NMSA405: exercise 5 – symmetric simple random walk

Definition 2.6: Let $X_1, X_2, \ldots$ be an iid random sequence such that $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. We call the corresponding random walk (S_n) the *symmetric simple random walk*.

Exercise 5.1: (Proposition 2.9) (Reflection principle) Let (S_n) be a symmetric simple random walk. Consider the stopping time T, the first hitting time of the set $\{a\}$ by the random walk for a given $a \in \mathbb{N}$. Denote

$$S_k^r = 2S_{\min\{k,T\}} - S_k, \quad k \in \mathbb{N}.$$

Then

$$(S_1^r, S_2^r, \ldots) \stackrel{d}{=} (S_1, S_2, \ldots).$$

Exercise 5.2: (Proposition 2.10) (Maxima of the symmetric simple random walk) For a symmetric simple random walk (S_n) denote $M_n = \max_{k=1,\ldots,n} S_k$, $n \in \mathbb{N}$. Consider the stopping time T, the first hitting time of the set $\{a\}$ by the random walk for a given $a \in \mathbb{N}$. Then

$$\mathbb{P}(T \le n) = \mathbb{P}(M_n \ge a) = 2\mathbb{P}(S_n \ge a) - \mathbb{P}(S_n = a) \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}(M_n \ge a) = 1.$$

NMSA405: exercise 6 – martingales

Definition 2.10: Let $\{\mathcal{F}_n\}$ be a filtration and let $X = (X_1, X_2, \ldots)$ be a sequence of integrable random variables. We say that X is an $\mathcal{F}_n$-*martingale* if it is $\mathcal{F}_n$-adapted and $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ a.s. for all $n \in \mathbb{N}$. If $\{\mathcal{F}_n\}$ is the canonical filtration of X, we call X simply a *martingale* and it satisfies $\mathbb{E}[X_{n+1}|X_1, X_2, \ldots, X_n] = X_n$ a.s. for all $n \in \mathbb{N}$. If the equality sign is replaced by $\ge$, X is called $\mathcal{F}_n$-*submartingale* or *submartingale*, respectively. If the equality sign is replaced by $\le$, X is called $\mathcal{F}_n$-*supermartingale* or *supermartingale*, respectively.

Exercise 6.1: (Proposition 2.18) Let (X_n) be a sequence of independent integrable random variables. Denote $S_n = X_1 + \ldots + X_n$ for $n \in \mathbb{N}$.

- c) If $\mathbb{E}X_n = 1$ for all $n \in \mathbb{N}$ then $Z_n = \prod_{j=1}^n X_j$ is a martingale.
- d) If $\mathbb{P}(X_n = -1) = q$ and $\mathbb{P}(X_n = 1) = p$ where $p \in (0,1)$ and $p + q = 1$ then $Y_n = (q/p)^{S_n}$ is a martingale.

Exercise 6.2: Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda|_{[0,1]})$, a finite measure $\mu \ll \lambda$ on $([0,1], \mathcal{B}([0,1]))$ and an increasing sequence of sets $\{0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = 1\}$ such that

$$\max_{k\in\{0,1,\ldots,k_n-1\}} |t_{k+1}^n - t_k^n| \to 0.$$

Denote $B_k^n = [t_k^n, t_{k+1}^n)$ and

$$D_n(x) = \frac{\mu(B_k^n)}{\lambda(B_k^n)}, \quad x \in B_k^n.$$

Show that (D_n) is an $(\mathcal{F}_n)$-martingale where $\mathcal{F}_n = \sigma(B_1^n, \ldots, B_{k_n}^n)$. What is the a.s. limit of D_n for $n \to \infty$?

Exercise 6.3: Let Y be an integrable random variable and let $(\mathcal{F}_n)$ be a filtration. Consider the sequence $X_n = \mathbb{E}[Y \mid \mathcal{F}_n]$, $n \in \mathbb{N}$, and show that (X_n) is a $\mathcal{F}_n$-martingale.

Exercise 6.4: (Pólya urn model) Consider an urn which at time $n = 0$ contains b black and w white balls, $b, w \in \mathbb{N}$. At each time $n \in \mathbb{N}$ we draw a ball from the urn at random, write down its color and put it back together with $\Delta \in \mathbb{N}$ new balls of the same color. Denote X_n the relative frequency of the white balls in the urn at time n (i.e. the ratio of the number of white balls to the number of all balls in the urn at the given time). Show that (X_n) is a martingale. Consider also the case with $\Delta = 0$ or $\Delta = -1$.

