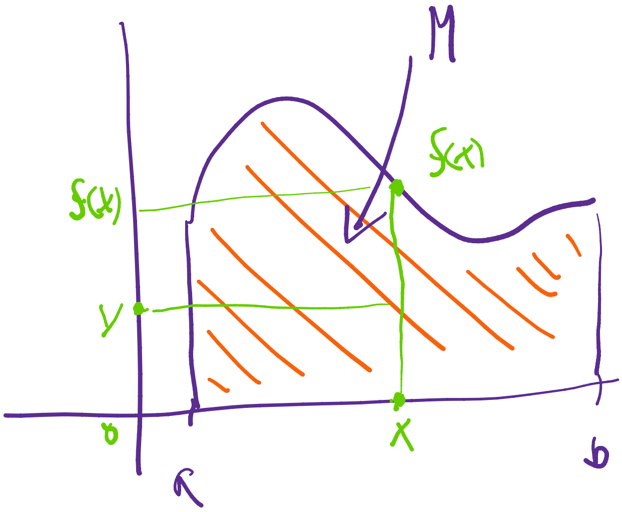
Cviceni 12.3.

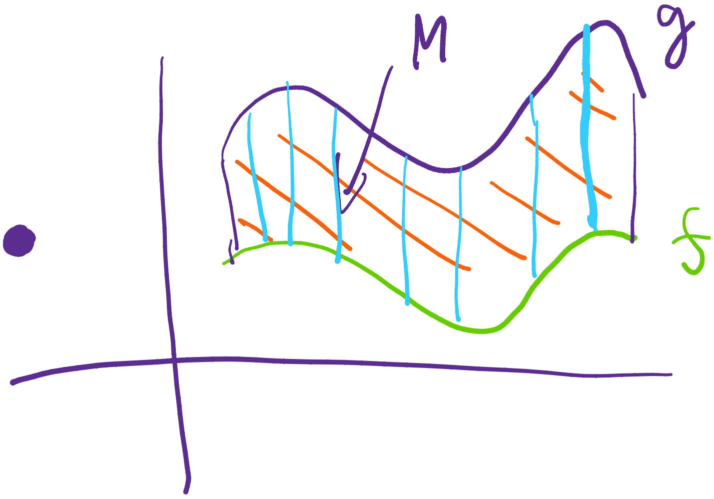
pátek 12. března 2021 11:31

$$P(M) = \int_a^b f \quad , \quad f \geq 0$$



$$M = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b \,,\, 0 \leq y \leq f(x)\}$$

