Řešení 19.11.

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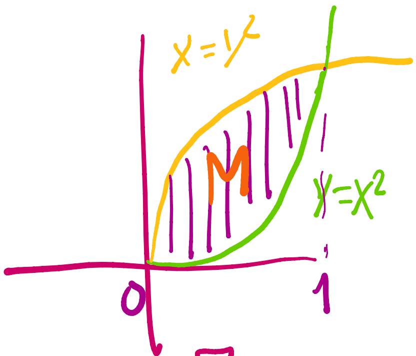
(2) $M = \{(x,y) : x+2 \geq y \geq x^2\}$

$$\lambda_2(M) = \int \chi_M = \int_{-1}^{2} \int_{x^2}^{x+2} 1 \, dy \, dx = \int_{-1}^{2} \left[y \right]_{y=x^2}^{y=x+2} dx = \int_{-1}^{2} x+2-x^2$$

$x+2 \geq x^2 \iff x \in [-1,2]$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = 2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3} = \underline{\underline{\frac{9}{2}}}$$

(5) $M = \{(x,y) : x \geq y^2,\ y \geq x^2\}$, $f(x,y) = x^2 + y$



$$\int_M f = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 + y \, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 x^{\frac{5}{2}} + \frac{x}{2} - x^4 - \frac{x^4}{2} \, dx = \left[\frac{2}{7} x^{\frac{7}{2}} + \frac{x^2}{4} - \frac{3}{2} \cdot \frac{1}{5} x^5 \right]_0^1$$

$$= \frac{2}{7} + \frac{1}{4} - \frac{3}{10} = \underline{\underline{\frac{33}{140}}}$$

(7) $M=\{(x,y): x^2+y^2 \leq x\}$ $\quad f=\frac{1}{\sqrt{x^2+y^2}}$

Použijeme polární souřadnice $x=r\cos\alpha,\ y=r\sin\alpha$, tak dostaneme podmínku

$$r^2 \leq r\cos\alpha \rightarrow \quad 0<r<\cos\alpha$$

Pro $\varphi(r,\alpha)=(r\cos\alpha, r\sin\alpha)$ platí

$J_\varphi = r$, $\{(x,y): 0<x^2+y^2<x\}=\varphi(U)$,

kde $U=\{(r,\alpha): 0<r<\cos\alpha\}$

Tedy
$$\int_M f = \int_{\varphi(U)} f = \int_U J_\varphi \cdot f\circ\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos\alpha} r\frac{1}{r}\,dr\,d\alpha$$

$\varphi(U)\subseteq M$

$\lambda_2(M\setminus\varphi(U))=0$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\alpha = \underline{\underline{2}}$$

(9) M omezená plochami $x=y^2, y=x^2, z=0, z=xy$

$f(x,y,z)=xyz$

$$M=\{(x,y,z)\in\mathbb{R}^3 : \; x^2\leq y\leq\sqrt{x},\; 0\leq z\leq xy\}$$

$\Downarrow$

$x\in[0,1]$

$$\int_M f=\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{xy} xyz\,dz\,dy\,dx$$

$$=\int_0^1\int_{x^2}^{\sqrt{x}}\left[\frac{xyz^2}{2}\right]_{z=0}^{z=xy}dy\,dx$$

$$=\int_0^1\int_{x^2}^{\sqrt{x}}\frac{x^3y^3}{2}\,dy\,dx=\int_0^1\left[\frac{x^3y^4}{8}\right]_{y=x^2}^{y=\sqrt{x}}dx$$

$$=\int_0^1\frac{x^5}{8}-\frac{x^{11}}{8}\,dx=\left[\frac{x^6}{48}-\frac{x^{12}}{96}\right]_0^1=\underline{\underline{\frac{1}{96}}}$$

(11) $M = \{(x,y):\ y \leq x^2 \leq 4y,\ 2x \leq y^2 \leq 3x\}$

$$1 < \underbrace{\frac{x^2}{y}}_{=u} < 4, \quad 2 < \underbrace{\frac{y^2}{x}}_{=v} < 3$$

$$\phi : (x,y) \to \left(\frac{x^2}{y}, \frac{y^2}{x}\right)$$

$$u = \frac{x^2}{y} \qquad v = \frac{y^2}{x}$$

$$\frac{u}{v} = \frac{\frac{x^2}{y}}{\frac{y^2}{x}} = \left(\frac{x}{y}\right)^3 \qquad u \cdot v = xy$$

$$\frac{x}{y} = \left(\frac{u}{v}\right)^{\frac{1}{3}} \qquad xy = uv$$

$$x^2 = \left(\frac{u}{v}\right)^{\frac{1}{3}} \cdot uv = u^{\frac{4}{3}} v^{\frac{2}{3}} \to \boxed{x = u^{\frac{2}{3}} v^{\frac{1}{3}}}$$

$$y^2 = \frac{xy}{\frac{x}{y}} = \frac{uv}{\left(\frac{u}{v}\right)^{\frac{1}{3}}} = u^{\frac{2}{3}} v^{\frac{4}{3}} \to \boxed{y = u^{\frac{1}{3}} v^{\frac{2}{3}}}$$

$$\phi^{-1}(u,v) = \left(u^{\frac{2}{3}} v^{\frac{1}{3}}, u^{\frac{1}{3}} v^{\frac{2}{3}}\right)$$

