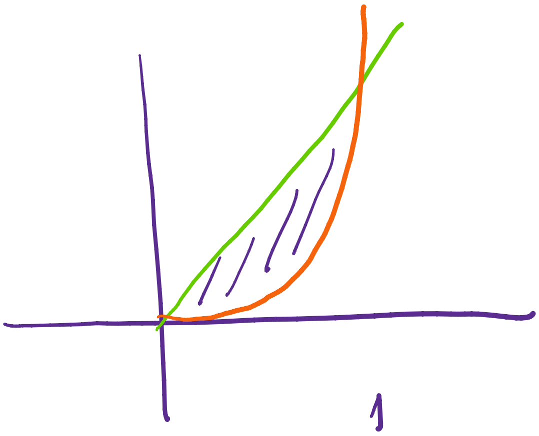
(1)



$x = x^4 \iff x = 0, 1$

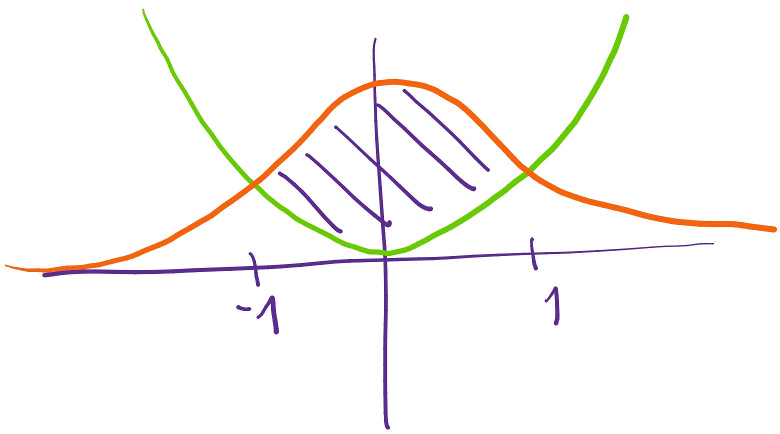
Chceme tedy $\int_0^1 x - x^4 \, dx = \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} - \frac{1}{5} = \underline{\underline{\frac{3}{10}}}$

(2)

$\frac{2}{1+x^2} = x^2 \iff 2 = x^2 + x^4$

$t = x^2 \Rightarrow t^2 + t - 2 = 0 \quad t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = -2, 1$

$x = \pm 1$



Chceme tedy $\int_{-1}^{1} \frac{2}{1+x^2} - x^2 = \left[2 \operatorname{arctg} x - \frac{x^3}{3} \right]_{-1}^{1} = \underline{\underline{\pi - \frac{2}{3}}}$

(3) $f(x) = \arcsin x + \sqrt{1-x^2}$ $\qquad \varphi(x) = (x, f(x))$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-2x}{2\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}}$$

$$\ell(\varphi) = \int_{-1}^{1} \sqrt{1+(f'(x))^2}\, dx = \int_{-1}^{1} \sqrt{1 + \frac{(1-x)^2}{1-x^2}}\, dx = \int_{-1}^{1} \sqrt{\frac{1+x+1-x}{1+x}}$$

$$= \int_{-1}^{1} \sqrt{\frac{2}{1+x}} = \sqrt{2}\left[2\sqrt{1+x}\right]_{-1}^{1} = \underline{\underline{4}}$$

(4) $f(x) = \log(\cos x)$ $\qquad f'(x) = \frac{-\sin x}{\cos x} = -\operatorname{tg} x$

$\varphi(x) = (x, f(x)),\ x \in [0, \frac{\pi}{6}]$

$$\ell(\varphi) = \int_0^{\frac{\pi}{6}} \sqrt{1 + \operatorname{tg}^2 x}\, dx = \int_0^{\frac{\pi}{6}} \sqrt{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}}\, dx = \int_0^{\frac{\pi}{6}} \frac{1}{\cos x}\, dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos x}{1-\sin^2 x}\, dx = \left| \begin{matrix} t = \sin x & dt = \cos x\, dx \\ x = 0 & \Rightarrow t = 0 \\ x = \frac{\pi}{6} & \Rightarrow t = \frac{1}{2} \end{matrix} \right| = \int_0^{\frac{1}{2}} \frac{1}{1-t^2}\, dt$$

$$= \left[\operatorname{argtgh} t\right]_0^{\frac{1}{2}} = \operatorname{argtgh} \frac{1}{2} = \underline{\underline{\log\sqrt{3}}}$$

