Příklady 22.12.

úterý 29. prosince 2020 16:27

(2) $f(x)=e^{\sin x}$ $n=5$ $x=0$

$$T^5_{0,e^x}(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}$$

$$T^5_{0,\sin x}(x)=x-\frac{x^3}{6}+\frac{x^5}{120}$$

$$T^5_{0,e^{\sin x}}(x)=1+x-\frac{x^3}{6}+\frac{x^5}{120}+\frac{1}{2}\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^2+\frac{1}{6}\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^3$$

$$+\frac{1}{24}\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^4+\frac{1}{120}\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^5+o(x^5)$$

$$=1+x+\frac{1}{2}x^2+x^3\left(-\frac{1}{6}+\frac{1}{6}\right)+x^4\left(\frac{1}{2}\cdot\left(-\frac{1}{6}\right)\cdot 2+\frac{1}{24}\right)$$

$$+x^5\left(\frac{1}{120}+\frac{1}{6}\left(-\frac{1}{6}\right)\cdot 3+\frac{1}{120}\right)$$

$$=1+x+\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^5}{15}$$

(4) $T^3_{0,\sin x}(x) = x - \dfrac{x^3}{6}$

$$T^3_{0,\sin(\sin x)}(x) = x - \frac{x^3}{6} - \frac{1}{6}\left(x - \frac{x^3}{6}\right)^3 + o(x^3)$$

$$= x + x^3\left(-\frac{1}{6} - \frac{1}{6}\right) = x - \frac{1}{3}x^3$$

$$T^3_{0,\sin(\sin(\sin x))}(x) = x - \frac{1}{3}x^3 - \frac{1}{6}\left(x - \frac{1}{3}x^3\right)^3 + o(x^3)$$

$$= x + x^3\left(-\frac{1}{3} - \frac{1}{6}\right) = x - \frac{1}{2}x^3$$

$$T^4_{0,\sin(\sin(\sin(\sin x)))}(x) = x - \frac{1}{2}x^3 - \frac{1}{6}\left(x - \frac{1}{2}x^3\right)^3 + o(x^3)$$

$$= x + x\left(-\frac{1}{2} - \frac{1}{6}\right) = \underline{\underline{x - \frac{2}{3}x^3}}$$

(5) $T^n_{0,\sin}(x) - \sin x = \dfrac{\sin^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ pro nějaké $\xi \in (0,x)$

Tedy $\left|T^n_{0,\sin}(1) - \sin 1\right| \leq \dfrac{|\sin^{(n+1)}(\xi)|}{(n+1)!}1^{n+1} \leq \dfrac{1}{(n+1)!}$.

Stačí tedy aby $10^5 \leq (n+1)!$ což je pro $n \geq 8$

(9)

$$(a^x)' = \log a \cdot a^x \quad (a^x)'' = \log^2 a \cdot a^x$$

tedy $T^2_{0,a^x}(x) = 1 + \log a \cdot x + \frac{\log^2 a}{2} x^2$

$$(a^{-x})' = -\log a \cdot a^x \quad (a^{-x})'' = \log^2 a \cdot a^x$$

tedy $T^2_{0,a^{-x}}(x) = 1 - \log a \cdot x + \frac{\log^2 a}{2} x^2$

$$\lim_{x \to 0} \frac{a^x + a^{-x} - 2}{x^2} = \lim_{x \to 0} \frac{T^2_{0,a^x}(x) + T^2_{0,a^{-x}}(x) + o(x^2) - 2}{x^2}$$

$$= \lim_{x \to 0} \frac{\log^2 a \cdot x^2 + o(x^2)}{x^2} = \underline{\underline{\log^2 a}}$$

(10) $(\arctan x)' = \dfrac{1}{1+x^2}$, $(\arctan x)'' = -\dfrac{2x}{(1+x^2)^2}$

$(\arctan x)''' = \dfrac{6x^2-2}{(1+x^2)^3}$ (po úpravách)

tedy $T^3_{0,\arctan x}(x) = x - \dfrac{x^3}{3}$

a
$$\lim_{x\to 0} \frac{\arctan x - x}{\sin x - x} = \lim_{x\to 0} \frac{T^3_{0,\arctan x}(x) - x + o(x^3)}{T^3_{0,\sin x}(x) - x + o(x^3)}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{3} + o(x^3)}{-\frac{x^3}{6} + o(x^3)} = \underline{\underline{2}}$$

(11)
$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2} x \sin x}{\log^4(1+x)}$$

$$T^4_{0,x\sin x}(x) = x \cdot T^4_{0,\sin x} + o(x^4) = x\left(x - \frac{x^3}{6}\right) + o(x^3) = x^2 - \frac{x^4}{6}$$

Dále použijeme $T^4_{1,\log x}(x) = x-1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}$

$$T^4_{0,\log(1+x)}(x) = T^4_{1,\log x}(1+x)$$
$$= 1+x-1-\frac{(1+x-1)^2}{2}+\frac{(1+x-1)^3}{3}-\frac{(1+x-1)^4}{4}$$
$$= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$$

$$T^4_{0,\log^4(1+x)}(x) = \left[T^4_{1,\log x}(1+x)\right]^4 + o(x^4)$$
$$= \left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\right)^4 + o(x^4) \quad = x^4$$

Protože $T^4_{0,\cos x}(x) = 1-\frac{x^2}{2}+\frac{x^4}{24}$, dostáváme

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x\cos x}{\log^4(1+x)} = \lim_{x\to 0} \frac{1-\frac{x^2}{2}+\frac{x^4}{24}-1+\frac{1}{2}\left(x^2-\frac{x^4}{6}\right)+o(x^4)}{x^4+o(x^4)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{24}x^4+o(x^4)}{x^4+o(x^4)} = \underline{\underline{-\frac{1}{24}}}$$