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(1) $$\lim_{n\to\infty} \int_0^1 \underbrace{\frac{nx}{1+n^2x^2}}_{= f_n(x)}$$

Dokážeme $|f_n(x)| \leq 1 = g(x)$, $x \in [0,1]$, $n \in \mathbb{N}$.

$$\frac{nx}{1+n^2x^2} \leq 1 \iff 0 \leq 1+n^2x^2-nx$$

$$1+n^2x^2-nx \geq 1-2nx+n^2x^2 = (1-nx)^2 \geq 0$$

Protože $$\int_0^1 |g| = 1 < \infty$$

Máme podle Lebesgueovy věty o majorantě, že

$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 \lim_{n\to\infty} f_n = \int_0^1 0 = 0$$

(2)

$$\lim_{n\to\infty}\int_0^\infty e^{-x^n} \quad f_n(x)$$

$$|f_n(x)| \leq \begin{cases} 1 & x\in[0,1] \\ e^{-x} & x\in(1,\infty) \end{cases} = g(x)$$

$$\int_0^\infty |g| = \int_0^1 1 + \int_1^\infty e^{-x} = 1 + \left[-e^{-x}\right]_1^\infty = 1 + \frac{1}{e} < \infty$$

Tedy

$$\lim_{n\to\infty}\int_0^\infty f_n = \int_0^\infty \lim_{n\to\infty} f_n = \int_0^1 1 + \int_1^\infty 0 = 1$$

(3) $f_n(x) = nx\, e^{-nx^2}$

- na $(0,1)$: snadno spočteme $\int_0^1 f_n = 1 - e^{-n}$

  zároveň $f_n \to 0$ na $(0,1)$

Neplatí tedy $\lim_{n\to\infty} \int_0^1 f_n = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n$

- na $(1,\infty)$: můžeme opět spočítat $\int_1^\infty f_n = \frac{e^{-n}}{2}$

  zároveň $f_n \to 0$ na $(1,\infty)$.

  Tedy $\lim_{n\to\infty} \int_1^\infty f_n = 0 = \int_1^\infty \lim_{n\to\infty} f_n$

Můžeme argumentovat i Lebesgueovou větou

Pro $a \geq 1$ platí $a \leq 2e^{\frac{a}{2}}$, tedy

$$n x e^{-nx^2} \leq n x e^{-nx} \overset{a=nx}{\leq} 2 e^{\frac{nx}{2}} \cdot e^{-nx} = 2e^{-\frac{nx}{2}} \leq \underbrace{2e^{-\frac{x}{2}}}_{g(x)}$$

Dále $\int_1^\infty |g(x)| < \infty$.

(4) $\displaystyle\int_0^\infty \frac{\log(x+n)}{n} e^{-x} \cos x$ $\quad /\!/ f_n(x)$

$\log(x+n-1+1) \leq x+n-1$

$$\frac{\log(x+n)}{n} \leq \frac{x+n-1}{n} \leq 2x \quad x \geq 1$$

$$\leq 1 \quad x \in (0,1]$$

Celkově $\displaystyle\frac{\log(x+n)}{n} \leq 2x+1$, $x \in (0,\infty)$

Tedy $|f_n(x)| \leq (2x+1)\cdot e^{-x} =: g(x)$

Dále $\displaystyle\int_0^\infty g < \infty$. Tedy můžeme použít Lebesgueovu větu. Dostaneme

$$\lim_{n\to\infty} \int_0^\infty f_n = \int_0^\infty \lim_{n\to\infty} f_n = \int_0^\infty 0 = 0.$$

(5) $$\lim_{n\to\infty} \int_0^\infty \frac{1}{\left(1+\frac{x}{n}\right)^n \cdot \sqrt[n]{x}} \quad =: f_n(x)$$

Pro $x \in (0,1]$ platí

$$|f_n(x)| \leq \frac{1}{\sqrt[n]{x}} \leq \frac{1}{\sqrt{x}} =: g_1(x), \quad \int_0^1 |g_1| < \infty$$

Pro $x \in (1,\infty)$ platí

$$|f_n(x)| \leq \frac{1}{\left(1+\frac{x}{n}\right)^n} \leq \frac{1}{cx^2} = g_2(x)$$

$$\left(1+\frac{x}{n}\right)^n = 1 + n\cdot\frac{x}{n} + \frac{n(n-1)}{2}\cdot\frac{x^2}{n^2} + \underbrace{Q}_{\geq 0}$$

$$\geq \frac{n(n-1)}{2n^2}x^2 \geq C\cdot x^2$$

$$\int_1^\infty \frac{1}{cx^2} < \infty.$$ Můžeme tedy použít

Lebesgueovu větu s majorantou

$$g(x) = \begin{cases} g_1 & \text{na } (0,1] \\ g_2 & \text{na } (1,\infty) \end{cases}$$

