Příklady 27.10.

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1) $$\lim_{x\to0}\frac{\operatorname{tg}x-\sin x}{x^3}=\lim_{x\to0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}$$

$$=\lim_{x\to0}\frac{\sin x}{x}\cdot\left(\frac{1}{\cos x}-1\right)\frac{1}{x^2}$$

$$=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2}\cdot\frac{1}{\cos x}$$

$$=\lim_{x\to0}\frac{\sin x}{x}\cdot\lim_{x\to0}\frac{1-\cos x}{x^2}\cdot\lim_{x\to0}\frac{1}{\cos x}$$

$$=1\cdot\frac{1}{2}\cdot1=\frac{1}{2}$$

- známé limity
- $\cos x$ spojitá (a tedy i $\frac{1}{\cos x}$)

2) $$\lim_{x\to1}\frac{\sin\pi x}{1-x}=\lim_{x\to1}\frac{\sin(\pi(x-1)+\pi)}{1-x}$$

$$=\lim_{x\to1}\frac{\sin(\pi(x-1))\cdot\overset{=-1}{\cos\pi}+\cos(\pi(x-1))\cdot\overset{=0}{\sin\pi}}{1-x}$$

$$=\lim_{x\to1}\frac{-\sin(\pi(x-1))}{1-x}=\pi\cdot\lim\frac{\sin(\pi(x-1))}{\pi(x-1)}=\pi$$

$f(x)=\pi(x-1)$ $\quad g(x)=\frac{\sin x}{x}$ $\quad g\circ f(x)=\frac{\sin(\pi(x-1))}{\pi(x-1)}$ $\quad =(P)$

$$\underbrace{f(x)\neq0 \quad x\in P(1,\delta)}_{(P)}$$

$\lim_{x\to1=A} f(x)=0=B$ $\qquad \lim_{x\to0} g(x)=1=C$

3) $$\lim_{x\to 0} \frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}, \quad a\in\mathbb{R}$$

$2\sin a - \sin a$

$$= \lim_{x\to 0} \frac{\sin a\cos 2x+\cos a\sin 2x-2\sin a\cos x-2\cos a\sin x+\sin a}{x^2}$$

$$= \lim_{x\to 0} 2\sin a\frac{1-\cos x}{x^2} - \sin a\frac{1-\cos 2x}{x^2} + \cos a\frac{\sin 2x-2\sin x}{x^2}$$

$$\overset{?}{=} 2\sin a\cdot\lim_{x\to 0}\frac{1-\cos x}{x^2} - \sin a\cdot\lim_{x\to 0}\frac{1-\cos 2x}{x^2} + \cos a\lim_{x\to 0}\frac{\sin 2x-2\sin x}{x^2}$$

$$= 2\sin a\cdot\frac{1}{2} - \sin a\cdot 2 + \cos a\cdot 0 = \underline{\underline{-\sin a}}.$$

$(2x=0 \Leftrightarrow x=0)\Rightarrow(P)$

VOLSK

Protože $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$, $\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0} 4\cdot\frac{1-\cos 2x}{(2x)^2} = 2$

$$\lim_{x\to 0}\frac{\sin 2x-2\sin x}{x^2} = \lim_{x\to 0}\frac{2\sin x(\cos x-1)}{x^2} = \lim_{x\to 0} 2\sin x\cdot\frac{\cos x-1}{x^2}$$

$$\overset{?}{=} 2\cdot\lim_{x\to 0}\sin x\cdot\lim_{x\to 0}\frac{\cos x-1}{x^2} = 2\cdot 0\cdot\frac{1}{2} = 0$$

? na těchto místech jsme použili aritmetiku limit. Je dobré si uvědomit, že v tu chvíli jsme ale nevěděli, zda ji vůbec můžeme použít. Zjistili jsme to až v okamžiku, kdy se ukázalo, že limity napravo existují.

6) $\lim\limits_{x\to 0} \dfrac{e^{\alpha x} - e^{\beta x}}{\sin\alpha x - \sin\beta x}$ $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$

$$= \lim_{x\to 0} e^{\beta x} \cdot \frac{e^{(\alpha-\beta)x} - 1}{\sin\alpha x - \sin\beta x} = \lim_{x\to 0} e^{\beta x} \cdot \frac{\dfrac{e^{(\alpha-\beta)x} - 1}{x}}{\dfrac{\sin\alpha x}{x} - \dfrac{\sin\beta x}{x}}$$

$$= \lim_{x\to 0} e^{\beta x} \cdot \frac{\dfrac{e^{(\alpha-\beta)x} - 1}{(\alpha-\beta)x} \cdot (\alpha-\beta)}{\dfrac{\sin\alpha x}{\alpha x} \cdot \alpha - \dfrac{\sin\beta x}{\beta x} \cdot \beta} \qquad \alpha \neq 0, \beta \neq 0$$

$$\overset{?}{=} \lim_{x\to 0} e^{\beta x} \cdot \frac{(\alpha-\beta) \cdot \lim\limits_{x\to 0} \dfrac{e^{(\alpha-\beta)x} - 1}{(\alpha-\beta)x}}{\alpha \cdot \lim\limits_{x\to 0} \dfrac{\sin\alpha x}{\alpha x} - \beta \cdot \lim\limits_{x\to 0} \dfrac{\sin\beta x}{\beta x}} = 1 \cdot \frac{(\alpha-\beta)\cdot 1}{\alpha\cdot 1 - \beta\cdot 1} = 1$$

Protože $e^{\beta x}$ je spojitá,

a $\lim\limits_{x\to 0} \dfrac{\sin\gamma x}{\gamma x}$ $\gamma \neq 0$ pro $\gamma = \alpha-\beta, \alpha, \beta$

Pokud $\alpha = 0$ nebo $\beta = 0$ (nejde oboji zároveň, protože $\alpha \neq \beta$), pak $\sin\alpha x = 0 = \alpha$, $\sin\beta x = 0 = \beta$, a dostaneme stejný výsledek.

