Cvičení 21.4.

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(1) $$\sum_{n=1}^{\infty} \frac{(z-3)^n}{n5^n} \qquad \underbrace{z_0 = 3 \quad a_n = \frac{1}{n5^n}}$$

$$\sum_{n=1}^{\infty} a_n (z-z_0)^n$$

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

$$= \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \leftarrow \text{pokud limity existují}$$

$$a_n = \frac{1}{n5^n}$$

$$\sqrt[n]{\frac{1}{n5^n}} = \frac{1}{\sqrt[n]{n}\cdot 5} \to \frac{1}{5} \to \underline{R=5}$$

$$\frac{|a_n|}{|a_{n+1}|} = \frac{\frac{1}{n5^n}}{\frac{1}{(n+1)5^{n+1}}} = 5\,\frac{n+1}{n} \to \underline{R=5}$$

$$\overset{5}{\overset{\shortparallel}{R}} > |z-z_0| = |z-3| \to \text{mocninná řada KA}$$

$$|z-3|>5 \to \text{— || — D}$$

Co $|z-3|=5 \qquad w = z-3$

$$\sum_{n=1}^{\infty} \frac{w^n}{n5^n} \qquad w = 5(\cos\varphi + i\sin\varphi)$$

$$\sum_{n=1} n5^n$$

$$|w| = 5$$

$$w^n = 5^n(\cos n\varphi + i \sin n\varphi)$$

Re

$$\sum_{n=1}^{\infty} \frac{\cos n\varphi}{n}, \quad \sum_{n=1}^{\infty} \frac{\sin n\varphi}{n} \quad \leftarrow \text{Im}$$

konverguje pro $\varphi \neq 2k\pi$.

konverguje pro $\varphi \in \mathbb{R}$

$\sum \cos n\varphi$ má OČS

$$\sum \frac{w^n}{n5^n}$$ konverguje pro $w = 5(\cos\varphi + i\sin\varphi)$ právě tehdy, když $\varphi \neq 2k\pi, k \in \mathbb{Z}$.

Nemůže ale konvergovat absolutně (srovnáme s $\frac{1}{n}$)

(2) $$\sum_{n=1}^{\infty} a^{n^2} z^n, \quad a \in \mathbb{R}^+$$

$R = ?$ v závislosti na a

$$a_n = a^{n^2} \quad \frac{a_n}{\ldots} = \frac{a^{n^2}}{\ldots} = \frac{1}{\ldots} \to \ldots$$

$R = \infty \quad a<1$

$1 \quad a=1$

$$a_n = a^{n^2} \to \frac{a_n}{a_{n+1}} = \frac{a^{n^2}}{a^{n^2+2n+1}} = \frac{1}{a^{2n+1}} \to \begin{cases} 1 & a=1 \\ 0 & a>1 \end{cases}$$

Konvergence na hranici

$a<1$ není co řešit

$a>1 \to$ konvergence v 0

$a=1 \to \sum z^n$ - diverguje pro $|z|=1$

(4) $$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} (z-1)^n \qquad w = z-1$$

$R = ?$ $\qquad a_n = \left(1+\frac{1}{n}\right)^{n^2}$

$$\sqrt[n]{a_n} = \left(1+\frac{1}{n}\right)^n \to e \to R = \frac{1}{e}$$

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} w^n \qquad |w| = \frac{1}{e}$$

$$\downarrow |w|^n$$

$$w = \frac{1}{e}(\cos\varphi + i\sin\varphi)$$

$$w^n = \frac{1}{e^n}(\cos n\varphi + i\sin n\varphi)$$

$$\sum^{\infty} \boxed{\left(1+\frac{1}{n}\right)^{n^2} \cdot \frac{1}{e^n}} (\cos n\varphi + i\sin n\varphi)$$

$$\sum_{n=1}^{\infty} \underbrace{\boxed{\left(1+\frac{1}{n}\right)^{n^2} \cdot \frac{1}{e^n}}}_{\alpha_n} \left(\cos n\varphi + i \sin n\varphi\right)$$

$|a_n w^n| \not\to 0$

$$\lim_{n\to\infty} \alpha_n = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x^2} e^{-x}$$

$$= \lim_{x\to\infty} e^{\boxed{x^2 \log\left(1+\frac{1}{x}\right) - x}}$$

$$\lim_{x\to\infty} x^2 \log\left(1+\frac{1}{x}\right) - x = \lim_{x\to 0^+} \frac{1}{x^2} \log(1+x) - \frac{1}{x}$$

$\log(1+x) = x - \frac{x^2}{2} + o(x^2)$

$$= \lim_{x\to 0^+} \frac{\log(1+x) - x}{x^2} = \lim_{x\to 0^+} \frac{\cancel{x} - \frac{x^2}{2} - \cancel{x} + o(x^2)}{x^2}$$

$$= -\frac{1}{2}$$

$$\lim \alpha_n = e^{-\frac{1}{2}} \neq 0$$

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(5) $\displaystyle\sum_{n=0}^{\infty} n x^n = f(x)$

$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in \mathbb{R}$

$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n \quad |x|<1$

$$\frac{1}{1-x} = \sum_{n=0} x^n \qquad |x|<1$$

$$f(x)=\sum_{n=0}^{\infty} a_n x^n \longrightarrow f'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$g(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n \qquad g'(x)=\sum_{n=1}^{\infty} n x^{n-1}$$

$$x g'(x)=\sum_{n=1}^{\infty} n x^n = f(x)$$

$$f(x)=x\cdot\left(\frac{1}{1-x}\right)' = x\cdot\frac{+1}{(1-x)^2}=\frac{x}{(1-x)^2}$$

$$f(x)=\sum_{n=1}^{\infty} n x^n = x\sum_{n=1}^{\infty} n x^{n-1} = x\left(\sum_{n=1}^{\infty} x^n\right)'$$

$$= x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2} \qquad |x|<1$$

(6)

