Cvičení 24.3.

středa 24. března 2021 12:04

$$xy' + y = y^2 \rightarrow y' = \frac{y^2 - y}{x} \qquad y' = F(x,y)$$

$y_0 = 0$, $y_1 = 1$ $\quad x \in (-\infty, 0), (0, \infty)$

$y_3 = \frac{1}{1+e^c x}$, $x \in (-\infty, -e^{-c})$

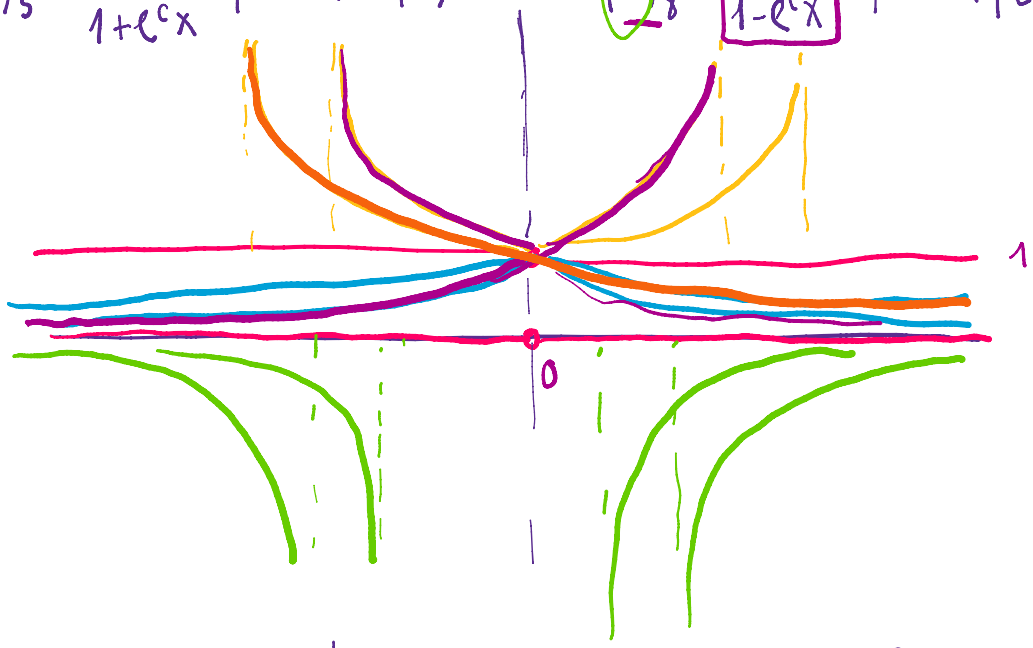
$y_4 = \boxed{\frac{1}{1-e^c x}}$, $x \in (-\infty, 0)$

$y_5 = \frac{1}{1+e^c x}$, $x \in (-e^{-c}, 0)$

$y_6 = \frac{1}{1-e^c x}$, $x \in (e^{-c}, \infty)$

$y_7 = \frac{1}{1+e^c x}$, $x \in (0, \infty)$

$y_8 = \boxed{\frac{1}{1-e^c x}}$, $x \in (0, e^{-c})$



$$\left(\frac{1}{1-e^c x}\right)' = -\frac{-e^c}{(1-e^c x)^2} = \frac{e^c}{(1-e^c x)^2} \xrightarrow{x \to 0} \boxed{e^c} > 0$$

$$\left(\frac{1}{1+e^c x}\right)' = \frac{-e^c}{(1+e^c x)^2} \xrightarrow{x \to 0} \boxed{-e^c} < 0$$

$$xy' + y = y^2 \quad \leftarrow \quad \boxed{\frac{1}{1 \pm e^c x}} \qquad x \in (-\infty, e^{-c})$$

