Příklady 24.11.

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(1) $$\int \frac{1}{x^2-x+2}\,dx = \int \frac{1}{(x-\frac{1}{2})^2+\frac{7}{4}}\,dx = \frac{4}{7}\int \frac{1}{\left(\frac{2x-1}{\sqrt{7}}\right)^2+1}\,dx \qquad x\in\mathbb{R}$$

$$= \left| \begin{array}{l} t \doteq \frac{2x-1}{\sqrt{7}} \\ dt = \frac{2}{\sqrt{7}}\,dx \end{array} \right| = \frac{2}{\sqrt{7}}\int \frac{1}{t^2+1}\,dt \overset{c}{=} \frac{2}{\sqrt{7}}\,\mathrm{arctg}\, t = \underline{\underline{\frac{2}{\sqrt{7}}\,\mathrm{arctg}\left(\frac{2x-1}{\sqrt{7}}\right)}}$$

(3) $$\int \sin(3x-5)\,dx = \left| \begin{array}{l} t=3x-5 \\ dt=3\,dx \end{array} \right| = \frac{1}{3}\int \sin t\,dt \overset{c}{=} \underline{\underline{-\frac{1}{3}\cos(3x-5)}} \qquad x\in\mathbb{R}$$

(4) $$\int x e^{-x^2}\,dx = -\frac{1}{2}\int -2x e^{-x^2}\,dx = \left| \begin{array}{l} t=-x^2 \\ dt=-2x\,dx \end{array} \right| = -\frac{1}{2}\int e^t\,dt$$

$$\overset{c}{=} -\frac{1}{2}e^t = \underline{\underline{-\frac{1}{2}e^{-x^2}}} \qquad x\in\mathbb{R}$$

(5) $$\int x^3 a^{-x^2}\,dx = \left| \begin{array}{l} t=x^2 \\ dt=2x\,dx \end{array} \right| = \frac{1}{2}\int t\,a^{-t}\,dx$$

$\frac{1}{2}$ $\quad$ 2 $\quad$ $x\cdot x^2$

$$= \left| \begin{array}{ll} f=t & f'=1 \\ g'=a^{-t} & g=-\frac{a^{-t}}{\log a} \end{array} \right| = -\frac{1}{2}\cdot\frac{t a^{-t}}{\log a} + \frac{1}{2}\cdot\frac{1}{\log a}\int a^{-t}\,dt$$

$$= -\frac{1}{2}\cdot\frac{t a^{-t}}{\log a} - \frac{1}{2}\,\frac{a^{-t}}{\log^2 a} = \underline{\underline{-\frac{1}{2}\cdot\frac{x^2 a^{-x^2}}{\log a} - \frac{1}{2}\,\frac{a^{-x^2}}{\log^2 a}}} \qquad x\in\mathbb{R}$$

(7)

$$\int \frac{1}{\sqrt{1-x^2}(\arcsin x)^2}dx = \left| \begin{matrix} t = \arcsin x \\ dt = \frac{1}{\sqrt{1-x^2}}dx \end{matrix} \right| = \int \frac{1}{t^2} \overset{c}{=} -\frac{1}{t} = -\frac{1}{\arcsin x}$$

$x \in (-1,1)$

(8)

$$\int \sin^7 x\,dx = \int \sin x(\sin^2 x)^3 dx = \int \sin x(1-\cos^2 x)^3 dx$$

$$= \left| \begin{matrix} t = \cos x \\ dt = -\sin x\,dx \end{matrix} \right| = -\int (1-t^2)^3 dx = -\int 1-3t^2+3t^4-t^6 dx$$

$$\overset{c}{=} -t + t^3 - \frac{3t^5}{5} + \frac{t^7}{7} = -\cos x + \cos^3 x - \frac{3\cos^5 x}{5} + \frac{\cos^7 x}{7}$$

(9)

$$\int \arccos x\,dx = \left| \begin{matrix} f = \arccos x & f' = \frac{-1}{\sqrt{1-x^2}} \\ g' = 1 & g = x \end{matrix} \right| = x\arccos x \overset{-\frac{1}{2}}{+} \int \frac{\overset{-2}{x}}{\sqrt{1-x^2}}dx$$

$$\overset{c}{=} x\arccos x - \frac{1}{2}\cdot 2 \cdot \sqrt{1-x^2} = x\cdot\arccos x - \sqrt{1-x^2} \quad x\in(-1,1)$$

