Přednáška 18.10.

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Legendrova transformace:

$f: \mathbb{R}^d \to \mathbb{R}$, $\underbrace{\text{striktně konvexní}, C^2, \nabla f(\mathbb{R}^d) = \mathbb{R}^d}_{\nabla f \text{ je bijekce na } \mathbb{R}^d}$ (*)

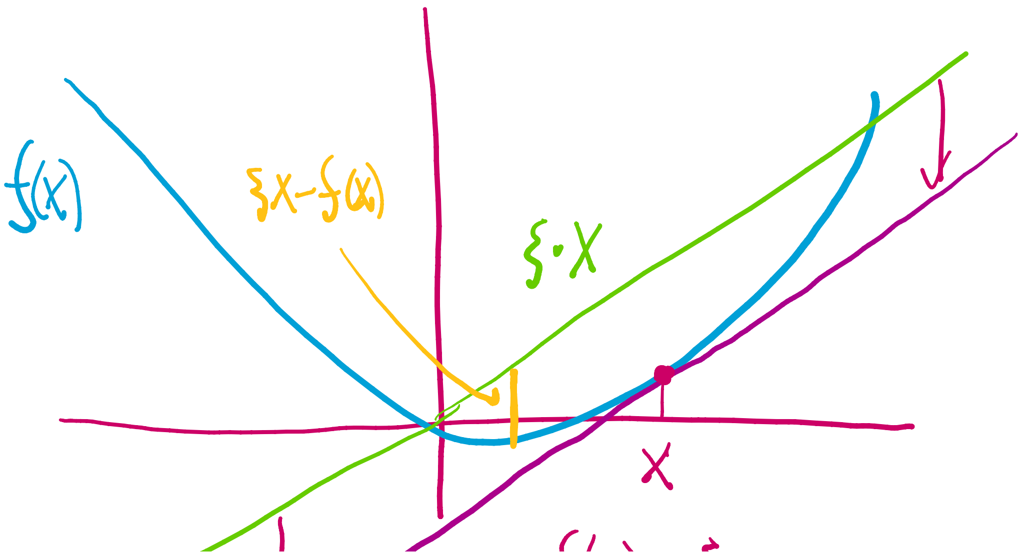
Cílem je zkonstruovat (duální, konjugovanou) funkci f^* (opět splňující (*)), že platí

$$\xi = \nabla f(x) \iff x = \nabla f^*(\xi)$$

$$f(x) = \frac{x^4}{4} \to f^*(x) = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} \quad \left(\frac{1}{4} + \frac{1}{\frac{4}{3}} = 1\right)$$

$$\xi = f'(x) = x^3 \to x = \sqrt[3]{\xi} = (f^*)'(\xi)$$

$$f^*(\xi) = \max_x \left[\xi \cdot x - f(x)\right]$$



Maximizující x musí splňovat $0=\nabla\left[\xi x - f(x)\right]$, tedy

$$\nabla f(x)=\xi \Rightarrow x=(\nabla f)^{-1}(\xi)$$

(jde určitě o maximum protože $\xi\cdot x - f(x)$ je konkávní)

$$f^*(\xi)=\xi\cdot(\nabla f)^{-1}(\xi) - f\left((\nabla f)^{-1}(\xi)\right)$$

Pro $d=1$: $x=(f')^{-1}(\xi)=h(\xi) \to h'(\xi)=\dfrac{1}{f''\left((f')^{-1}(\xi)\right)}>0$

$$f^*(\xi)=\xi\cdot h(\xi)-f(h(\xi))$$

$$(f^*)'(\xi)=h(\xi)+\xi h'(\xi)-\overset{=\xi}{f'(h(\xi))}\cdot h'(\xi)=h(\xi)$$

$$\boxed{x=(f^*)'(\xi) \Rightarrow \xi=f'(x)}$$

$$(f^*)''(\xi)=h'(\xi)>0 \to f^* \text{ striktně konvexní}$$

$$(f^*)^*(x)=x\cdot(\nabla f^*)^{-1}(x)-f^*\left((\nabla f^*)^{-1}(x)\right)$$

$$(f^*)^*(x) = x\cdot(\nabla f^*)^{-1}(x) - f^*\left((\nabla f^*)^{-1}(x)\right)$$

