Cvičení 26.5

středa 26. května 2021 12:22

$f: U \to \mathbb{R}$ $\qquad f \in C^2(U)$

$U \subset \mathbb{R}^d$ otevřená

nutná podmínka $\nabla f(a) = 0$

post. podmínka $H_f(a) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(a)_{i,j=1,\dots,d}$

PD $\to$ minimum

ND $\to$ maximum

IND $\to$ není extrém (sedlový bod)

$f(x,y,z) = x^2+y^2+z^2+2x+4y-6z$

$x^2+2x = (x+1)^2 - 1$

$y^2+4y = (y+2)^2 - 4$

$(z-3)^2 \dots$

$\nabla f(x,y,z) = (2x+2, 2y+4, 2z-6)$

$\nabla f = 0 \Rightarrow$

$2x+2=0$

$2y+4=0$

$2z-6=0$

$\boxed{x=-1,\ y=-2,\ z=3}$

$\boxed{(-1,-2,3)}$ je jediný stac. bod

$$H_f(x,y,z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \to PD \to (-1,-2,3) \text{ je lokální minimum}$$

(i globální)

$f(x,0,0) = x^2+2x \qquad \lim_{x\to\infty} x^2+2x = \infty$

- f nemá glob. maximum (ani lokální)

(2) $f(x,y,z) = 2xy^2 - 4xy + x^2 + z^2 - 4z \qquad (x,y,z) \in \mathbb{R}^3$

$$\nabla f(x,y,z) = \left(2y^2-4y+2x,\ 4xy-4x,\ 2z-4\right)$$

- $2y^2-4y+2x=0$
- $4xy-4x=0 \rightarrow x(y-1)=0 \rightarrow x=0 \rightarrow 2y^2-4y=0$, $y=1$
- $2z-4=0 \rightarrow z=2$

- $x=0 \rightarrow 2y^2-4y=0 \rightarrow y=0$, $2y-4=0 \rightarrow y=2$
- $y=1 \rightarrow -2+2x=0 \rightarrow x=1$

$$(0,0,2),\ (0,2,2),\ (1,1,2)$$

$$\nabla f = \left(2y^2-4y+2x,\ 4xy-4x,\ 2z-4\right)$$

$$\begin{pmatrix} 2 & 4y-4 & 0 \\ 4y-4 & 4x & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$H_f(0,0,2) = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$D_1=2,\ D_2=-16,\ D_3=-32$

IND $\rightarrow$ sedlový bod (není extrém)

$$H_f(0,2,2) = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$D_1=2,\ D_2=-16,\ D_3=-32$

IND $\rightarrow$ sedlový bod

$$H_f(0,1,2) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

IND → sedlový bod (není extrém)

$$H_f(1,1,2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow PD \rightarrow \underline{(1,1,2)} \text{ je bodem lok. minima}$$

$$f(x,y,z) = 2xy^2 - 4xy + x^2 + z^2 - 2z \qquad f(x,x,0) = 2x^3 - 3x^2 \xrightarrow[x \to -\infty]{} -\infty$$

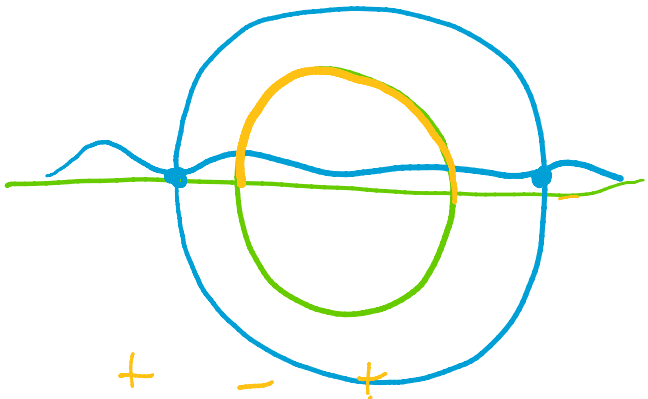
- glob. maximum neexistuje

$$f(1,1,2) = 2 - 4 + 1 + 4 - 4 = -1$$

- f není zdola omezená → glob. minimum neexistuje

(4) $\quad f(x,y) = \underline{(x^2+y^2)\, e^{-(x^2+y^2)}} \qquad (x,y) \in \mathbb{R}^2$

$$\underline{f(r\cos\alpha, r\sin\alpha)} = \underline{r^2 e^{-r^2}}$$



$$g(r) = \underline{r^2 e^{-r^2}}$$

$$g'(r) = 2r e^{-r^2} - 2r^3 e^{-r^2} = \underline{2r}\,\underline{(1-r^2)}\,\underline{e^{-r^2}}$$

$$g'(r) = 0 \iff r = 0, 1, -1$$

$\underline{r=0}$ je bodem glob. minima g

$r=0$ je bodem glob. minima g

$r=\pm 1$ je bodem glob. maxima g

$\hookrightarrow$ $(0,0)$ je bodem glob. minima f

body $\{(x,y): x^2+y^2=1\}$ jsou body glob. maxima f

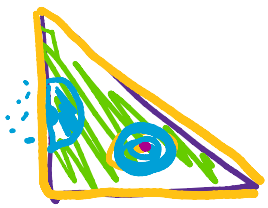
(5)