$$f \leq g$$

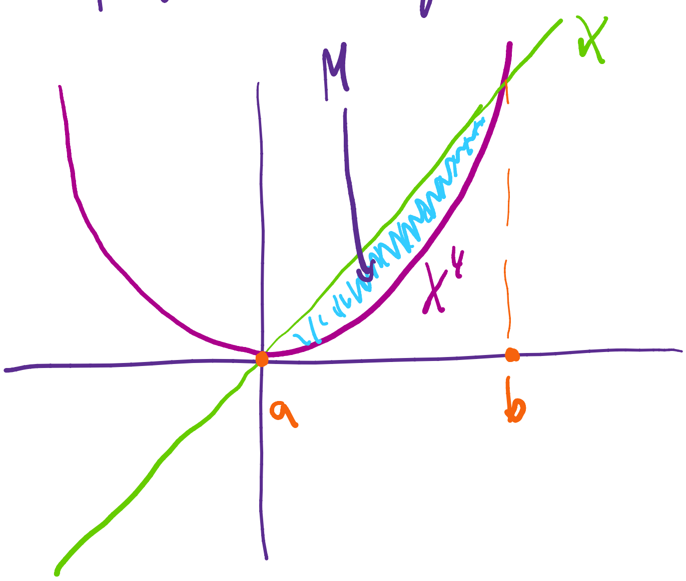


$$M = \{(x,y) : 0 \leq x \leq b \,,\, f(x) \leq y \leq g(x)\}$$

$$P(f,g,a,b) =$$

$$= P(M) = \int_a^b g - f$$

(1) plocha mezi grafy $f(x) = x$ $g(x) = x^4$



$$x = x^4 \iff x = 0,1$$

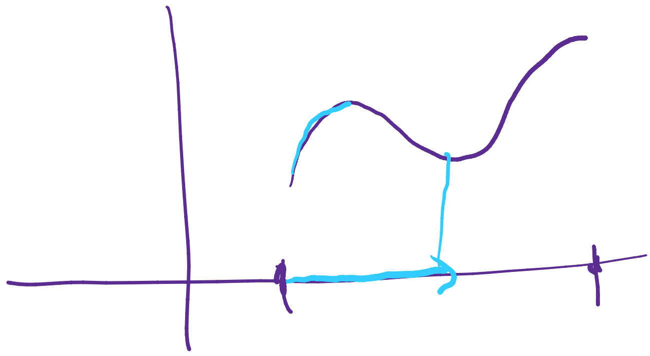
$\uparrow$

$f(x) = g(x)$

$$P(M) = \int_0^1 x - x^4 \, dx = \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

(3) délka grafu $f(x) = \arcsin x + \sqrt{1-x^2}$, $x \in [-1,1]$

$$\varphi(t) = (t, f(t))$$



$$\ell(\varphi) = \int_{-1}^{1} \sqrt{1 + (f'(x))^2} \, dx$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-2x}{2\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} = \sqrt{\frac{(1-x)^2}{1-x^2}}$$

$$= \sqrt{\frac{1-x}{1+x}}$$

$$\ell(\varphi) = \int_{-1}^{1} \sqrt{1 + \frac{1-x}{1+x}} \, dx = \int_{-1}^{1} \sqrt{\frac{1+\cancel{x}+1-\cancel{x}}{1+x}} \, dx$$

$$= \int_{-1}^{1} \sqrt{\frac{2}{1+x}} \, dx = \sqrt{2} \int_{-1}^{1} \frac{1}{\sqrt{1+x}} \, dx = \sqrt{2} \int_{-1}^{1} (1+x)^{-\frac{1}{2}} \, dx$$

$$= \left| \begin{array}{ll} t = 1+x & dt = dt \\ x = -1 \to t = 0 & \end{array} \right| = \sqrt{2} \int^{2} t^{-\frac{1}{2}} \, dt = \sqrt{2} \left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right]^2$$

$$= \left| \begin{array}{l} x=-1 \to t=0 \\ x=1 \to t=2 \end{array} \right| = \sqrt{2}\int_0 t^{-\frac{1}{2}}\,dt = \sqrt{2}\left[\frac{\;}{\frac{1}{2}}\right]_0$$

$$= \sqrt{2}\left[2\sqrt{t}\right]_0^2 = \underline{\underline{4}}$$

(5)

$$\varphi(t) = (\cos t + t\sin t,\ \sin t - t\cdot\cos t),\quad t\in[0,2\pi]$$

$$\ell(\varphi) = \int_a^b |\underbrace{\varphi'(t)}_{(\varphi_1'(t),\,\varphi_2'(t))}|\,dt = \int_a^b \sqrt{(\varphi_1'(t))^2 + (\varphi_2'(t))^2}\,dt$$

$$(t)' = 1$$

$$\varphi_1'(t) = -\sin t + \sin t + t\cos t = t\cos t$$

$$\varphi_2'(t) = \cos t - \cos t + t\sin t = t\sin t$$

$$\ell(\varphi) = \int_0^{2\pi}\sqrt{(t\cos t)^2 + (t\sin t)^2}\,dt = \int_0^{2\pi} t\underbrace{\sqrt{\cos^2 t + \sin^2 t}}_{=1}\,dt$$

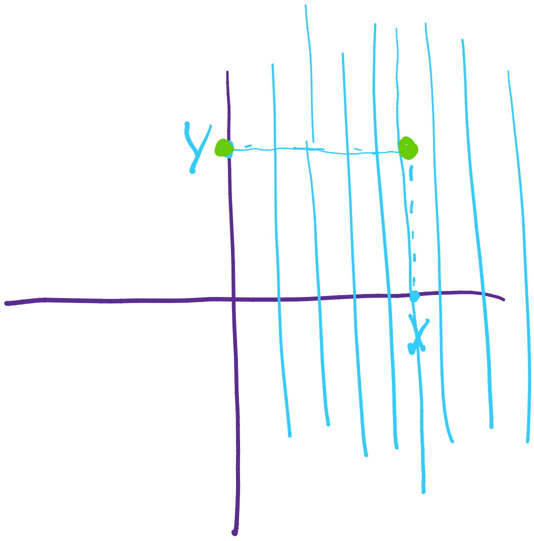
$$= \int_0^{2\pi} t\,dt = \left[\frac{t^2}{2}\right]_0^{2\pi} = \underline{\underline{2\pi^2}}$$