(5) $\varphi(t) = (\cos t + t\sin t, \sin t - t\cos t) \quad t \in [0, 2\pi]$

$$\varphi'(t) = (-\sin t + t\cos t + \sin t, \cos t + t\sin t - \cos t)$$
$$= (t\cos t, t\sin t)$$

$$\ell(\varphi) = \int_0^{2\pi} |\varphi'(t)| = \int_0^{2\pi} \sqrt{(\varphi_1'(t))^2 + (\varphi_2'(t))^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t}\, dt = \int_0^{2\pi} t\, dt = \left[\frac{t^2}{2}\right]_0^{2\pi} = \underline{\underline{2\pi^2}}$$

(6) $\varphi(t) = (e^{-t}\cos t, e^{-t}\sin t), \quad t \in [0, \infty).$

$$\varphi'(t) = (-e^{-t}\cos t - e^{-t}\sin t, -e^{-t}\sin t + e^{-t}\cos t)$$

$$\ell(\varphi) = \int_0^{\infty} \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2}\, dt$$
$$= \int_0^{\infty} e^{-t} \sqrt{\cos^2 t + \underline{\cos t\sin t} + \sin^2 t + \sin^2 t \underline{- \sin t\cos t} + \cos^2 t}\, dt$$
$$= \int_0^{\infty} \sqrt{2}\, e^{-t} dt = \sqrt{2}\left[-e^{-t}\right]_0^{\infty} = \sqrt{2}(0 - (-1)) = \underline{\underline{\sqrt{2}}}$$

(7) $\varphi(t) = (\cos t, \sin t, t) \quad t \in [0, T]$

$\varphi'(t) = (-\sin t, \cos t, 1)$

$$\ell(\varphi) = \int_0^T |\varphi'(t)| = \int_0^T \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \int_0^T \sqrt{2} = \underline{\underline{\sqrt{2}\, T}}$$

(8) $f(x) = \sqrt{R^2 - x^2}, \quad x \in (-R, R),$

$$f'(x) = \frac{-2x}{2\sqrt{R^2 - x^2}} = -\frac{x}{\sqrt{R^2 - x^2}}$$

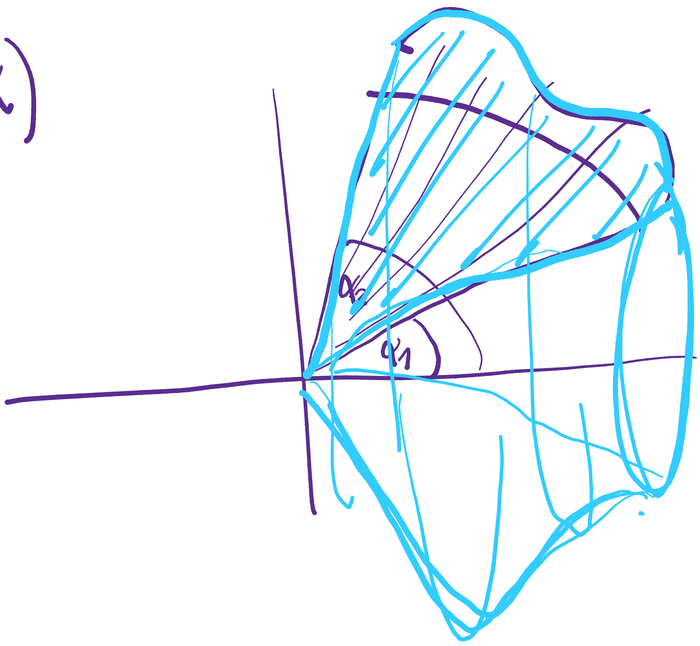
$$V = \pi \int_{-R}^{P} (f(x))^2 dx = \pi \int_{-R}^{R} R^2 - x^2 = \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{R}$$

$$= \pi \left(R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3} \right) = \frac{4}{3} \pi R^3$$

$$S = 2\pi \int_{-R}^{R} f(x) \sqrt{1 + (f'(x))^2}\, dx = 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx$$

$$= 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2} \cdot \sqrt{\frac{R^2}{R^2 - x^2}}\, dx = 2\pi \int_{-R}^{R} R = 2\pi \left[Rx \right]_{-R}^{R} = \underline{\underline{4\pi R^2}}$$