$$0\, y'(0) = y(0)^2 - y(0) = 0 \ \checkmark$$

$y(x) = 0 \qquad x \in (-\infty, \infty)$

$y'(x) = 0 \ \checkmark$

$y(x)=0 \qquad x\in(-\infty,\infty)$

$y'(x)=0$ ✓

$y(x)=1 \qquad x\in(-\infty,\infty)$

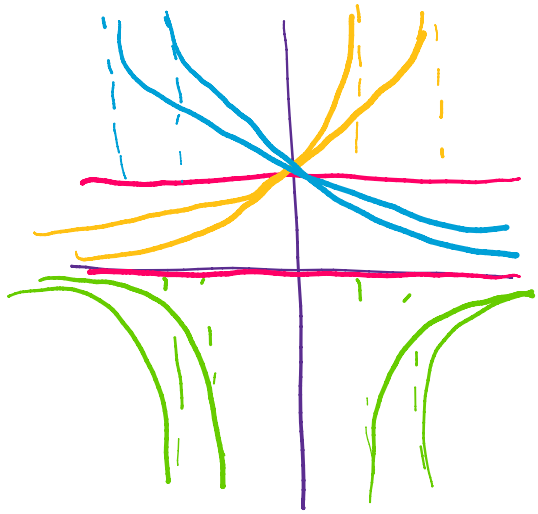
$y'(x)=0$

$$\tilde{y}_1(x) = 0 \qquad x\in\mathbb{R}$$

$$\tilde{y}_2(x) = 1 \qquad x\in\mathbb{R}$$

$$\hat{y}_3(x) = \frac{1}{1-e^c x} \qquad x\in(-\infty, e^{-c})$$

$$\tilde{y}_4(x) = \frac{1}{1+e^c x} \qquad x\in(-e^{-c}, \infty)$$



y_3, y_4

(2) podobně

(3) $\boxed{xy' = 2x - y} \rightarrow \boxed{y' = 2 - \frac{y}{x}}$ ✓

$\frac{\lambda y}{\lambda x} = \frac{y}{x}$

$F(\frac{1}{x}x, \frac{1}{x}y) = F(1, \frac{y}{x})$

Homogenní rovnice $\rightarrow$ $y' = F(x,y)$, $F(\lambda x, \lambda y) = F(x,y)$ $\lambda \neq 0$

$\boxed{z = \boxed{\frac{y}{x}}} \rightarrow \boxed{y = xz}$, $\boxed{y'} = z + xz'$

$$z + xz' = 2 - z \rightarrow z' = \underbrace{\boxed{(1-z)}}_{g(z)} \underbrace{\boxed{\frac{2}{x}}}_{S(x)}$$

(1) $I_1 = (-\infty, 0)$, $I_2 = (0, \infty)$

(1) $z_1 = 1$, $z_2 = -1$

(2) $J_1 = (-\infty, 1)$, $J_2 = (1, \infty)$

(3) $z_1 = 1$, na I_1, I_2

(4) $F(x) = \log(x^2)$ $\quad G(z) = -\log|1-z|$

$G(z) = -\log(1-z) \quad z \in J_1$

$G(z) = -\log(z-1) \quad z \in J_2$

(5) $G(J_1) = G((-\infty, 1)) = (-\infty, \infty)$

$G(J_2) = (-\infty, \infty)$

$G^{-1}(t) = 1 - e^{-t} \to J_1 \quad G^{-1}(t) = 1 + e^{-t} \to J_2$

$t = -\log(1-z) \to e^{-t} = 1 - z \to z = 1 - e^{-t}$

$t = -\log(z-1) \to e^{-t} = z - 1 \to z = 1 + e^{-t}$

$z_2(x) = 1 - e^{-(\log x^2 + c)} = 1 - \frac{1}{e^c x^2} \quad x \in (-\infty, 0)$, $x \in (0, \infty)$