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = f(x) \qquad |x|<1$$

$$f'(x)=\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right)' = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \left(\sum_{n=1} x^n\right)' = \sum_{n=1} n x^{n-1} = \sum_{n=0} x^n$$

$$= \frac{1}{1-x} \qquad f(x) \overset{c}{=} \int \frac{1}{1-x} = -\log|1-x|$$

$-\log|1-0| = 0$

$$f(0) = 0 \rightarrow f(x) = -\log|1-x|$$

$$\left(\sum_{n=1}^{\infty} \frac{x^{4n+1}}{4n+1}\right)' = \sum_{n=1}^{\infty} x^{4n} = \sum_{n=1}^{\infty} (x^4)^n$$

$$= x^4 \sum_{n=0}^{\infty} (x^4)^n = x^4 \frac{1}{1-x^4}$$

(7) $$\sum_{n=2}^{\infty} n(n-1)x^n = x \sum_{n=2}^{\infty} n(n-1)x^{n-1}$$

$$= x\left(\sum_{n=2}^{\infty} (n-1)x^n\right)'$$

(8) $$\sum_{n=1}^{\infty} \frac{1}{n2^n} \leftarrow \sum_{n=1}^{\infty} \frac{1}{n} x^n = f(x) \qquad f\left(\tfrac{1}{2}\right)$$

$x \quad \left(\tfrac{1}{2}\right)^n$

$= x^n$

$\tfrac{1}{2}$

$$R = 1 > \tfrac{1}{2}$$

$$f(x) = -\log|1-x| \to f\left(\tfrac{1}{2}\right) = -\log\tfrac{1}{2}$$

$$|x|<1$$

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \underline{\underline{\log 2}}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n} \qquad g(1) = ?$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{2}\right)^n \quad \ldots$$

(9)

$$\sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} x^{n-1} = f(x)$$

$$R = \infty$$

$$= \left(\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}\right)'$$

$$x \sum_{n=0}^{\infty} \frac{x^n}{n!} = xe^x$$

$$f(x) = (xe^x)' = e^x + xe^x = (1+x)e^x$$

$$f(x) = (xe^x)' = e^x + xe^x = (1+x)e^x$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n!} = f(1) = \underline{\underline{2e}}$$

$$\left(x^{2n-1}\right)' = (2n-1)x^{2n-2}$$

(10) $$f(x) \quad \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} x^{2n-1}\right)' = \sum_{n=1}^{\infty} (-1)^n x^{2n-2}$$

$2(n-1)$

$k = n-1$

$n=1 \rightarrow k=0$

$(-1)(-1)^n \quad (x^2)^n$

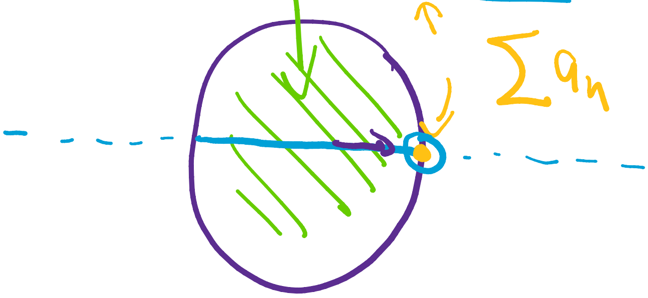
$$= \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n} = -\sum_{n=0}^{\infty} (-x^2)^n = \underline{\underline{\frac{-1}{1+x^2}}}$$

$$f(x) \overset{c}{=} \int -\frac{1}{1+x^2} = -\operatorname{arctg} x \quad x=0 \rightarrow 0$$

$$f(0) = 0 \rightarrow f(x) = -\operatorname{arctg} x \leftarrow |x|<1$$

? $$f(1) = -\operatorname{arctg} 1 = -\frac{\pi}{4} \qquad R = 1$$

$\sum a_n z^n = f(z)$



$\sum a_n$ konvergije

$$\lim_{x \to 1-} f(x) = \sum a_n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \longrightarrow \frac{1}{2n-1} \searrow 0$$

$$(-1)^n$$

řada konverguje (podle Leibnizova krit)

$$\sum \frac{(-1)^n}{2n-1} = -\frac{\pi}{4}$$

x^{n+1} $R=?$ $\sum k?$

(11) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} x^{n+1} \to f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

$x=0 \to 0$

$$= \sum_{n=1}^{\infty} \frac{(-x)^n}{n} = -\log(1+x)$$

$$f'(x) = -\log(1+x)$$

$$f(x) \stackrel{c}{=} -\int 1 \cdot \log(1+x) = \left| \begin{array}{ll} f = \log(1+x) & f' = \frac{1}{1+x} \\ g' = 1 & g = 1+x \end{array} \right|$$

$$f(x) = -\int 1\cdot\log(1+x) = \left| \begin{matrix} & \dots 1+x \\ g' = 1 & g = 1+x \end{matrix} \right|$$

$$= -(1+x)\log(1+x) + \int 1 \overset{c}{=} -(1+x)\log(x+1) + x$$

$x = 0$ $\qquad \downarrow$ 0

$$f(x) = -(1+x)\log(1+x) + x$$

$x \to 1-$

$$\sum \frac{(-1)^n}{n(n+1)} = \underline{\underline{-2\log(2) + 1}}$$