- $\boxed{}$ použití poznámky o rovnosti na okolí
- $\boxed{}$ využití spojitosti (aritmetiky spojitosti atd.)

5)
$$\lim_{x\to 0} \frac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}} = \lim_{x\to 0} \frac{x^2}{x^2}\cdot\frac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}}$$

$$\boxed{=} \lim_{x\to 0} \frac{2+x^2}{\sqrt{3-6x^2+5x^4}} \boxed{=} \frac{2+0}{\sqrt{3-0+0}} = \frac{2}{\sqrt{3}}$$

6)
$$\lim_{x\to 1} \frac{x+x^2+\dots+x^n-n}{x-1}, \quad n\in\mathbb{N}.$$

Nejdříve si z čitatele vytkneme $(x-1)$.

$$\begin{aligned} x+x^2+\dots+x^n-n &= (x-1)+(x^2-1)+ \dots +(x^n-1) \\ &= (x-1)\big[\underline{1}+\underline{(x+1)}+\underline{(x^2+x+1)} \\ &\quad + \dots + \underline{(x^{n-1}+\dots+x+1)}\big] \\ &= (x-1)\big[n+(n-1)x+\dots+x^{n-1}\big] \end{aligned}$$

Tedy

$$\lim_{x\to 0} \frac{x+x^2+\dots+x^n-1}{x-1} \boxed{=} \lim_{x\to 1} n+(n-1)x+\dots+x^{n-1}$$

$$\boxed{=} n+(n-1)+\dots+2+1$$

vzoreček z 2. cvičení

$$= \frac{n(n+1)}{2}$$

7) $\lim\limits_{x\to 0} \dfrac{(1+mx)^n-(1+nx)^m}{x^2}$, $n,m\in\mathbb{N}$.

Čitatel upravíme pomocí binomické věty

$$(1+mx)^n = 1+\binom{n}{1}mx+\binom{n}{2}(mx)^2+\sum_{k=3}^{n}\binom{n}{k}(mx)^k$$

$$= 1+nmx+\frac{n(n-1)}{2}m^2x^2+x^3\underbrace{\sum_{k=3}^{n}\binom{n}{k}m^k x^{k-3}}_{P(x)\text{ - polynom}}$$

analogicky

$$(1+nx)^m = 1+mnx+\frac{m(m-1)}{2}n^2x^2+x^3\underbrace{\sum_{k=3}^{n}\binom{m}{k}n^k x^{k-3}}_{Q(x)\text{ - polynom}}$$

Tedy

$$(1+mx)^n-(1+nx)^n = 1+nmx+\frac{n(n-1)}{2}m^2x^2+x^3P(x)$$

$$-\left(1+mnx+\frac{m(m-1)}{2}n^2x^2+x^3Q(x)\right)$$

$$= \frac{n(n-1)m^2-m(m-1)n^2}{2}x^2+\big(P(x)-Q(x)\big)x^3$$

$$= \frac{mn(n-m)}{2}x^2+\big(P(x)-Q(x)\big)x^3$$

Celkové

$$\lim_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2} = \lim_{x\to 0}\frac{mn(n-m)}{2}+\left(P(x)-Q(x)\right)x$$

$$= \frac{mn(n-m)}{2}+\left(P(0)-Q(0)\right)\cdot 0$$

$$= \underline{\underline{\frac{mn(n-m)}{2}}}\,.$$

10) $$\lim_{x\to 0^+}\frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}-1}}{x} = \lim_{x\to 0^+}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}$$

$x\to 0^+ \Rightarrow \sqrt{x^2}=x$

$$= \lim_{x\to 0^+}\frac{1+x^2-(1-x^2)}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$$

$$= \lim_{x\to 0^+}\frac{2}{\sqrt{1+x^2}+\sqrt{1-x^2}}$$

$$= \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = \underline{\underline{1}}$$

$$a-b=\frac{a^2-b^2}{a+b}$$

11) $$\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4} = \lim_{x\to 16}\frac{\sqrt{x}-4}{(\sqrt{x}-4)(\sqrt[4]{x}+2)}$$

$$= \lim_{x\to 16}\frac{1}{\sqrt[4]{x}+2} = \frac{1}{\sqrt[4]{16}+2} = \underline{\underline{\frac{1}{4}}}$$

12) $$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}-\sqrt{1-x}}$$

Za použití vzorečku $a^6-b^6=(a-b)(a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5)$

upravíme na tvary

$$\sqrt{1+x}-\sqrt[3]{1-x}=\frac{(1+x)^3-(1-x)^2}{(1+x)^{\frac{5}{2}}+(1+x)^{\frac{4}{2}}(1-x)^{\frac{1}{3}}+(1+x)^{\frac{3}{2}}(1-x)^{\frac{2}{3}}+(1+x)^{\frac{2}{2}}(1-x)^{\frac{3}{3}}+(1+x)^{\frac{1}{2}}(1-x)^{\frac{4}{3}}+(1-x)^{\frac{5}{3}}}$$

$$=\frac{(1+x)^3-(1-x)^2}{f(x)} \qquad f \text{ spojitá v } 0 \qquad f(0)=6$$

a podobně

$$\sqrt[3]{1+x}-\sqrt{1-x}=\frac{(1+x)^2-(1-x)^3}{g(x)} \qquad g \text{ spojitá v } 0 \qquad g(0)=6$$

Celkem tedy dostáváme

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+1}-\sqrt{1-x}}=\lim_{x\to 0}\frac{(1+x)^2-(1-x)^3}{(1+x)^3-(1-x)^2}\cdot\frac{g(x)}{f(x)}$$

$$=\lim_{x\to 0}\frac{1+2x+x^2-1+3x-3x^2+x^3}{1+3x+3x^2+x^3-1+2x-x^2}\ \frac{g(x)}{f(x)}$$

$$=\lim_{x\to 0}\frac{x^3-2x^2+5x}{x^3+2x^2+5x}\cdot\frac{f(x)}{g(x)}\ \boxed{=}\ \lim_{x\to 0}\frac{x^2-2x+5}{x^2+2x+5}\cdot\frac{f(x)}{g(x)}$$

$$\boxed{=}\ \frac{5}{5}\cdot\frac{6}{6}=\underline{\underline{1}}$$

13) $$\lim_{x\to 0}\frac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}} = \lim_{x\to 0}\frac{27+x-(27-x)}{x\left(1+2x^{\frac{1}{3}}\right)\underbrace{\left((27+x)^{\frac{2}{3}}+(27+x)^{\frac{1}{3}}(27-x)^{\frac{1}{3}}+(27-x)^{\frac{2}{3}}\right)}_{f(x)}}$$

$$a-b=\frac{a^3-b^3}{a^2+ab+b^2}$$

f spojitá v 0 $\quad f(0)=27$

$$=\lim_{x\to 0}\frac{2x}{x\left(1+2x^{\frac{1}{3}}\right)\,f(x)} \boxed{=} \lim_{x\to 0}\frac{2}{\left(1+2x^{\frac{1}{3}}\right)\cdot f(x)}$$

$$\boxed{=}\ \frac{2}{(1+0)f(0)}=\frac{2}{27}$$