloys in a control volume σ at time t is equal to

$$m(\sigma, t) = \int_\sigma \rho(x, t)c(x, t)\ \mathrm{d}x.$$

Let us consider the flow in a domain $\Omega \subset \mathbb{R}^N$ and time interval $(0, T)$. Let $v_i, \rho \in C^1(\Omega \times (0, T))$, $c \in C^2(\Omega \times (0, T))$.

Physical pospulate: Let $\sigma(t)$ be a control volume, $\sigma(t) \subset \overline{\sigma(t)} \subset \Omega$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $t_0 \in (0, T)$. Let $\sigma(t)$ is formed by the same fluid particles for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Rate of change of the total amount of the alloy in the control volume $\sigma(t)$ is equal to the production of the alloy due to sources and flux of the alloy through $\partial\sigma(t)$.

4.1 Mathematical formulation

Total amount of the alloy in $\sigma(t)$:

$$m(\sigma(t),t) = \int_{\sigma(t)} \rho(x,t)c(x,t)\ \mathrm{d}x. \tag{4.1.1}$$

Production due to sources:

$$m_{pr}(\sigma(t),t) = \int_{\sigma(t)} \rho(x,t)q(x,t)\ \mathrm{d}x, \tag{4.1.2}$$

where q is the density of sources (related to the unit of mass of the fluid). Flux through $\partial\sigma(t)$ is given by

$$Flux = m_{fl}(\sigma(t),t) = -\int_{\partial\sigma(t)} \vec{q}(x,t)\cdot\vec{n}(x,t)\ \mathrm{d}x, \tag{4.1.3}$$

where $\vec{n}(x,t)$ is unit outer normal to $\partial\sigma(t)$ and $\vec{q}(x,t)$ denotes the flux at $x \in \partial\sigma(t)$ and time t.

Fourier law: $\vec{q}(x,t) = -k\nabla c(x,t)$, where $k \geq 0$ is the diffusion coefficient .

Then the physical postulate can be formulated mathematically as the identity

$$\frac{d}{dt}\int_{\sigma(t)} \rho(x,t)c(x,t)\ \mathrm{d}x = \int_{\sigma(t)} \rho(x,t)q(x,t)\ \mathrm{d}x + \int_{\partial\sigma(t)} k\nabla c(x,t)\cdot\vec{n}(x,t)\ \mathrm{d}S.$$

If we use the transport theorem and Green's theorem, we get

$$\int_{\sigma(t)} \left(\frac{\partial\rho c}{\partial t} + \mathrm{div}\,(\rho c\vec{v})\right)(x,t)\ \mathrm{d}x = \int_{\sigma(t)} (\rho q)(x,t)\ \mathrm{d}x + \int_{\sigma(t)} \mathrm{div}\,(k\nabla c)(x,t)\ \mathrm{d}S. \tag{4.1.4}$$

Since equation (4.1.4) holds for arbitrary time $t = t_0$ and arbitrary control volume $\sigma = \sigma(t_0)$, (4.1.4) implies the differential equation

$$\frac{\partial\rho c}{\partial t} + \mathrm{div}\,(\rho c\vec{v}) = \rho q + \mathrm{div}\,(k\nabla c) \quad \text{in } Q_T = \Omega\times(0,T). \tag{4.1.5}$$

Initial condition: $c(x,0) = c^0(x), \quad x \in \Omega.$
Boundary conditions on $\partial\Omega = \Gamma_D(t)\cup\Gamma_N^1(t)\cup\Gamma_N^2(t)$, $t \in (0,T)$:

$$\begin{aligned} c(x,t) &= c_D(x,t), \quad x \in \Gamma_D(t),\\ k\frac{\partial c}{\partial\vec{n}}(x,t) \equiv k\nabla c\cdot\vec{n}(x,t) &= \varphi_N^1(x,t), \quad x \in \Gamma_N^1(t),\\ \left(k\frac{\partial c}{\partial\vec{n}} - \rho c\vec{v}\cdot\vec{n}\right)(x,t) &= \varphi_N^2(x,t), \quad x \in \Gamma_N^2(t). \end{aligned}$$

In this problem we assume that the data ρ, $\vec{v}$, k, q are known. The quantities ρ, $\vec{v}$ are obtained from the equations describing the fluid flow:

- continuity equation
- equation of motion
    - inviscid Euler equations or
    - viscous Navier-Stokes equations
- energy equation and thermodynamical relations.