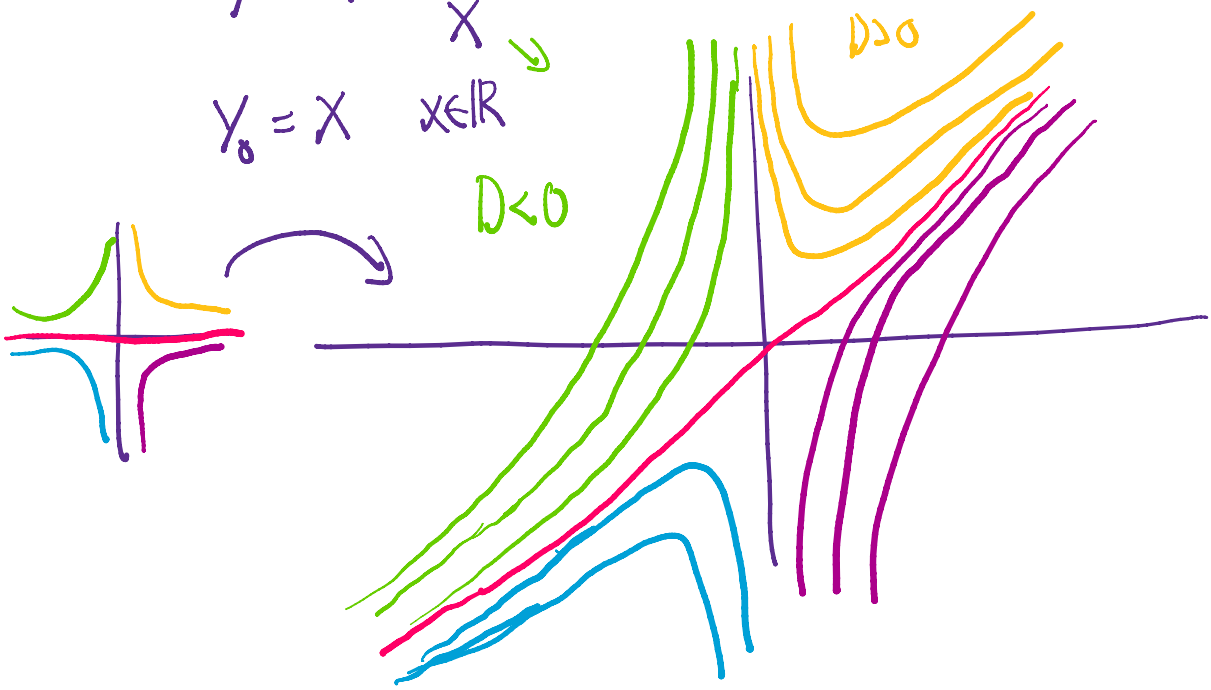
$z_3(x) = 1 + \frac{1}{e^c x^2} \quad x \in (-\infty, 0)$, $x \in (0, \infty)$

$$\boxed{\tilde{z} = 1 + \frac{D}{x^2} \qquad x \in (-\infty, 0), (0, \infty)}$$

$y = x + \frac{D}{x} \quad x \in (0, \infty), (-\infty, 0)$

$$y = x + \frac{D}{x} \qquad x \in (0,\infty), (-\infty,0)$$

$$y_0 = x \qquad x \in \mathbb{R}$$



(4) $\boxed{x^2 y' = y(x-y)} \rightarrow \boxed{y' = \frac{y}{x} - \left(\frac{y}{x}\right)^2}$

$x \neq 0$ $\quad z = \frac{y}{x} \qquad y = xz \qquad y' = z + xz'$

$$z + xz' = z - z^2 \rightarrow z' = -\frac{z^2}{x}$$

$g(z) = -z^2$

$f(x) = \frac{1}{x}$

(1) $I_1 = (-\infty,0)$, $I_2 = (0,\infty)$

(2) $J_1 = (-\infty,0)$, $J_2 = (0,\infty)$

(3) $z_0 = 0$, na I_1, I_2

(4) $F(x) = \log|x|$, $G(z) = \frac{1}{z}$

$F(x) = \log(-x) \quad x \in I_1$

$F(x) = \log(x) \quad x \in I_2$

$G(z) = \frac{1}{z} \quad z \in I$

$F(x) = \log(x) \quad x \in I_2$

$G(z) = \frac{1}{z} \quad x \in J_1, J_2$

(5) $G^{-1}(t) = \frac{1}{t}$

$G(J_1) = (-\infty, 0), \quad G(J_2) = (0, \infty)$

$F(x) + C \in (-\infty, 0)$

$I_1, J_1 \Rightarrow \quad \log(-x) + C \in (-\infty, 0) \Leftrightarrow \log(-x) \in (-\infty, -C)$