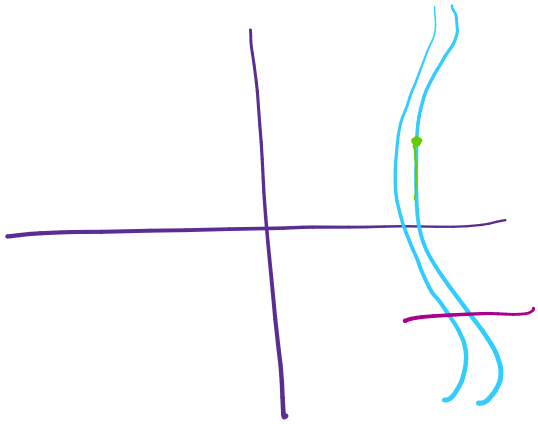
$$\varphi = (\varphi_1, \dots, \varphi_n) \qquad \varphi: \underset{[a,b]}{I} \to \mathbb{R}^n$$

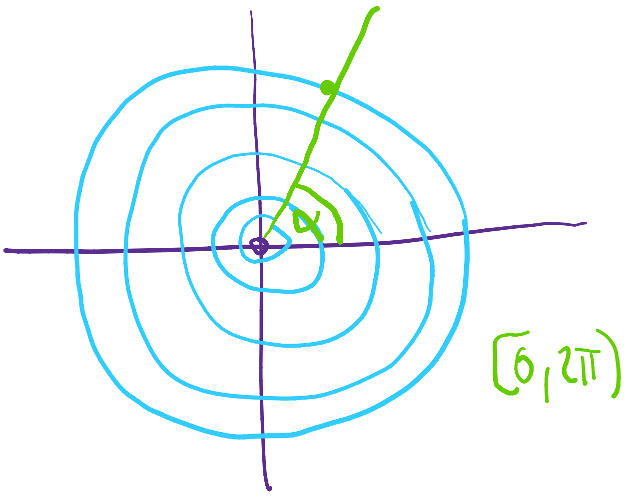
$$\ell(\varphi) = \int_a^b \sqrt{(\varphi_1')^2 + \dots + (\varphi_n')^2}$$

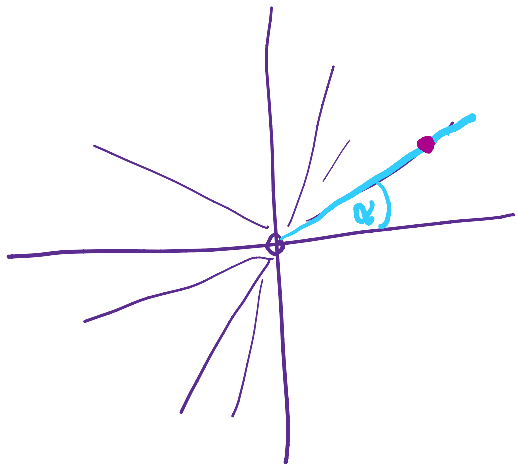
(6)

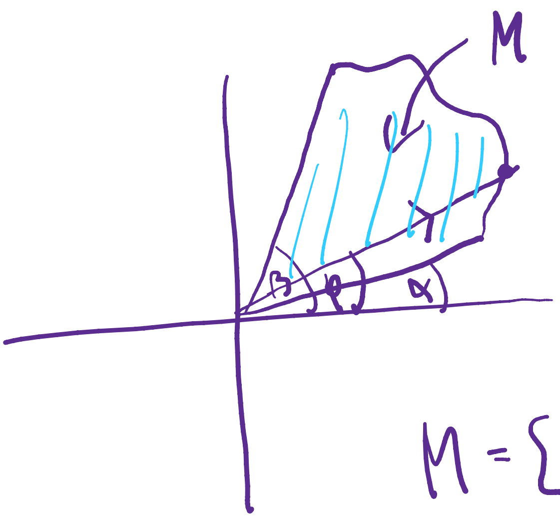
(8)