Since the velocity of air in atmosphere is small, is possible to apply the incompresible model formed by the continuity equation

$$\operatorname{div} \vec{v} = 0$$

and the Navier-Stokes equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\ \nabla)\vec{v} = \vec{f} - \frac{\nabla p}{\rho} + \nu \Delta \vec{v},$$

equipped with the initial condition $\vec{v}(x,0) = \vec{v}^0(x)$, $x \in \Omega$ and the boundary condition $\vec{v}|_{\Gamma_D(t)} = \vec{v}_D$.
If the concentration c influences the flow, then it is necessary to use the Boussinesq approximation of the Navier-Stokes equations:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\ \nabla)\vec{v} = \vec{f} - \frac{\nabla p}{\rho} + \nu \Delta \vec{v} + \vec{g}c, \tag{4.1.6}$$

where $\vec{g} = (0,0,-g)$, if the axis x_3 is orthogonal to the Earth.

5

MODELLING OF COMPRESSIBLE FLOW

5.1 Inviscid flow

Let us consider inviscid, barotropic flow. It is described by the continuity equation Continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0, \tag{5.1.1}$$

equations of motion (Euler equations)

$$\frac{\partial \rho v_i}{\partial t} + \operatorname{div}(\rho v_i \vec{v}) = \rho f_i - \frac{\partial p}{\partial x_i}, \; i = 1, \ldots, N, \tag{5.1.2}$$

and the equation of state of the barotropic flow

$$p = f(\rho), \; f \in C^1(0, +\infty).$$

We shall assume that the flow is stationary ($\frac{\partial}{\partial t} = 0$). Then from equations (5.1.1) and (5.1.2) we have

$$\rho(\vec{v} \cdot \nabla)\vec{v} = \rho \vec{f} - \nabla p. \tag{5.1.3}$$

The outer volume force $\vec{f}$ is usually neglected.
Let us suppose that $\rho_0 = const > 0$ is a reference density. We choose it in such a way that it is the density at zero velocity ($\vec{v} = 0$). We define the so-called pressure function

$$P(\rho) := \int_{\rho_0}^{\rho} \frac{f'(\tau)}{\tau} \, \mathrm{d}\tau. \tag{5.1.4}$$

Then we find that

$$\begin{aligned} \nabla P(\rho) &= \nabla (P \circ \rho) = P'(\rho) \nabla \rho \\ &= \frac{f'(\rho)}{\rho} \nabla \rho = \frac{1}{\rho} \nabla (f(\rho)) = \frac{\nabla p}{\rho} \end{aligned}$$

Let us note that

$$(\vec{v} \cdot \nabla)\vec{v} = -\vec{v} \times \operatorname{rot} \vec{v} + \nabla \left(\frac{1}{2} |\vec{v}|^2 \right).$$

In what follow we shall assume that assume that the flow is irrotational, which means that $\operatorname{rot} \vec{v} = 0$. Then equation (5.1.3) becomes

$$\nabla \left(\frac{1}{2} |\vec{v}|^2 \right) = -\nabla P(\rho)$$

or

$$\nabla \left(P(\rho) + \frac{1}{2} |\vec{v}|^2 \right) = 0 \quad \text{in } \Omega. \tag{5.1.5}$$

This is satisfied if and only if

$$P(\rho) + \frac{1}{2} |\vec{v}|^2 = \text{const} \quad \text{in } \Omega. \tag{5.1.6}$$

This is a generalization of the Bernoulli equation for compressible flow, called the Saint-Venant equation.