Dostáváme

$$\lim_{n\to\infty} \int_0^\infty f_n = \int_0^\infty \lim_{n\to\infty} f_n = \int_0^\infty e^{-x} = 1 .$$

(10) $\displaystyle\int_0^1 \frac{\log(1-x)}{x} \overset{?}{=} -\frac{\pi^2}{6}$

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad x \in (0,1)$$

Počítáme tedy

$$\int_0^1 \frac{\log(1-x)}{x} = -\int_0^1 \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n} = -\int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$$

Levi pro řady

$$= -\sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n} = -\sum_{n=1}^{\infty} \left[\frac{x}{n^2}\right]_0^1$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{6}$$

(11) $$\int_0^\infty \frac{x}{e^x-1} = \int_0^\infty e^{-x}\frac{x}{1-e^{-x}} = \int_0^\infty e^{-x}\cdot x\cdot \sum_{n=0}^\infty e^{-nx}$$

Levi pro řady

$$= \int_0^\infty \sum_{n=0}^\infty x\cdot e^{-(n+1)x}$$

$$= \sum_{n=0}^\infty \int_0^\infty x e^{-(n+1)x}$$

$$\left[\int_0^\infty x e^{-(n+1)x} = \left[-x\,\frac{e^{-(n+1)x}}{n+1}\right]_0^\infty + \frac{1}{n+1}\int_0^\infty e^{-(n+1)x}\right.$$

$$\left. = 0 + \frac{1}{n+1}\left[-\frac{e^{-(n+1)x}}{n+1}\right]_0^\infty = \frac{1}{(n+1)^2}\right]$$

$$= \sum_{n=0}^\infty \frac{1}{(n+1)^2} = \frac{\pi^2}{6}$$

(12) $\int_0^\infty \log(1-e^{-x})$

Postup 1. $\int_0^\infty \log(1-e^{-x})$

$$= \left| \begin{array}{ll} t=e^{-x} & x=0 \to t=1 \\ dx=-\frac{1}{t}dt & x=\infty \to t=0 \end{array} \right|$$

$$= \int_1^0 -\frac{1}{t}\cdot\log(1-t) = \int_0^1 \frac{\log(1-t)}{t} = -\frac{\pi^2}{6}$$

podle (10)

Postup 2 $\log(1-x) = -\sum_{n=1}^\infty \frac{x^n}{n}$

Tedy $\int_0^\infty \log(1-e^{-x}) = -\int_0^\infty \sum_{n=1}^\infty \frac{e^{-nx}}{n}$

Levi pro řady

$$= -\sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-nx}}{n}$$

$$= -\sum_{n=1}^{\infty} \left[- \frac{e^{-nx}}{n^2} \right]_0^{\infty}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{6}$$

(13) $$\int_0^\infty e^{-ax}\sin bx = \int_0^\infty e^{-ax}\sum_{n=0}^\infty (-1)^n \cdot \frac{(bx)^{2n+1}}{(2n+1)!}.$$

$$\left| e^{-ax}\sum_{n=0}^N (-1)^n \frac{(bx)^{2n+1}}{(2n+1)!} \right| \leq e^{-ax}\sum_{n=0}^\infty \frac{(|b|x)^n}{n!} \cdot$$

$$= e^{-ax}\cdot e^{|b|x} = e^{-(a-|b|)x} \quad ,, g(x)$$

Pro $|b| < a$ platí $\int_0^\infty g < \infty$

Podle Lebesgueovy věty tedy

$$\int_0^\infty e^{-ax}\sin bx = \sum_{n=0}^\infty (-1)^n \cdot \int_0^\infty e^{-ax}\frac{(bx)^{2n+1}}{(2n+1)!}$$

per partes

$$= \sum_{n=0}^\infty (-1)^n \cdot \frac{b^{2n+1}}{a^{2n+2}}$$

$$= \frac{b}{a^2} \cdot \sum_{n=0}^{\infty} \left(- \frac{b^2}{a^2} \right)^n$$

$$= \frac{b}{a^2} \cdot \frac{1}{1 + \frac{b^2}{a^2}} = \frac{b}{a^2 + b^2}$$