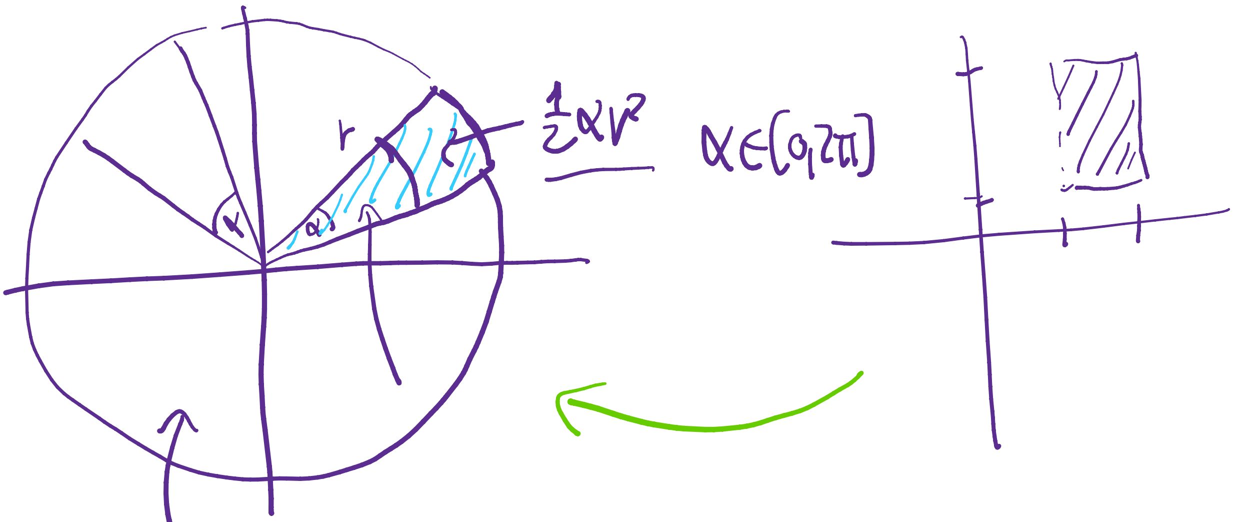






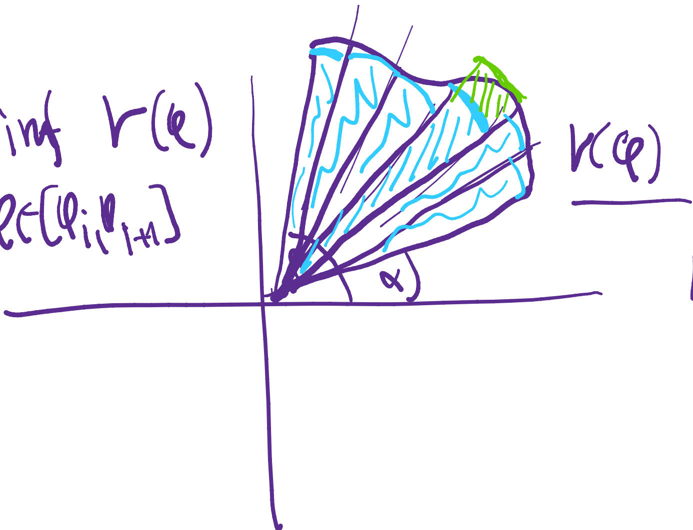
$r(\varphi)\,,\ \varphi \in [\alpha, \beta]$