- dokázat existenci glob. extrémů
- najít podezřelé body $\rightarrow$ konečně mnoho (chceme dosadit)

$f(x,y)=x^2+y$

$M=\{(x,y)\in\mathbb{R}^2,\ x,y\geq 0,\ x+y\leq 1\}$

hledáme glob. extrémy f vzhledem k M



a je bodem glob. maxima f vzhledem k M, pokud platí $x\in M \Rightarrow f(x)\leq f(a)$

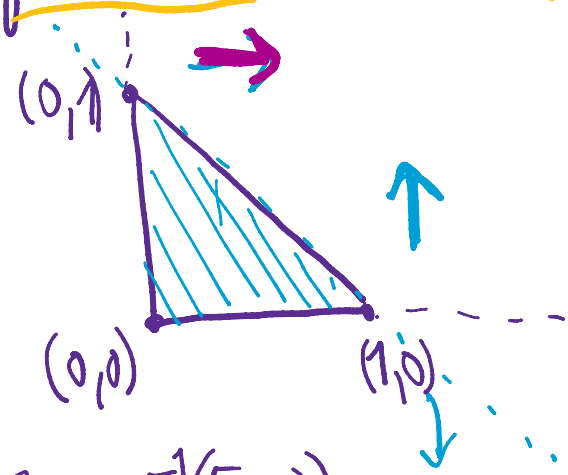
V: Nechť $M\subseteq\mathbb{R}^d$ je uzavřená a omezená, a f je spojitá vzhledem k M, potom

a f je spojitá vzhledem k M, potom

f nabývá glob. maxima i glob. minima vzhledem k M.

$x, y \geq 0$, $x+y \leq 1$

$y \leq 1-x$



$g_1 : (x,y) \to y$ $\quad \{(x,y),\ y \geq 0\} = g_1^{-1}([0,\infty))$

uzavřená mn.

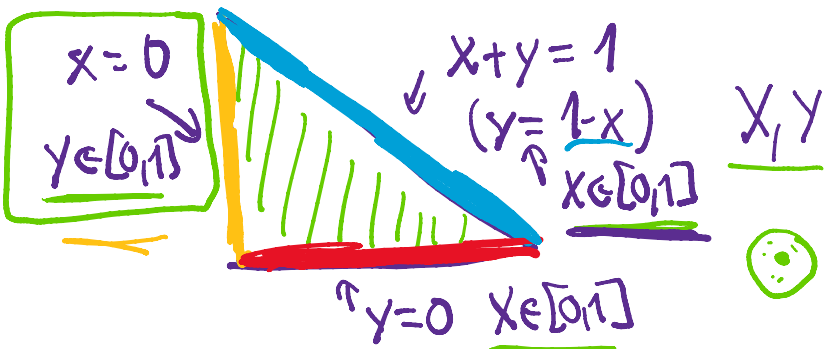
$\to x$ $\quad \{(x,y) : x \geq 0\}$

$\{(x,y),\ \underbrace{x+y-1 \leq 0}_{\text{spoj fce}}\} = g_3^{-1}([0,\infty))$

$g_3(x,y) = x+y-1$

$(x,y) \in M \Rightarrow |x|, |y| \leq 1 \qquad x^2+y^2 \leq 2$

$M \subseteq U(0,3)$ $\qquad$ M je omezená



$x, y > 0$, $x+y < 1$ $\to$ vnitřek M (M°)