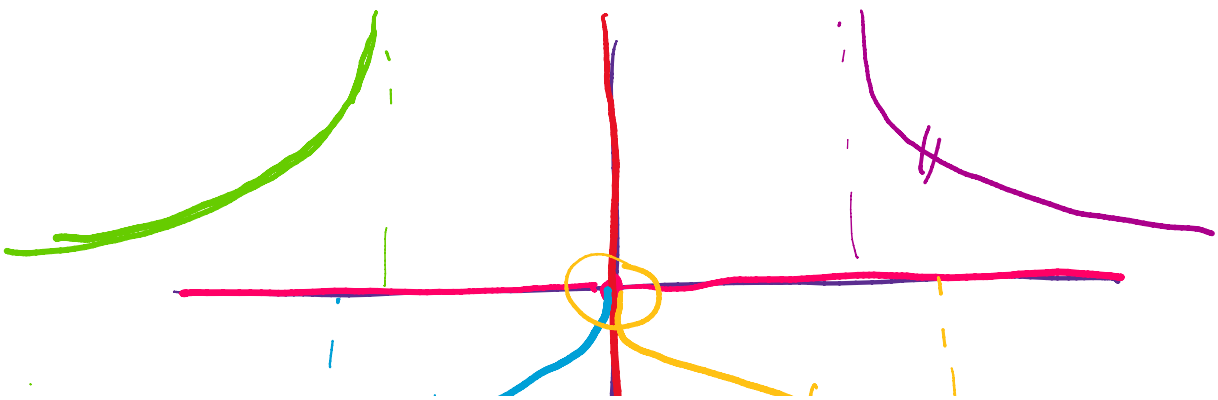
$\Leftrightarrow -x \in (0, e^{-C}) \Leftrightarrow x \in (-e^{-C}, 0)$

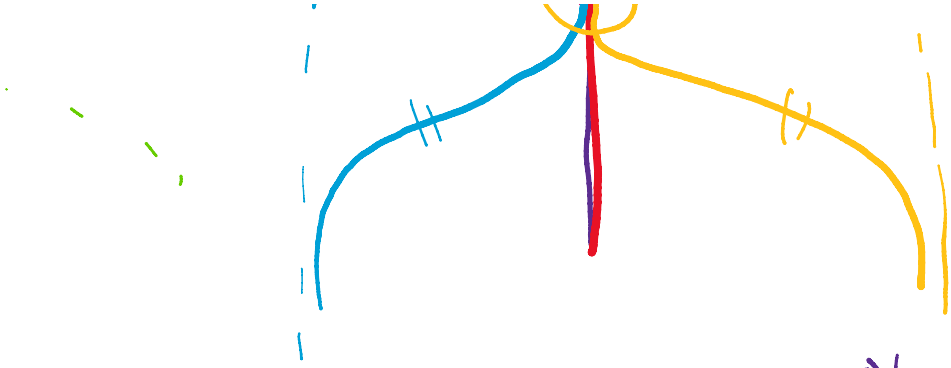
- $z_1 = \dfrac{1}{\log(-x) + C} \qquad x \in (-e^{-C}, 0)$

$I_1, J_2 \quad \log(-x) + C \in (0, \infty) \Leftrightarrow -x \in (e^{-C}, \infty), x \in (-\infty, -e^{-C})$

- $z_2 = \dfrac{1}{\log(-x) + C} \qquad x \in (-\infty, -e^{-C})$
- $z_3 = \dfrac{1}{\log x + C} \qquad x \in (0, e^{-C})$

