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(1) $F(y) = \int_0^\pi (y')^2 \quad y(0) = y(\pi) = 0 \,, \int_0^\pi y^2 = 1$

$G(y) = \int_0^\pi y^2 \qquad f = z^2 \quad g = y^2$

$h = f - \lambda g = z^2 - \lambda y^2 \qquad h_y = -2\lambda y \quad h_z = 2z$

E-L $\quad -2\lambda y - [2y']' = 0$

$-2\lambda y - 2y'' = 0$

$y'' = -\lambda y \rightarrow y = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x$

$0 = y(0) = C$

$0 = y(\pi) = D\sin\sqrt{\lambda}\pi \Rightarrow \lambda = n^2 \quad n \in \mathbb{N}$

$1 = \int_0^\pi D\cdot\sin^2 nx\,dx = D\frac{\pi}{2} \rightarrow D = \frac{2}{\pi}$

$\int_0^\pi (y')^2 = \int_0^\pi \frac{2}{\pi}n^2\cos^2 nx\,dx = \frac{2n^2}{\pi} \rightarrow n = 1\ (= \lambda)$

$y = \frac{2}{\pi} \sin x$ je kandidát na minimum

Uvažme pomocný funkcionál

$$H(y) = \int_0^{\pi} (y')^2 - \lambda \int_0^{\pi} y^2 + \lambda L = \int_0^{\pi} (y')^2 - y^2 \, dx + 1$$

Potom $H = F$ na $\tilde{M}$.

$h = 1 - y^2 + z^2 \qquad h_y = -2y \quad h_z = 2z$

$h_{yy} = -2 \quad h_{yz} = 0, \; h_{zz} = 2$

$P = 2 \quad Q = -2$

$(J) \to -2h - (2h')' = 0 \to h'' = -h$

Bohužel, zde π je konjugovaný bod $(h = \sin x)$.

(2) $F(y)=\int_0^1 (y')^2 \quad , \quad y(0)=1, \; y(1)=6, \; \int_0^1 y = 3$

$h = f - \lambda g = z^2 - \lambda y \qquad h_y = -\lambda, \; h_z = 2z$

$-\lambda - [2y']' = 0 \; \to \; y'' = \frac{\lambda}{2}$

Tedy $\quad y = \frac{\lambda}{4}x^2 + Cx + D$

$1 = y(0) = D \qquad 6 = y(1) = \frac{\lambda}{4} + C + 1$

$C = 5 - \frac{\lambda}{4}$

$y = \frac{\lambda}{4}x^2 + \left(5 - \frac{\lambda}{4}\right)x + 1$

$3 = \int_0^1 \frac{\lambda}{4}x^2 + \left(5 - \frac{\lambda}{4}\right)x + 1 = \frac{\lambda}{12} + \frac{5}{2} - \frac{\lambda}{8} + 1$

$24 \cdot \left(-\frac{1}{2}\right) = 2\lambda - 3\lambda \qquad \boxed{\lambda = 12, \; y = 3x^2 + 2x + 1}$

$$H(y) = \int_0^1 (y')^2 - \lambda\left[\int_0^1 y - 3\right]$$

$$= \int_0^1 (y')^2 - 12y + 36 \quad = H(y)$$

$h = z^2 - 12y + 36 \quad H_h = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ PSD

y je bodem minima H, zároveň na $\tilde{M}$ platí $H = F$, tedy y je bodem minima F vzhledem k $\tilde{M}$.

(3) $F(y)=\int_0^1 (y')^2$, $y(0)=0$, $y(1)=\frac{1}{4}$, $\int_0^1 y-(y')^2=\frac{1}{12}$

$h=f-\lambda g=(1+\lambda)z^2-\lambda y$

$h_y=-\lambda$, $h_z=2(1+\lambda)z$

$(E-L)\Rightarrow \quad -\lambda-\left[2(1+\lambda)y'\right]'=0$

$$-\lambda-2(1+\lambda)y''=0$$

$2(1+\lambda)y''=-\lambda \quad (\lambda\neq-1)\to y''=\frac{\lambda}{2(\lambda+1)}$

$$y=\frac{\lambda}{4(\lambda+1)}x^2+Cx+D$$

$y(0)=0 \to D=0$

$\frac{1}{4}=y(1)=\frac{\lambda}{4(\lambda+1)}+C \quad \to C=\frac{1}{4}-\frac{\lambda}{4(\lambda+1)}$

$$y = \frac{\lambda}{4(\lambda+1)} x^2 + \left[\frac{1}{4} - \frac{\lambda}{4(\lambda+1)}\right] x$$

$$= \frac{\lambda}{4(\lambda+1)} x^2 + \frac{1}{4(\lambda+1)} x$$

$$\int_0^1 y - (y')^2 = \dots = \frac{3+4\lambda}{48(1+\lambda)^2}$$

$$\frac{3+4\lambda}{48(1+\lambda)^2} = \frac{1}{12} \rightarrow 3+4\lambda = 4+8\lambda+4\lambda^2$$

$$(2\lambda+1)^2 = 4\lambda^2+4\lambda+1 = 0 \rightarrow \lambda = -\frac{1}{2}$$

$$\boxed{y = -\frac{x^2}{4} + \frac{x}{2}}$$

$$H(y) = \int_0^1 (y')^2 - \frac{1}{2} y + \frac{1}{2}(y')^2 - \frac{1}{12}$$

opět konvexní ve druhých dvou souřadnicích → minimum