$$= x\cdot(\nabla f^*)^{-1}(x) - (\nabla f^*)^{-1}(x)\cdot \overbrace{\nabla f^*\left((\nabla f^*)^{-1}(x)\right)}^{x} + f(x)$$

$$= f(x)$$

Pohyb N hmotných bodů

$$L : \underset{\substack{x\\ \uparrow\\ \text{čas}}}{\mathbb{R}} \times \underset{\substack{y\\ \uparrow\\ \text{pozice}}}{\mathbb{R}^{3N}} \times \underset{\substack{z\\ \uparrow\\ \text{rychlosti}}}{\mathbb{R}^{3N}} \to \mathbb{R} \qquad \left[L(x,y,z) = \sum_{i=1}^{3N} \frac{1}{2} m_i (\dot{x}_i)^2 - U(t,x)\right]$$

$$L\Big(t, \underbrace{\overbrace{x_1^1(t), x_2^1(t), x_3^1(t)}^{x^1(t)}, \dots, x_1^N(t), x_2^N(t), x_3^N(t)}_{x(t) = \vec{x}(t)},$$

$$\underbrace{\dot{x}_1^1(t), \dot{x}_2^1(t), \dot{x}_3^1(t), \dots, \dot{x}_1^N(t), \dot{x}_2^N(t), \dot{x}_3^N(t)}_{\dot{x}(t) = \vec{\dot{x}}(t)}\Big) = L(t, x, \dot{x})$$

Okrajové podmínky $\to$ pozice bodů na začátku $(t=a)$
a na konci $(t=b)$

$$F(x) = \int_a^b L(t, x, \dot{x})$$

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$$0 = D_h \Phi(u) = \int_a^b \nabla_y L(t, x, \dot{x}) \cdot h + \nabla_z L(t, x, \dot{x}) \cdot h'$$

$h = (h_1, \dots, h_{3N})$ [použijeme $h = (0, \dots, 0, h_k, 0, \dots, 0)$]

Tedy $\int_a^b L_{y_k}(t, x, \dot{x})\, h_k + L_{z_k}(t, x, \dot{x})\, h_k'$

a $$L_{y_k}(t, x, \dot{x}) - \left[L_{z_k}(t, x, \dot{x})\right]' = 0$$

$$\left[-\frac{\partial U}{\partial y_k}(t, x) - \left[m_k \dot{x}_k\right]' = 0 \right]$$

$$L_y(t, q, \dot{q}) = \left[L_z(t, q, \dot{q})\right]'$$

Nechť $L^{x,y}(z) = L(x, y, z)$ $[\mathbb{R} \to \mathbb{R}]$

splňuje předpoklady na Legendrovu transformaci splňuje předpoklady na Legendrovu transformaci

$$h^{x,y} = \left[\left(L^{x,y}\right)'\right]^{-1}$$

$$H(x,y,z) = \left(L^{x,y}\right)^*(z) = z\cdot h^{x,y}(z) - L^{x,y}\left(h^{x,y}(z)\right)$$

$$H(t,q,p) = p\cdot\dot{q} - L(t,q,\dot{q})$$

$$h^{t,q}(p) = \left[\left(L^{t,q}\right)'\right]^{-1}\left(\left(L^{t,q}\right)'(\dot{q})\right) = \dot{q}$$

$$H(x,y,z) = z\cdot\xi(y,z) - L(x,y,\xi(y,z))$$

$$H_y(x,y,z) = \underline{z\cdot\xi_y(y,z)} - L_y(x,y,\xi(y,z))$$
$$\underline{-L_z(x,y,\xi(y,z))\cdot\xi_y(y,z)}$$

$$\xi = \dot{q}(q,p)$$

$$H_y(t,q,p) \to L_z(t,q,\dot{q}) = p$$

$$\frac{\partial H}{\partial y}(t,q,p) = -L_y(t,q,\dot{q}) = -\dot{p}$$

$$\frac{\partial H}{\partial y}(t,q,p) = -L_y(t,q,\dot{q}) = -p$$

$$\frac{\partial H}{\partial z}(t,q,p) = \dot{q} \qquad ?$$

$$z = \left[\left(L^{x,y}\right)^*\right]'(u) \iff u = \left(L^{x,y}\right)'(z)$$

$$\dot{q} = \left[\left(L^{t,q}\right)^*\right]'(p) \iff p = \left(L^{t,q}\right)'(\dot{q})$$

definicja p