Cvičení 7.4.

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$$\sum_{n=1}^{\infty} a_n \qquad \boxed{a_n \geq 0} \qquad \bullet\ S_n = \sum_{k=1}^{n} a_n$$

$$S_n \nearrow S \in [0,\infty] \qquad \boxed{\begin{array}{ll} \sum a_n < \infty & \text{(konverguje)} \\ \sum a_n = \infty & \text{(diverguje)} \end{array}}$$

$$\frac{1}{n^\alpha}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \qquad \boxed{\frac{n^2}{n^3+1} \sim \frac{1}{n}}$$

$$\frac{a_n}{b_n} \to L \in (0,\infty) \Rightarrow \left(\sum a_n < \infty \Leftrightarrow \sum b_n < \infty \right)$$

$$\frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \frac{n^3}{n^3+1} \to 1$$

$\boxed{q \geq 0}$

$$\sum q^n < \infty \Leftrightarrow \underline{q < 1}$$

$$\sum \frac{1}{n^\alpha} < \infty \Leftrightarrow \alpha > 1$$

$\angle \; n^{\alpha}$

$$\sum \frac{n^2}{n^3+1} < \infty \iff \sum \frac{1}{n} < \infty$$

$$\frac{n^2}{n^3+\cancel{1}} \leq \frac{n^2}{n^3} = \frac{1}{n}$$

$$\sum \frac{n^2}{n^3+1} = \infty$$

$$\sum \frac{3^n+n}{5^n+4^n} < \infty \iff \sum \left(\frac{3}{5}\right)^n < \infty$$

$$q = \frac{3}{5} < 1$$

$$\frac{3^n+n}{5^n+4^n} \sim \left(\frac{3}{5}\right)^n$$

$$\frac{\frac{3^n+n}{5^n+4^n}}{\frac{3^n}{5^n}} = \frac{1+\overset{\nearrow 0}{\frac{n}{3^n}}}{1+\underset{\searrow 0}{\left(\frac{4}{5}\right)^n}} \to 1 \in (0,\infty)$$

Výsledek:

$$\sum \frac{3^n+n}{5^n+4^n} < \infty$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

- $(L > 0) \Rightarrow \left(\sum a_n < \infty \Rightarrow \sum b_n < \infty\right)$
- $(L < \infty) \Rightarrow \left(\sum b_n < \infty \Rightarrow \sum a_n < \infty\right)$

(2)

$$\sum \frac{n^7}{4^n} \qquad \frac{1}{n^\alpha} \qquad q^n$$

$$\frac{\frac{n^7}{4^n}}{\frac{1}{3^n}} = \frac{n^7}{\left(\frac{4}{3}\right)^n} \to 0 < \infty$$

$$q = \frac{4}{3} > 1$$

$$\sum b_n = \sum \frac{1}{3^n} < \infty \Rightarrow \sum \frac{n^7}{4^n} < \infty$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \boxed{L} \in [0, \infty]$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \boxed{L} \in [0, \infty]$$

Potom:

$$L < 1 \Rightarrow \sum a_n < \infty$$

$$L<1 \Rightarrow \sum a_n < \infty$$

$$L>1 \Rightarrow \sum a_n = \infty$$

a_n obsahuje $n!$, $\binom{n}{k}$, ...

$$\sum \frac{n^7}{4^n} \; ? \qquad \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^7}{4^{\cancel{n+1}}}}{\frac{n^7}{4^{\cancel{n}}}} = \left(\frac{n+1}{n}\right)^7 \cdot \frac{1}{4} \to \frac{1}{4}$$

$$\frac{1}{4} < 1 \Rightarrow \sum \frac{n^7}{4^n} < \infty$$

$$\sum \boxed{\frac{1}{n^2}} < \infty \qquad \frac{\frac{1}{(n+1)^2}}{\frac{1}{n}} \to 1$$