$M = \{(r, \varphi) : \alpha \leq \varphi \leq \beta\,,\ 0 < r \leq r(\varphi)\}$



$$S = \pi r^2 = \frac{1}{2} 2\pi r^2 \to \frac{1}{2} \alpha r^2$$

$r_i = \inf r(\varphi)$
$\varphi \in [\varphi_i, \varphi_{i+1}]$



$D \in \mathcal{D}([\alpha, \beta])$

$\alpha = \varphi_0 < \varphi_1 < \dots < \varphi_n = \beta$

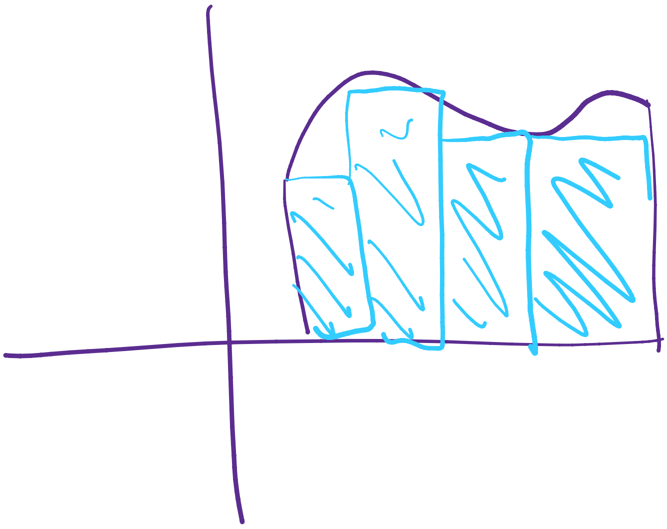
$$s(D) = \sum \frac{1}{2} r_i^2 (\varphi_{i+1} - \varphi_i) = s(g, D)$$

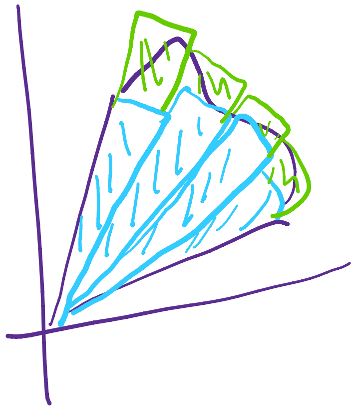
$$S(D) = \sum \frac{1}{2} R_i^2 (\varphi_{i+1} - \varphi_i) = S(g, D)$$

- $g(\varphi) = \frac{1}{2} (r(\varphi))^2$

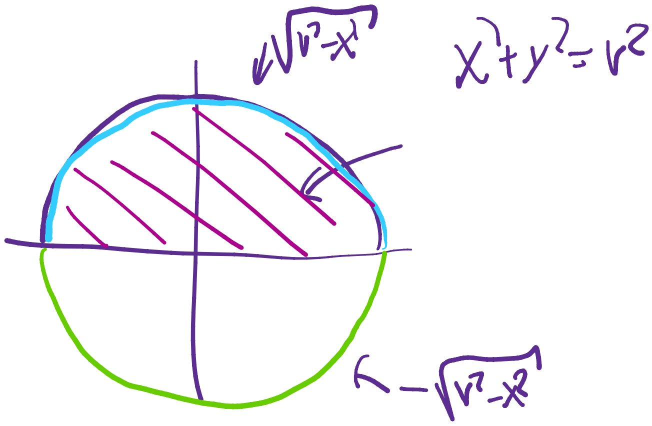
$$S = \int_\alpha^\beta g(\varphi)\, d\varphi = \frac{1}{2} \int_\alpha^\beta (r(\varphi))^2\, d\varphi$$