$z_4 = \dfrac{1}{\log x + C} \qquad x \in (e^{-C}, \infty)$





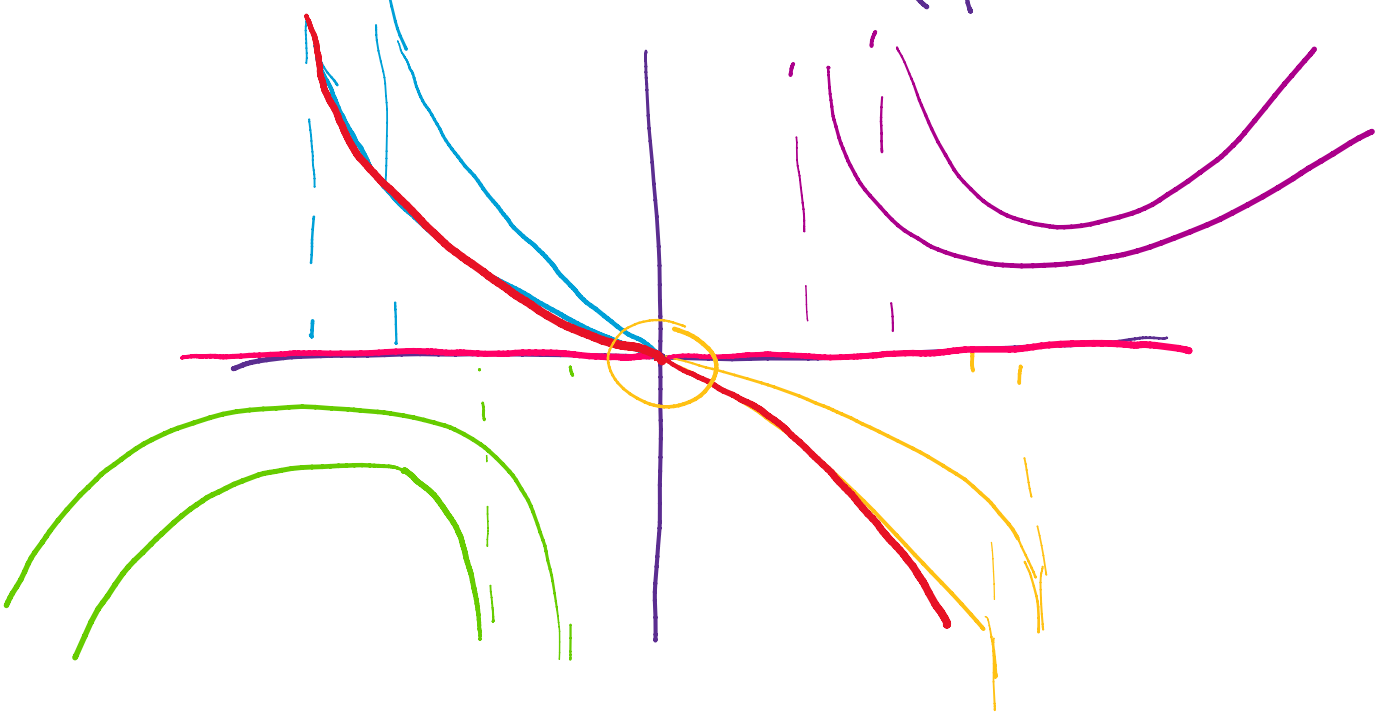
$$\left(\frac{1}{\log x + c}\right)' = -\frac{\frac{1}{x}}{(\log x + c)^2} \xrightarrow{x \to 0^+} -\infty$$

$$\lim_{x \to 0^+} x \log^2 x = 0$$

$y = xz$

$$\boxed{y = \frac{x}{\log|x| + c}}$$

$x \in (-\infty, -e^{-c}), (-e^{-c}, 0), (0, e^{c}], (e^{c}, \infty)$



$$\left(\frac{x}{\log x + c}\right)' = \frac{\log x + c - 1}{(\log x + c)^2} \xrightarrow{x \to 0^+} 0$$

$$\left(\frac{\quad}{\log x + C}\right)' = \frac{\quad}{(\log x + C)^2} \to 0$$

$$\left(\frac{x}{\log(-x) + C}\right)' \xrightarrow{x \to 0-} 0$$

$$\tilde{y} = \begin{cases} \dfrac{x}{\log(-x) + C} & x \in (-e^{-C}, 0) \\ 0 & x = 0 \\ \dfrac{x}{\log x + D} & x \in (0, e^{-D}) \end{cases}$$

potom $\tilde{y}$ je diferencovatelná na $(-e^{-C}, e^{-D})$

$$\tilde{y}'(0) = 0$$

$$x^2 y' = y(x-y)$$

$x=0$ $\quad 0 = -y^2(0) = 0$ ✓

Lineární rovnice 1. řádu

- $y' + a(x)y = b(x)$ (na I)