$q=1$

$$\sum \boxed{\frac{1}{n}} = \infty \qquad \frac{\frac{1}{n+1}}{\frac{1}{n}} \to 1$$

$$a_n = \frac{n^7}{4^n} \qquad \sqrt[n]{a_n} = \frac{\sqrt[n]{n^7}}{4} \to \frac{1}{4} < 1$$

$$\sum \frac{n^7}{4^n} < \infty$$

$$\sum \frac{n}{4^n} < \infty$$

$\{a_n\}$ nerostoucí s nezápornými členy, potom

$$\sum a_n < \infty \iff \sum 2^n a_{2^n} < \infty$$

$\alpha > 0$

$$\sum \frac{1}{n^\alpha} < \infty \iff \sum 2^n \frac{1}{2^{n\alpha}} < \infty$$

$$\iff \sum \frac{1}{2^{n\alpha - n}} < \infty$$

$$\iff \sum \frac{1}{2^{n(\alpha-1)}} < \infty$$

$$\iff \sum \frac{1}{(2^{\alpha-1})^n} < \infty$$

$$\iff \alpha > 1$$

$\alpha > 0$

$$\sum \frac{1}{n^\alpha} < \infty \iff \int_1^\infty \frac{1}{x^\alpha} < \infty \iff \alpha > 1$$

$\log 2^n \cdot \log 2^n$

$$\sum^\infty \frac{1}{\ldots} < \infty \iff \sum^\infty 2^n \cdot \frac{1}{\ldots}$$

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < \infty \iff \sum_{n=2} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \log^2(2^n)} < \infty$$

$$\iff \sum \frac{1}{n^2} \cdot \frac{1}{\log^2 2} < \infty$$

$$\sum \frac{1}{x \log^2 x} \qquad t = \log x$$

(3) $$\sum_{n=1}^{\infty} \frac{2^n}{n!} \quad ?$$

- $\lim_{n \to \infty} a_n = 0$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1} \to 0 < 1 \Rightarrow \sum \frac{2^n}{n!} < \infty$$

$$\sqrt[n]{a_n} = \frac{2}{\sqrt[n]{n!}} \to 0 < 1$$

$\to \infty$

$$\sum \frac{n!}{2^n} = \infty$$

$$\sum \frac{n!}{2^{(n^2)}} = a_n$$

n^3+2n+1

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{2^{(n+1)^2}}}{\frac{n!}{2^{n^2}}} = \frac{n+1}{2^{2n+1}} \xrightarrow{n\to\infty} 0 < 1$$

$$\Rightarrow \sum a_n < \infty$$

a^2-b^2

(5)

$$\sum \left(\sqrt{n^3+1} - \sqrt{n^3-1}\right)$$

- $\lim\limits_{n\to\infty} a_n = 0$? ✓

$$\lim_{n\to\infty} \boxed{a_n} = \lim_{n\to\infty} \sqrt{n^3+1} - \sqrt{n^3-1} =$$

$$= \lim \frac{n^3+1-(n^3-1)}{\sqrt{} \quad \sqrt{}}$$

$$= \lim_{n\to\infty} \frac{n^3+1-(n^3-1)}{\sqrt{n^3+1}+\sqrt{n^3-1}}$$

$$= \lim_{n\to\infty} \boxed{\frac{2}{\sqrt{n^3+1}+\sqrt{n^3-1}}} = 0$$

$$\frac{2}{\sqrt{n^3+1}+\sqrt{n^3-1}} \sim \frac{1}{n^{\frac{3}{2}}} \qquad \alpha = \frac{3}{2} > 1 \Rightarrow \sum a_n < \infty$$

(8)

$$\sum \frac{4n^4+2n}{(n^2+n+111)^2} = \infty \qquad a_n \to 4 \neq 0$$

$$\sum \frac{n!}{n^n} \longrightarrow \frac{a_{n+1}}{a_n} = \frac{\frac{\cancel{(n+1)!}}{(n+1)^{n+1}}}{\frac{\cancel{n!}}{n^n}} = \frac{\cancel{n+1}}{\cancel{(n+1)}\left(\frac{n+1}{n}\right)^n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e}$$

$$\to e$$