$$J_{\phi^{-1}}(u,v) = \det \begin{pmatrix} \frac{2}{3}\left(\frac{v}{u}\right)^{\frac{1}{3}} & \frac{1}{3}\left(\frac{u}{v}\right)^{\frac{2}{3}} \\ \frac{1}{3}\left(\frac{v}{u}\right)^{\frac{2}{3}} & \frac{2}{3}\left(\frac{u}{v}\right)^{\frac{1}{3}} \end{pmatrix}$$

$$= \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

$$J_\phi = \det \begin{pmatrix} 2\frac{x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & 2\frac{y}{x} \end{pmatrix}$$

$$= 4 - 1 = 3$$

$$M^\circ = \phi^{-1}(\underbrace{(1,4) \times (2,3)}_{= U})$$

$$\int_{\phi^{-1}(U)} 1 = \int_U J_{\phi^{-1}} \cdot 1 \circ \phi$$

$$= \int_2^3 \int_1^4 \frac{1}{3} = \underline{\underline{1}}$$

(12) $M = \{(x,y) : 1 \leq x+y \leq 2,\ 2x \leq y \leq 3x\}$

$u = x+y \qquad v = \frac{y}{x} \qquad \phi(x,y) = \left(x+y, \frac{y}{x}\right)$

$u \in (1,2),\ v \in (2,3)$

$u - x = vx \to u = (1+v)x \to \boxed{x = \frac{u}{1+v}}$

$u - y = \frac{y}{v} \to u = \left(\frac{1}{v}+1\right)y \to \boxed{y = \frac{uv}{1+v}}$

$$\phi^{-1}(u,v) = \left(\frac{u}{1+v}, \frac{uv}{1+v}\right)$$

$$J_{\phi^{-1}}(u,v) = \det \begin{pmatrix} \frac{1}{1+v} & -\frac{u}{(1+v)^2} \\ \frac{v}{1+v} & \frac{u}{(1+v)^2} \end{pmatrix}$$

$$= u\left(\frac{1}{(1+v)^3} + \frac{v}{(1+v)^3}\right) = \underline{\underline{\frac{u}{(1+v)^2}}}$$

$$\int_{M^\circ} 1 = \int_U J_{\phi^{-1}} = \int_1^2 \int_2^3 \frac{u}{(1+v)^2}\, dv\, du$$

$$= \int_1^2 \left[\frac{-u}{1+v}\right]_{v=2}^{v=3} du = \int_1^2 -u\left[\frac{1}{4} - \frac{1}{3}\right] du$$

$$= \frac{1}{12}\int_1^2 u\, du = \frac{1}{12}\left[\frac{u^2}{2}\right]_1^2 = \frac{1}{12}\left[2 - \frac{1}{2}\right] = \underline{\frac{1}{8}}$$

(15) $M=\{(x,y,z):\ x^2+y^2+z^2<2z,\ x^2+y^2<z^2\}$

$$x = r\cos\alpha\cos\beta \qquad \beta\in\left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right)$$

$$y = r\sin\alpha\cos\beta \qquad \alpha\in(0,2\pi)$$

$$z = r\sin\beta$$

$$\phi:(r,\alpha,\beta)\to(r\cos\alpha\cos\beta,\ r\sin\alpha\cos\beta,\ r\sin\beta)$$

$$J_\phi(r,\alpha,\beta)=\det\begin{pmatrix}\cos\alpha\cos\beta & -r\sin\alpha\cos\beta & -r\cos\alpha\sin\beta\\ \sin\alpha\cos\beta & r\cos\alpha\cos\beta & -r\sin\alpha\sin\beta\\ \sin\beta & 0 & r\cos\beta\end{pmatrix}$$

$$= r^2\cos^2\alpha\cos^3\beta + r^2\sin^2\alpha\cos\beta\sin^2\beta$$

$$+ r^2\cos^2\alpha\cos\beta\sin^2\beta + r^2\sin^2\alpha\cos^3\beta$$

$$= r^2\cos\beta$$

$$r^2<2r\sin\beta \qquad r^2\cos^2\beta<r^2\sin^2\beta$$

$$0<r<2\sin\beta \qquad \overbrace{\cos^2\beta-\sin^2\beta}^{=\cos 2\beta}<0$$

$\sin\beta>0$

$$U=\{(r,\alpha,\beta):\ \alpha\in(0,2\pi),\ \beta\in\cancel{\left(-\tfrac{\pi}{2},-\tfrac{\pi}{4}\right)}\cup\left(\tfrac{\pi}{4},\tfrac{\pi}{2}\right),\ 0<r<2\sin\beta\}$$

$$\int_{\phi(U)} 1=\int_U J_\phi=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\sin\beta} r^2\cos\beta\,dr\,d\beta\,d\alpha$$

$$=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[\frac{r^3}{3}\cos\beta\right]_{r=0}^{2\sin\beta}d\beta\,d\alpha$$

$$=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{8}{3}\sin^3\beta\cos\beta\,d\beta\,d\alpha$$

$$=\int_0^{2\pi}\left[\frac{8}{3}\,\frac{\sin^4\beta}{4}\right]_{\beta=\frac{\pi}{4}}^{\beta=\frac{\pi}{2}}d\alpha$$

$$=\int_0^{2\pi}\frac{8}{3}\left(\frac{1}{4}-\frac{1}{16}\right)d\alpha=\underline{\underline{\pi}}$$