α α

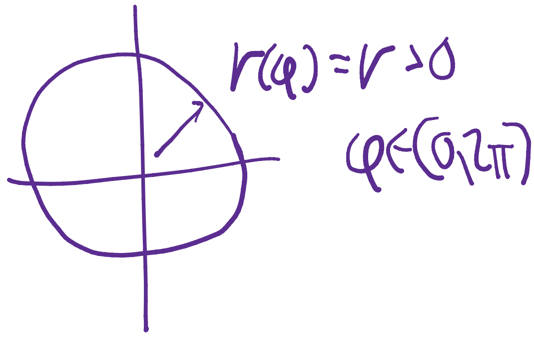




• $\displaystyle\int_0^{\pi} \sqrt{r^2-x^2}\,dx$

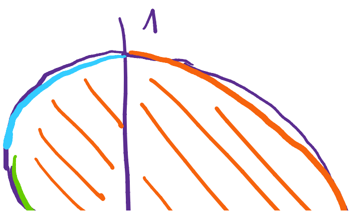


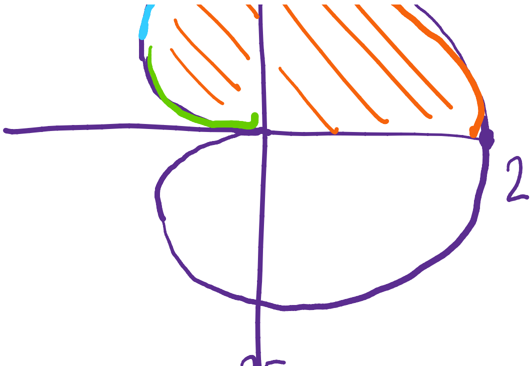
$$S=\frac{1}{2}\int_{\alpha}^{\beta} r(\varphi)^2\,d\varphi$$

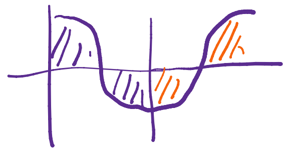


$$=\frac{1}{2}\int_0^{2\pi} r^2\,d\varphi=\frac{1}{2}r^2\cdot 2\pi=\underline{\pi r^2}$$

(8) $r(\varphi)=1+\cos\varphi$ $\quad\varphi\in[0,2\pi]$







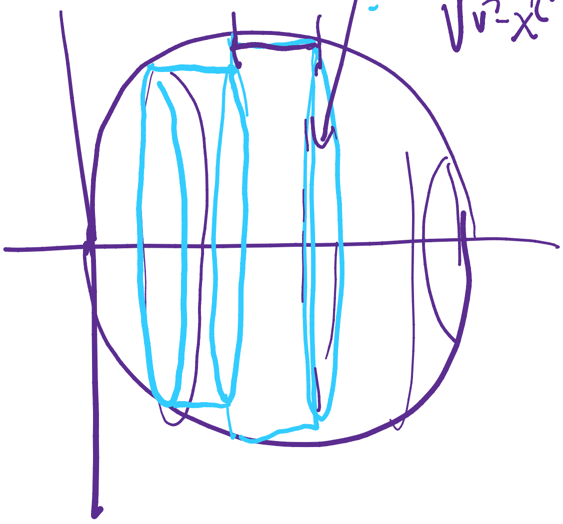
$$S=\frac{1}{2}\int_0^{2\pi}(1+\cos\varphi)^2\,d\varphi=\frac{1}{2}\int_0^{2\pi}1+2\cos\varphi+\cos^2\varphi\,d\varphi$$

$$=\frac{1}{2}\left(2\pi+0+\pi\right)=\frac{3\pi}{2}$$

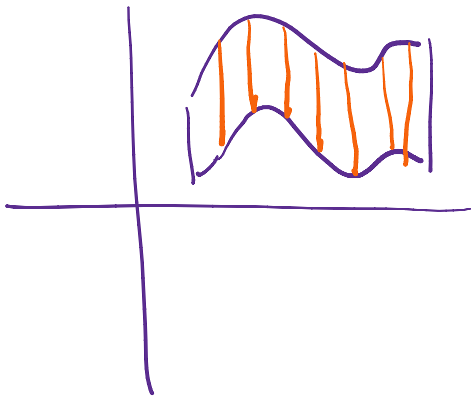
$\cos^2\varphi+\sin^2\varphi=1$

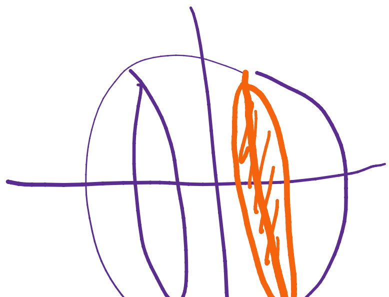
$\int_0^{2\pi}\sin^2\varphi=\int_0^{\pi}\cos^2\varphi$