Exercise 6.5: A deck of cards contains a black and b red cards. The deck has been shuffled randomly and we start drawing the cards from the top one after another. Denote X_n the relative number of black cards after drawing n cards where $n \in \{0, \ldots, a+b-1\}$. Let $X_n = X_{a+b-1}$ for $n \geq a+b$. Show that (X_n) is a martingale.

Exercise 6.6: Let (X_n) be a sequence of random variables such that the probability density function $f_n : \mathbb{R}^n \to (0, \infty)$ of the random vector $(X_1, \ldots, X_n)$ is positive on $\mathbb{R}^n$. Suppose we are given a consistent system of probability density functions (g_n), i.e. $g_n : \mathbb{R}^n \to [0, \infty)$ fulfills $\int_{\mathbb{R}^n} g_n(x)\,\mathrm{d}x = 1$ and $\int_{\mathbb{R}} g_{n+1}(x, y)\,\mathrm{d}y = g_n(x)$ for almost all $x \in \mathbb{R}^n$. We define the *likelihood ratio*

$$S_n = \frac{g_n(X_1, \ldots, X_n)}{f_n(X_1, \ldots, X_n)}, \quad n \in \mathbb{N}.$$

Show that (S_n) is a martingale.

Exercise 6.7: Let $(\mathcal{F}_n)$ be a filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and (Q_n) a consistent system of $\mathcal{F}_n$-probability measures, i.e. $Q_{n+1}|_{\mathcal{F}_n} = Q_n$ for $n \in \mathbb{N}$, such that $Q_n \ll \mathbb{P}|_{\mathcal{F}_n}$. We define $X_n = \frac{\mathrm{d}Q_n}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_n}}$. Show that (X_n) is a $\mathcal{F}_n$-martingale.

Exercise 6.8: Let $X_n : (\Omega, \mathcal{F}) \to (S_n, \mathcal{S}_n)$, $n \in \mathbb{N}$, be a sequence of random variables. Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ and (ν_n) a consistent system of probability distributions such that $\nu_n \ll P_{X_1,\ldots,X_n} =: \mu_n$. Similarly as above show that the likelihood ratio $T_n = \frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n}(X_1, \ldots, X_n)$ between $H_1 : (X_1, \ldots, X_n)^{\mathrm{T}} \sim \nu_n$ and $H_0 : (X_1, \ldots, X_n)^{\mathrm{T}} \sim \mu_n$ is a $\sigma(X_1, \ldots, X_n)$-martingale under the null hypothesis H_0.

Exercise 6.9: Let (X_n) be an iid random sequence. Let $\alpha \in \mathbb{R}$ be such that $\beta = \ln \mathbb{E}\mathrm{e}^{\alpha X_1} \in \mathbb{R}$. We define $Z_n = \exp\{\alpha S_n - \beta n\}$ where $S_n = X_1 + \ldots + X_n$. Show that (Z_n) is a martingale.

Exercise 6.10: Let (X_n) be a sequence of independent integrable random variables with zero mean. We define $M_n = \sum_{k=1}^{n} \prod_{i=1}^{k} X_i$ for $n \in \mathbb{N}$. Show that (M_n) is a martingale.

NMSA405: exercise 7 – Doob decomposition

Definition 2.11: Let $\{\mathcal{F}_n\}$ be a filtration. The random sequence $I_1, I_2, \ldots$ is $\mathcal{F}_n$-*predictable* if I_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{N}$, where we put $\mathcal{F}_0 = \{\emptyset, \Omega\}$, i.e. I_1 is a constant.

Theorem 2.20: Let $\{S_n\}$ be an $\mathcal{F}$-submartingale. Then there exists an $\mathcal{F}_n$-martingale $\{M_n\}$ and a non-decreasing $\mathcal{F}_n$-predictable sequence $\{I_n\}$ so that $S_n = M_n + I_n, n \in \mathbb{N}$. The summands M_n and I_n are a.s. uniquely determined under the additional condition $I_1 = 0$. The sequence $\{I_n\}$ is called the *compensator* of $\{S_n\}$.

Exercise 7.1: Let (X_n) be an iid random sequence with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = \sigma^2 \in (0, \infty)$ and $\mathbb{E}\exp\{X_1\} = \gamma < \infty$. Consider the corresponding random walk (S_n). Show that the following sequences are submartingales and determine their compensators:

- a) S_n^2,
- b) $V_n = X_1^2 + \ldots + X_n^2$,
- c) $\exp\{S_n\}$.

Exercise 7.2: Let (X_n) be a $\mathcal{F}_n$-martingale such that $X_n \in L_2$. Show that

$$I_n = \sum_{k=1}^{n} \operatorname{var}(X_k \mid \mathcal{F}_{k-1})$$

is the compensator of the sequence $Z_n = X_n^2$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$.