7) $\lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\operatorname{tg} x} = \lim\limits_{x\to\frac{\pi}{2}} e^{\operatorname{tg} x \cdot \log(\sin x)}$

VOLSF e^x spojitá v 0

$= \lim\limits_{x\to\frac{\pi}{2}} e^{\operatorname{tg} x \cdot \frac{\log(\sin x)}{\sin x - 1} \cdot (\sin x - 1)} = e^{0 \cdot 1} = e^0 = 1,$

protože

● $\lim\limits_{x\to\frac{\pi}{2}} \frac{\log(\sin x)}{\sin x - 1} = 1$ "C"

$f(x) = \sin x \quad g(x) = \frac{\log x}{x-1}$

$g\circ f(x) = \frac{\log(\sin x)}{\sin x - 1}$, $x \in P(\frac{\pi}{2}, \pi) \Rightarrow f(x) \neq 0$
"A" "B"

● $\lim\limits_{x\to\frac{\pi}{2}} \operatorname{tg} x(\sin x - 1) = \lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \frac{\sin^2 x - 1}{\sin x + 1}$ (a^2-b^2)

$= \lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x}{\sin x + 1} \cdot \frac{\cos^2 x}{\cos x} = \lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x \cos x}{\sin x + 1} = 0$

spojitá v $\frac{\pi}{2}$

8) $\lim_{x\to\frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x} = \lim_{x\to\frac{\pi}{4}} e^{\operatorname{tg}2x \log(\operatorname{tg} x)}$

$$= \lim_{x\to\frac{\pi}{4}} e^{\operatorname{tg}2x\cdot(\operatorname{tg}x-1)\cdot\frac{\log(\operatorname{tg}x)}{\operatorname{tg}x-1}}$$

e^x spojitá

$$= e^{(-1)\cdot 1} = e^{-1}$$

Protože

$$\lim_{x\to\frac{\pi}{4}} \operatorname{tg}2x\cdot(\operatorname{tg}x-1) = \lim_{x\to\frac{\pi}{4}} \frac{\sin 2x}{\cos 2x}\left(\frac{\sin x}{\cos x}-1\right)$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\sin 2x}{\cos^2 x-\sin^2 x}\,\frac{\sin x-\cos x}{\cos x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-\sin 2x}{\cos x(\cos x+\sin x)} = -1$$

$$\lim_{x\to\frac{\pi}{4}} \frac{\log(\operatorname{tg}x)}{\operatorname{tg}x-1} = 1 \qquad f(x)=\operatorname{tg}x \qquad g(x)=\frac{\log x}{x-1}$$

VOLSF

$$g\circ f(x)=\frac{\log(\operatorname{tg}x)}{\operatorname{tg}x-1}$$

$$\left(x\in P\left(\tfrac{\pi}{4},\tfrac{\pi}{4}\right)\Rightarrow \operatorname{tg}x\neq 0\right) \Rightarrow (P)$$

9) $\quad a, b \neq 0$

$$\lim_{x\to 0} \frac{\log(\cos ax)}{\log(\cos bx)} = \lim_{x\to 0} \frac{\log(\cos ax)}{\cos ax - 1} \cdot \frac{\cos bx - 1}{\log(\cos bx)} \cdot \frac{\cos ax - 1}{\cos bx - 1}$$

chceme použít $\lim_{x\to 1} \frac{\log x}{x-1}$

a teď ještě $\frac{1-\cos x}{x^2}$

$$= \lim_{x\to 0} \frac{\log(\cos ax)}{\cos ax - 1} \cdot \frac{\cos bx - 1}{\log(\cos bx)} \cdot \frac{\cos ax - 1}{(ax)^2} \cdot \frac{(bx)^2}{\cos bx - 1} \cdot \frac{(ax)^2}{(bx)^2}$$

$$\overset{?}{=} \frac{a^2}{b^2} \lim_{x\to 0} \frac{\log(\cos ax)}{\cos ax - 1} \cdot \lim_{x\to 0} \frac{\cos bx - 1}{\log(\cos bx)} \cdot \lim_{x\to 0} \frac{\cos ax - 1}{(ax)^2} \cdot \lim_{x\to 0} \frac{(bx)^2}{\cos bx - 1}$$

$$= \frac{a^2}{b^2} \cdot 1 \cdot 1 \cdot \left(-\frac{1}{2}\right) \cdot \frac{1}{\left(-\frac{1}{2}\right)} = \underline{\underline{\frac{a^2}{b^2}}}$$

VOLSF vnitřní funkce ax nebo bx, vnější $\frac{\log x}{x-1}$ a $\frac{\cos x - 1}{x^2}$

$$\underbrace{ax \neq (\Leftrightarrow)\ bx \neq 0 \Leftrightarrow x \neq 0}_{\Downarrow}$$

(P)

Pokud $a=0$, potom je limita triviálně rovna 0.