$y' = 1 - y \to \underline{y' + y = 1}$ $\quad a, b$ konstantní

- $\underline{A(x)}$ PF k a (na I)
- $B(x)$ PF k $\underline{b(x)e^{A(x)}}$

• $B(x)$ PF k $b(x)e^{A(x)}$

$$y(x) = B(x)e^{-A(x)} + Ce^{-A(x)} \qquad C \in \mathbb{R} \quad x \in I$$

$a(x) = 1 \rightarrow A(x) = x$

$b(x) = 1 \rightarrow B(x) = \int 1 \cdot e^{x} \overset{c}{=} e^{x}$

$$y(x) = e^{x} \cdot e^{-x} + C e^{-x} = 1 + Ce^{-x} \qquad C \in \mathbb{R},\ x \in \mathbb{R}$$

(3) $xy' = 2x - y \rightarrow y' + \frac{y}{x} = 2$

$a(x) = \frac{1}{x} \qquad b(x) = 2 \qquad (-\infty, 0), (0, \infty)$

$A(x) = \log|x| \qquad e^{A(x)} = |x|$

$B(x) \overset{c}{=} \int b(x) e^{A(x)} = \int 2 \cdot |x| \overset{c}{=} x \cdot |x|$

$$y(x) = B(x)e^{-A(x)} + Ce^{-A(x)} = x + \frac{C}{|x|} = x + \frac{C}{x}$$

$x \in (-\infty, 0), (0, \infty)$

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(2), (3), (5) podobné

(7) $y' = y + y^2$

$$y' + p(x)\,y = q(x)\,\boxed{y^\alpha} \qquad \alpha \neq 1$$

(Bernoulliho rovnice)

vynásobíme $(1-\alpha)\frac{1}{y^\alpha}$

$$\underline{y' - y = y^2} \rightarrow p(x) = -1,\ q(x) = 1,\ \alpha = 2$$

$1-\alpha = -1$

$$\boxed{-\frac{y'}{y^2}} + \boxed{\frac{1}{y}} = -1$$

$$\left(\frac{1}{y}\right)' = y' \cdot \left(-\frac{1}{y^2}\right) \qquad \boxed{z = \frac{1}{y}} \qquad \left(z = y^{1-\alpha}\right)$$

$$\underline{z' + z = -1}$$

$a(x) = 1 \quad b(x) = -1$

$A(x) = x \quad B(x) = \int (-1) \cdot e^x = -e^x$

$$z(x) = B(x) \cdot e^{-A(x)} + C \cdot e^{-A(x)} = -e^x \cdot e^{-x} + Ce^{-x}$$

$$= -1 + Ce^{-x}$$

$$y = \frac{1}{z} \longrightarrow y = \frac{1}{Ce^{-x} - 1} \quad \text{na } (-\infty$$

$$C > 0 \rightarrow Ce^{-x} = 1 \Leftrightarrow e^{-x} = \frac{1}{C} \rightarrow -x = \log\frac{1}{C}$$

$$\rightarrow x = \log C$$

$$C < 0 \rightarrow Ce^{-x} - 1 \neq 0$$

$\Rightarrow x = \log C$

$C \leq 0 \Rightarrow Ce^{-x} - 1 \neq 0$

- $y = \dfrac{1}{Ce^{-x} - 1}$ $\quad x \in \begin{cases} C > 0 & (-\infty, \log C), (\log C, \infty) \\ C \leq 0 & \mathbb{R} \end{cases}$

(9)

- $y' + 4\dfrac{y}{x} = 2x\sqrt{y} \rightarrow \boxed{\dfrac{1}{2}\dfrac{y'}{\sqrt{y}}} + \dfrac{2}{x}\boxed{\sqrt{y}} = x$

$\alpha = \dfrac{1}{2}$ $\qquad (\sqrt{y})' = \boxed{\dfrac{y'}{2}\dfrac{1}{\sqrt{y}}}$

$z = \sqrt{y} \rightarrow \bullet\ z' + \dfrac{2}{x} z = x$

$a(x) = \dfrac{2}{x}$ $\quad A(x) = \log(x^2)$ $\quad e^{A(x)} = x^2$

$b(x) = x$ $\quad B(x) \overset{\circ}{=} \int x \cdot x^2 \overset{c}{=} \dfrac{x^4}{4}$

$$z = \frac{x^2}{4} + C\frac{1}{x^2} = \frac{x^4 + 4C}{4x^2}$$

$$y = z^2 = \underbrace{\left(\frac{x^4 + 4C}{4x^2}\right)^2}_{>0} = \qquad x^4 + 4C > 0$$

$y \equiv 0$ je také řešení