Podobně pak $\int \operatorname{argsinh} x\,dx$ a $\int \operatorname{argtanh} x\,dx$

(12) $\int \frac{x}{\cos^2 x}dx = \left| \begin{matrix} f=x & f'=1 \\ g'=\frac{1}{\cos^2 x} & g=\operatorname{tg}x \end{matrix} \right| = x\operatorname{tg}x - \int \frac{\sin x}{\cos x}dx$

$= \underline{\underline{x\operatorname{tg}x + \log|\cos x|}}$ maximální intervaly $(-\frac{\pi}{2}, \frac{\pi}{2}) + k\pi$

(1) $\int \frac{x^3+1}{x^3-5x^2+6x}dx = \int \frac{x^3-5x^2+6x+5x^2-6x+1}{x^3-5x^2+6x}dx$

$= \int 1\,dx + \int \frac{5x^2-6x+1}{x(x-2)(x-3)}dx = x + \int \frac{\frac{1}{6}}{x} - \frac{\frac{9}{2}}{x-2} + \frac{\frac{28}{3}}{x-3}dx$

$\frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$

$A = \frac{5x^2-6x+1}{(x-2)(x-3)}\Big|_{x=0} = \frac{1}{6}$ $\qquad B = \frac{5x^2-6x+1}{x(x-3)}\Big|_{x=2} = \frac{9}{-2}$

$C = \frac{5x^2-6x+1}{x(x-2)}\Big|_{x=3} = \frac{28}{3}$

$$= x + \frac{1}{6}\int \frac{1}{x}\,dx - \frac{9}{2}\int \frac{1}{x-2}\,dx + \frac{28}{3}\int \frac{1}{x-3}\,dx$$

$$\stackrel{c}{=} \underline{\underline{x + \frac{1}{6}\log|x| - \frac{9}{2}\log|x-2| + \frac{28}{3}\log|x-3|}}$$

maximální intervaly $(-\infty,0), (0,2), (2,3), (3,+\infty)$.

(2) $$\int \frac{1}{(x^3+1)^2}\,dx = \int \frac{1}{(x+1)^2(x^2-x+1)^2}\,dx$$

$$\frac{1}{(x+1)^2(x^2-x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1} + \frac{Ex+F}{(x^2-x+1)^2} \quad / \cdot (x+1)^2(x^2-x+1)^2$$

$$0x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 1$$

$$1 = A(x+1)(x^2-x+1)^2 + B(x^2-x+1)^2 + (Cx+D)(x+1)^2(x^2-x+1) + (Ex+F)(x+1)^2$$

$$= A(x^5 - x^4 + x^3 + x^2 - x - 1) + B(x^4 - 2x^3 + 3x^2 - 2x + 1)$$

$$+ (Cx+D)(x^4 + x^3 + x + 1) + (Ex+F)(x^2 + 2x + 1)$$

$$= x^5(A+C) + x^4(-A+B+D+C) + x^3(A-2B+D+E)$$

$$+ x^2(A+3B+C+F+2E) + x(-A-2B+C+D+E+2F)$$

$$+ (A+B+D+F)$$

Tedy dostáváme soustavu

$$\left.\begin{aligned}
A+C&=0\\
-A+B+C+D&=0\\
A-2B+D+E&=0\\
A+3B+C+2E+F&=0\\
-A-2B+C+D+E+2F&=0\\
A+B+D+F&=1
\end{aligned}\right\}$$

řešení

$A=\frac{2}{9}$ $B=\frac{1}{9}$ $C=-\frac{2}{9}$

$D=\frac{1}{3}$ $E=-\frac{1}{3}$ $F=\frac{1}{3}$

$$\int\frac{1}{(x^3+1)^2}=\int\frac{\frac{2}{9}}{x+1}\,dx+\int\frac{\frac{1}{9}}{(x+1)^2}\,dx+\int\frac{-\frac{2}{9}x+\frac{1}{3}}{x^2-x+1}\,dx+\int\frac{-\frac{1}{3}x+\frac{1}{3}}{(x^2-x+1)^2}\,dx$$

Parciální zlomky už pak integrujeme standardním postupem.