(9)



$$\pi\int_a^b f^2(t)\,dt$$

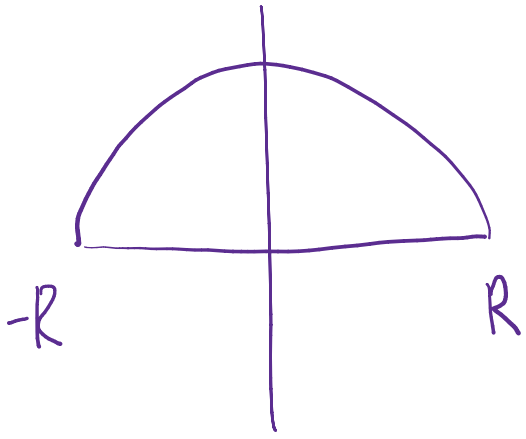




$f(x)=\sqrt{R^2-x^2}$

R

$$V = \pi \int_{-R}^{R} \left(\sqrt{R^2-x^2}\right)^2 dx$$



$$= \pi \int_{-R}^{R} R^2-x^2 \, dx = \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{R}$$

$$= \pi \left(R^3 - \frac{R^3}{3} - (-R)^3 + \frac{(-R)^3}{3} \right) = 2\pi \left(R^3 - \frac{R^3}{3} \right) = \boxed{\frac{4}{3}\pi R^3}$$

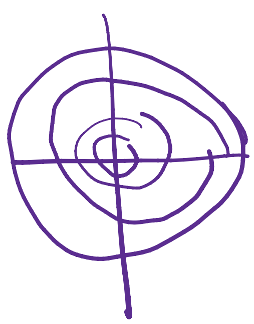
$$S = 2\pi \int_{-R}^{R} f(x) \sqrt{1 + (f'(x))^2} \, dx \qquad f(x) = \sqrt{R^2-x^2}$$

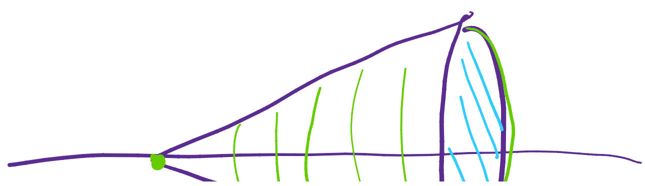
$$f'(x) = -\frac{x}{\sqrt{R^2-x^2}}$$

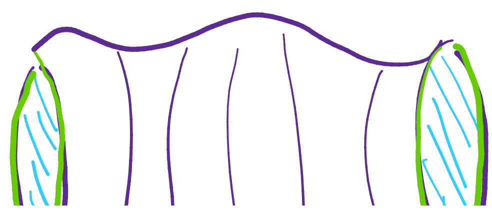
$$= 2\pi \int_{-R}^{R} \sqrt{R^2-x^2} \cdot \sqrt{1 + \frac{x^2}{R^2-x^2}}$$

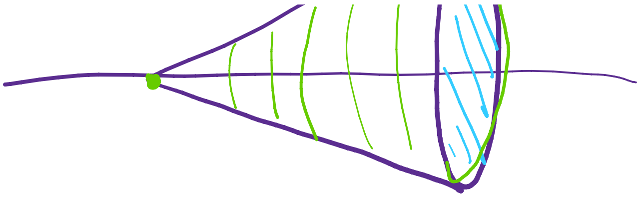
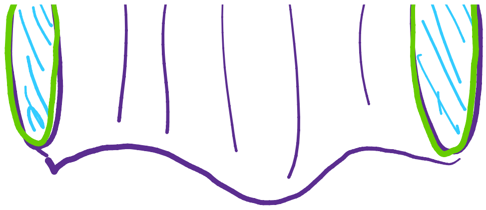
$$= 2\pi \int_{-R}^{R} \sqrt{R^2-x^2} \cdot \sqrt{\frac{R^2 - \cancel{x^2} + \cancel{x^2}}{R^2-x^2}} = 2\pi \int_{-R}^{R} R = 4\pi R^2$$

$$\left(\frac{4}{3} \pi R^3 \right)'$$

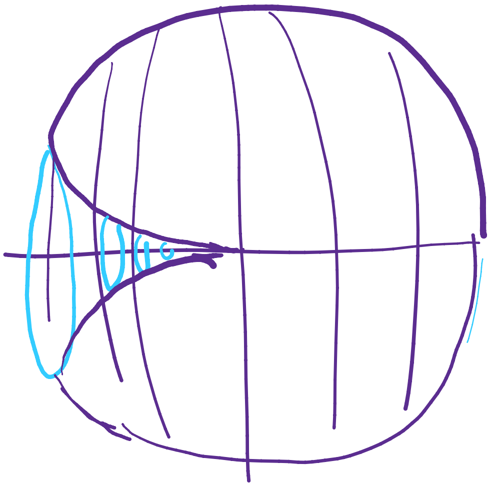






(10)