$f(x,y) = x^2+y$

$y=0 \quad x\in[0,1]$ $\qquad f(x,y)=x^2+y$

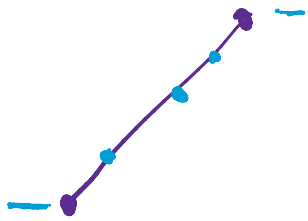
$\nabla f(x,y)=(2x,1) \qquad 1\neq 0$

tedy žádný bod M^0 nemůže být glob. extrém f vzhledem k M

- $f(0,y)=g(y) \quad y\in[0,1]$

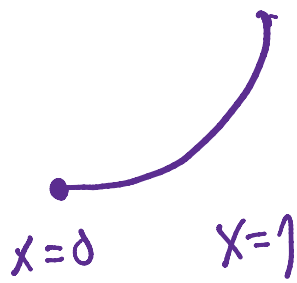
  $g(y)=y \quad y=0,1$

  $(0,0),(0,1)$



- $f(x,0)=x^2=g(x) \quad x\in[0,1]$

  $(0,0),(1,0)$

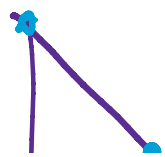


- $f(x,1-x)=x^2+1-x=g(x) \qquad x\in[0,1]$

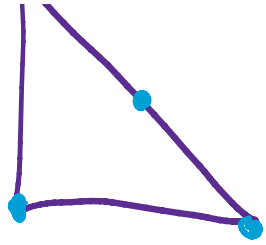
  $g'(x)=2x-1 \rightarrow g(x)=0 \Leftrightarrow x=\frac{1}{2}\in(0,1)$

  $\left(\frac{1}{2},\frac{1}{2}\right) \qquad f(x,y)=x^2+y$

Celkově: $(0,0),(0,1),(1,0),\left(\frac{1}{2},\frac{1}{2}\right)$



Celkově: $(0,0), (0,1), (1,0), (\tfrac{1}{2}, \tfrac{1}{2})$

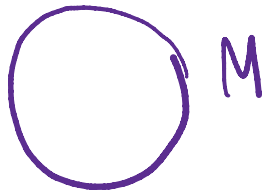


$f(0,0) = 0 \quad f(0,1) = 1, \quad f(1,0) = 1$

$f(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{3}{4}$

Tedy bod $(0,0)$ je bodem glob. minima a body $(0,1)$ a $(1,0)$ jsou body glob. maxima f vzhledem k M.

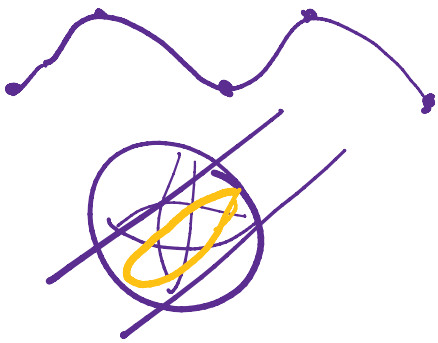
(6) $f(x,y) = x^2 + y^2 + \frac{2}{3}xy \qquad M = \{(x,y) : x^2 + y^2 = 1\}$



$$g(\alpha) = \cos^2\alpha + \sin^2\alpha + \tfrac{2}{3}\sin\alpha\cos\alpha$$

$$= 1 + \underline{\tfrac{1}{3}\sin 2\alpha} \qquad \alpha \in \mathbb{R}$$

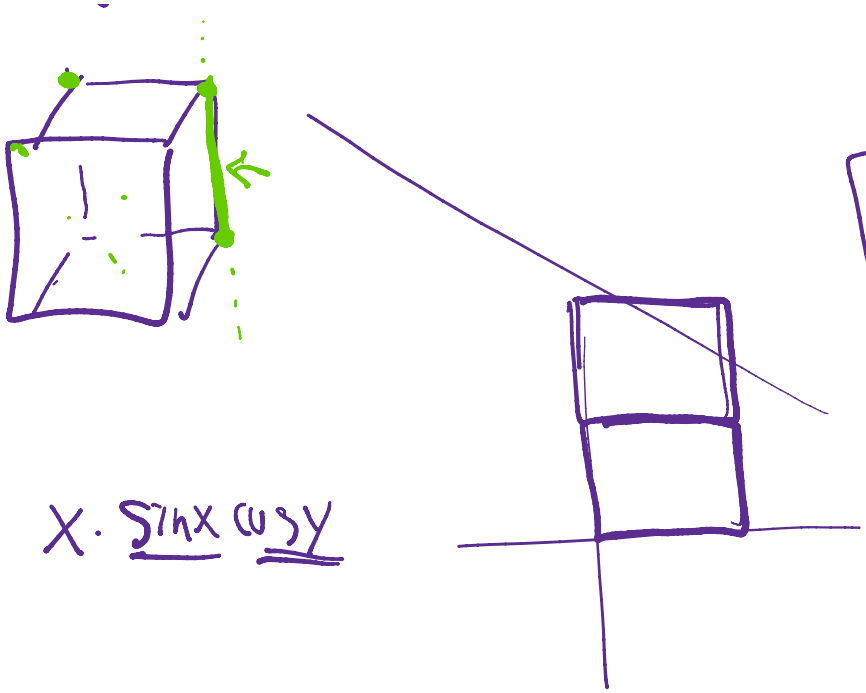
g'



(3) $f(x,y,z) = x^2 y - y e^z + 2x + z \qquad (x,y,z) \in \mathbb{R}^3$

lok. vyloh. extrémy

$(-1,1,0) \rightarrow$ není extrém



$x = 0$
$y = 1$

$x \cdot \underline{\sin x} \, \underline{\cos y}$