(3) $$\int \frac{\sin x \cos x}{1+\sin^3 x}\,dx = \left| \begin{matrix} t=\sin x \\ dt = \cos x\,dx \end{matrix} \right| = \int \frac{t}{1+t^3}\,dt$$

$$\frac{t}{1+t^3} = \frac{t}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$$

$$A = \left.\frac{t}{t^2-t+1}\right|_{t=-1} = \frac{-1}{3}$$

$$\frac{Bt+C}{t^2-t+1} = \frac{t}{(t+1)(t^2-t+1)} - \frac{-\frac{1}{3}}{t+1} = \frac{t+\frac{t^2}{3}-\frac{t}{3}+\frac{1}{3}}{(t+1)(t^2-t+1)}$$

$$= \frac{1}{3}\cdot\frac{t^2+2t+1}{(t+1)(t^2-t+1)} = \frac{1}{3}\cdot\frac{(t+1)^2}{(t+1)(t^2-t+1)} = \frac{1}{3}\,\frac{t+1}{t^2-t+1}$$

$$\int \frac{t}{1+t^3}\,dt = -\frac{1}{3}\int\frac{1}{t+1}\,dt + \frac{1}{3}\int\frac{t+1}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\int\frac{2t-1}{t^2-t+1}\,dt + \frac{1}{2}\int\frac{1}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log(t^2-t+1) + \frac{2}{3}\int\frac{1}{\left(\frac{2t-1}{\sqrt{3}}\right)^2+1}\,dt$$

$$\overset{c}{=} -\frac{1}{3}\log|t+1| + \frac{1}{6}\log(t^2-t+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2t-1}{\sqrt{3}}\right)$$

$$= -\frac{1}{3}\log|\sin x+1| + \frac{1}{6}\log(\sin^2 x-\sin x+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2\sin x-1}{\sqrt{3}}\right)$$

maximální intervaly $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)+2k\pi \quad k\in\mathbb{Z}$

(4)

$$\int \frac{\sin^3 x}{\cos^4 x}\,dx = \int \sin x \frac{\sin^2 x}{\cos^4 x}\,dx = \int \sin x \frac{1-\cos^2 x}{\cos^4 x}\,dx$$

$$= \left| \begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array} \right| = -\int \frac{1-t^2}{t^4}\,dt = -\int \frac{1}{t^4}\,dt + \int \frac{1}{t^2}\,dt$$

$$\overset{c}{=} \frac{1}{3}\cdot\frac{1}{t^3} - \frac{1}{t} = \underline{\underline{\frac{1}{3}\,\frac{1}{\cos^3 x} - \frac{1}{\cos x}}}$$

maximální intervaly $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) + k\pi \quad k \in \mathbb{Z}$.

(7)

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \end{array} \right| = \int \frac{\cos t}{(1-\sin^2 t)^{\frac{3}{2}}}\,dt$$

$$= \int \frac{1}{\cos^2 t}\,dt \overset{c}{=} \operatorname{tg} t = \operatorname{tg}(\arcsin x) \qquad x \in (-1,1).$$

(5) $$\int \sqrt{a^2+x^2}\,dx = \left| \begin{matrix} x = a\cdot \sinh t \\ dx = a\cdot \cosh t\, dt \end{matrix} \right|$$

$$= \int \sqrt{a^2+a^2\sinh^2 t}\cdot a\cdot \cosh t\, dt$$

$$= |a|\cdot a\cdot \int \cosh^2 t\, dt$$

$1+\sinh^2 t = \cosh^2 t$

$\cosh t > 0 \quad t\in\mathbb{R}$

$$\int \cosh^2 t\, dt = \left| \begin{matrix} f = \cosh t & f' = \sinh t \\ g' = \cosh t & g = \sinh t \end{matrix} \right|$$

$$= \sinh t \cosh t - \int \sinh^2 t\, dt = \sinh t \cosh t - \int \cosh^2 t - 1\, dt$$

$$= \sinh t\cdot \cosh t - \int \cosh^2 t\, dt + \int 1\, dt$$

Tedy $$\int \cosh^2 t\, dt \overset{c}{=} \frac{1}{2}\left(t + \sinh t \cosh t\right)$$

A celkově dostáváme

$$\int \sqrt{a^2+x^2}\,dx = \underline{\underline{|a|\cdot a\cdot \frac{1}{2}\left(\operatorname{argsinh}\frac{x}{a} + \frac{x}{a}\cdot \cosh\left(\operatorname{argsinh}\frac{x}{a}\right)\right)}}$$