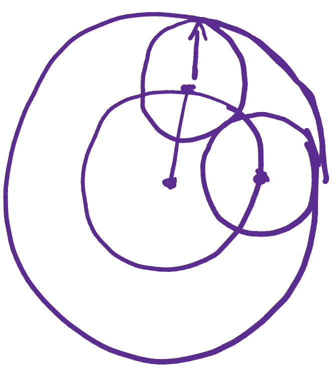
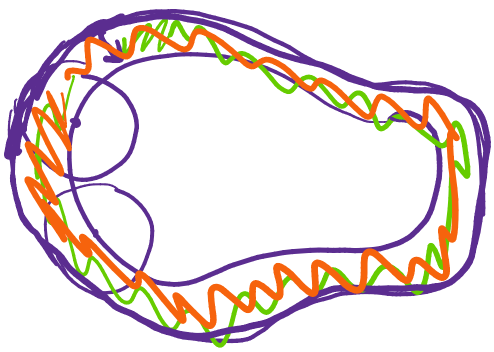
$$V = \frac{2}{3}\pi \int_{\alpha}^{\beta} r^3(\varphi)\sin\varphi \, d\varphi$$

$$r(\varphi) = 1+\cos\varphi$$

$$-\frac{2}{3}\pi \int_{0}^{\pi} \left(1+\cos\varphi\right)^3 (-\sin\varphi)\, d\varphi$$

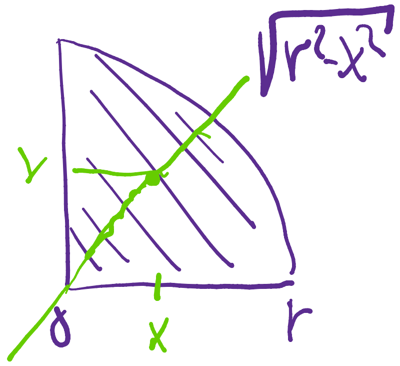
$$= \left| \begin{array}{ll} t = 1+\cos\varphi & dt = -\sin\varphi\, d\varphi \\ \varphi = 0 & \rightarrow t = 2 \\ \varphi = \pi & \rightarrow t = 0 \end{array} \right|$$

$$= -\frac{2}{3}\pi \int_{2}^{0} t^3\, dt = \frac{2}{3}\pi \int_{0}^{2} t^3\, dt = \frac{2}{3}\pi \left[\frac{t^4}{4} \right]_0^2 = \underline{\underline{\frac{8}{3}\pi}}$$

$(M_y \quad M_x)$

(11)



$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) \qquad \rho(x)=1$$

$$M_x = \frac{1}{2}\int_a^b f^2 \qquad M_y = \int_a^b x f(x)$$

$$M = \int_a^b f$$

$$M_x = \frac{1}{2}\int_0^r r^2-x^2\,dx \qquad M_y = \int_0^r x\sqrt{r^2-x^2}\,dx \qquad M = \int_0^r \sqrt{r^2-x^2}\,dx$$

$$M_y = \left|\begin{array}{ll} t=r^2-x^2 & dt=-2x\,dx \\ x=0 \to & t=r^2 \\ x=r \to & t=0 \end{array}\right| =$$

$$= -\frac{1}{2}\int_{r^2}^{0}\sqrt{t}\,dt = \frac{1}{2}\int_0^{r^2} t^{\frac{1}{2}}\,dt = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{3}t^{\frac{3}{2}}\right]_0^{r^2} = \frac{r^3}{3}$$

$$M = \int_0^r \sqrt{r^2-x^2}\,dx = \left|\begin{array}{ll} x=r\sin t & dx=r\cos t\,dt \\ x=0 \to & t=0 \\ x=r \to & t=\frac{\pi}{2} \end{array}\right|$$

$$= r^2\int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \qquad \frac{\pi r^2}{4}$$

$\underline{\overline{4}}$

$$T = \left(\frac{4}{3}\frac{r}{\pi}, \frac{4}{3}\frac{r}{\pi}\right).$$