(9) $\varphi(\alpha) = (1+\cos\alpha,\ \alpha)$



$$V = \frac{2}{3}\pi \int_{\alpha_1}^{\alpha_2} \rho^3(\alpha)\sin\alpha \, d\alpha$$

$$= \frac{2}{3}\pi \int_{0}^{\pi} (1+\cos\alpha)^3 \sin\alpha \, d\alpha = \left| \begin{array}{l} t = 1+\cos\alpha \quad dt = -\sin\alpha\, d\alpha \\ \alpha = 0 \to t = 2 \\ \alpha = \pi \to t = 0 \end{array} \right|$$

$$= \frac{2}{3}\pi \int_{2}^{0} -t^3 \, dt = \frac{2}{3}\pi \int_{0}^{2} t^3 \, dt = \frac{2}{3}\pi \left[\frac{t^4}{4} \right]_0^2 = \underline{\underline{\frac{8}{3}\pi}}$$

(10) $f(x) = \sqrt{r^2 - x^2} \qquad x \in [0, r]$

$$M_x = \frac{1}{2}\int_0^r f^2(x)\,dx = \frac{1}{2}\int_0^r r^2 - x^2 = \frac{1}{2}\left[r^2 x - \frac{x^3}{3}\right]_0^r = \frac{1}{3}r^3$$

$$M_y = \int_0^r x f(x) = \int_0^r x\sqrt{r^2-x^2}\,dx = \left| \begin{array}{ll} t = r^2 - x^2 & dt = -2x\,dx \\ x = 0 & \to t = r^2 \\ x = r & \to t = 0 \end{array} \right|$$

$$= -\frac{1}{2}\int_{r^2}^{0} \sqrt{t}\,dt = \frac{1}{2}\int_0^{r^2} \sqrt{t}\,dt = \frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^{r^2} = \underline{\underline{\frac{1}{3}r^3}}$$

$$M = \int_0^r \sqrt{r^2 - x^2}\,dx = \frac{1}{4}\pi r^2$$

Tedy $T = \left(\frac{M_x}{M}, \frac{M_y}{M}\right) = \left(\frac{\frac{1}{3}r^3}{\frac{1}{4}\pi r^2}, \frac{\frac{1}{3}r^3}{\frac{1}{4}\pi r^2}\right) = \left(\frac{4}{3}\cdot\frac{r}{\pi}, \frac{4}{3}\cdot\frac{r}{\pi}\right)$

(11) $f(x)=\sqrt{r^2-x^2}$, $x \in [0,r]$

$$M_{xz} = \pi \int_0^r x f^2(x)\,dx = \pi \int_0^r x(r^2-x^2)\,dx = \pi \left[\frac{r^2x^2}{2} - \frac{x^4}{4} \right]_0^r = \frac{\pi r^4}{4}$$

$$M = \frac{2}{3}\pi r^3$$

$$T = \left(\frac{M_{yz}}{M}, 0, 0 \right) = \left(\frac{\frac{\pi r^4}{4}}{\frac{2}{3}\pi r^3}, 0, 0 \right) = \left(\frac{3}{8}r, 0, 0 \right).$$

(12) $\varphi(t) = (\cos^3 t, \sin^3 t)$, $[0, \frac{\pi}{2}]$

$$\varphi'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t)$$

$$|\varphi'(t)| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= 3\cos t \sin t \sqrt{\cos^2 t + \sin^2 t} = 3\cos t \sin t$$

$$I_x = \int_0^{\frac{\pi}{2}} \varphi_2^2(t)\sqrt{(\varphi_1'(t))^2 + (\varphi_2'(t))^2} = 3\int_0^{\frac{\pi}{2}} \cos^7 t \sin t \, dt$$

$$= \left| \begin{array}{ll} u = \cos t & du = -\sin t \, dt \\ t = 0 & \Rightarrow u = 1 \\ t = \frac{\pi}{2} & \Rightarrow u = 0 \end{array} \right| = -3\int_1^0 u^7 du = 3\int_0^1 u^7 du = \underline{\underline{\frac{3}{8}}}$$

$$I_y = \int_0^{\frac{\pi}{2}} \varphi_1^2 \sqrt{(\varphi_1'(t))^2 + (\varphi_2'(t))^2} \, dt = 3\int_0^{\frac{\pi}{2}} \sin^7 t \cos t \, dt = \underline{\underline{\frac{3}{8}}}$$