Now we want to specify the constant *const* in equation (5.1.6). The function P is increasing and thus, there exists P_{-1} such that $P \circ P_{-1} = \mathbb{I}$. Since ρ_0 is the density at zero velocity ($\vec{v} = 0$), from equation (5.1.6) we get

$$P(\rho_0) = \text{const},$$

which yields the relation

$$P(\rho) + \frac{1}{2} |\vec{v}|^2 = P(\rho_0), \tag{5.1.7}$$

allowing to express the density in the form

$$\rho = P_{-1} \left(P(\rho_0) - \frac{1}{2} |\vec{v}|^2 \right).$$

In other words,

$$\rho = \rho \left(|\vec{v}|^2 \right). \tag{5.1.8}$$

5.2 Example of Adiabatic flow

In this case

$$p = p_0 \left(\frac{\rho}{\rho_0} \right)^{\gamma},$$

where $p_0 > 0$ is the pressure at $\vec{v} = 0$, $\rho_0 > 0$ is the density at $\vec{v} = 0$, $\gamma = \frac{c_P}{c_V} \in (1, 2)$ is the Poisson adiabatic constant. Then

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \gamma p_0 \frac{1}{\rho_0} \left(\frac{\rho}{\rho_0} \right)^{\gamma - 1} = \gamma \frac{p}{\rho}.$$

We define the local speed of sound as

$$a = \sqrt{f'(\rho)} = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}} = \sqrt{\gamma \frac{p}{\rho}},$$

and denote

$$a_0 = \sqrt{\gamma \frac{p_0}{\rho_0}}$$

the speed of sound at $\vec{v} = 0$.
Homework: Prove that

$$P(\rho) = \frac{a_0^2}{\gamma - 1}\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} - 1\right], \tag{5.2.9}$$

and

$$\rho = \rho_0 \left(1 - \frac{\gamma - 1}{2a_0^2}|\vec{v}|^2\right)^{\frac{1}{\gamma-1}}. \tag{5.2.10}$$

As a consequence of (5.2.10) we get the boundedness of the velocity.

We write

$$\rho(s) = \rho_0 \left(1 - \frac{\gamma - 1}{2a_0^2}s\right)^{\frac{1}{\gamma-1}}. \tag{5.2.11}$$

5.3 System of equations

- $\operatorname{div} \rho\vec{v} = 0$,
- $\operatorname{rot} \vec{v} = 0$,
- $\rho = \rho_0 \left(1 - \frac{\gamma-1}{2a_0^2}|\vec{v}|^2\right)^{\frac{1}{\gamma-1}}$ in Ω,
- boundary condition $\rho\vec{v}\cdot\vec{n} = \varphi_N$ on $\partial\Omega$.

Velocity potential: Let us assume that the velocity potential Φ exists: $\Phi \in C^2(\Omega)$, $\vec{v} = \nabla\Phi$ in Ω. Then from the continuity equation we get the equation

$$\operatorname{div}\left[\rho(|\nabla\Phi|^2)\nabla\Phi\right] = 0 \quad \text{in } \Omega, \tag{5.3.12}$$

which is a nonlinear second-order PDE. The boundary condition is transformed to

$$\left(\rho\,|\nabla\Phi|^2\right)\frac{\partial\Phi}{\partial\vec{n}} = \varphi_N \quad \text{on } \partial\Omega. \tag{5.3.13}$$

An interesting question is the type of the equation for Φ. Let us consider the case $N = 2$. The differentiation in (5.3.12) leads to the equation

$$\left(1 + 2\Phi_{x_1}^2\frac{\rho'}{\rho}\right)\frac{\partial^2\Phi}{\partial x_1^2} + 4\Phi_{x_1}\Phi_{x_2}\frac{\rho'}{\rho}\frac{\partial^2\Phi}{\partial x_1\partial x_2} + \left(1 + 2\Phi_{x_2}^2\frac{\rho'}{\rho}\right)\frac{\partial^2\Phi}{\partial x_2^2} = 0.$$

Let us denote the first term by A, the second term by$B$ and the third term by C. Using (5.2.11), we find that

$D = B^2 - AC = -1 + M^2$, where $M = \frac{|\vec{v}|}{a}$ is the Mach number.

Then we can see that

$D < 0 \Leftrightarrow M < 1$, i.e. elliptic equation $\Leftrightarrow$ subsonic flow,

$D > 0 \Leftrightarrow M > 1$, i.e. hyperbolic equation $\Leftrightarrow$ supersonic flow.

Thus, the equation for Φ is of the mixed of elliptic-hyperbolic type. There are **no** result on the existence of the solution, but there are numerical methods for the solution.

The above model is often used in the modelling of transonic flow past airplane wings. The obtained results are in a good agreement with reality provided the airfoils are thin, not curved too much and angles of attack are small.

6

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