Fractals 3.11.

úterý 3. listopadu 2020 11:57

$K_i + U(0, \frac{\varepsilon}{2^i})$

dimension

$\delta > 0 \quad s \geq 0 \quad K \subseteq \mathbb{R}^d$

- closed
- convex
- open

$$\mathcal{H}^s_\delta(K) = \inf\left\{ \sum \operatorname{diam}^s K_i : \bigcup_{i\in\mathbb{N}} K_i \supset K,\ \operatorname{diam} K_i \leq \delta \right\}$$

$\leq \varepsilon$

$0 < \varepsilon \leq \delta$

$$\mathcal{H}^s_\delta(K) \leq \mathcal{H}^s_\varepsilon(K)$$

$\delta \downarrow$

$$\mathcal{H}^s(K) = \sup_{\delta>0} \mathcal{H}^s_\delta(K) = \lim_{\delta\to 0+} \mathcal{H}^s_\delta(K)$$

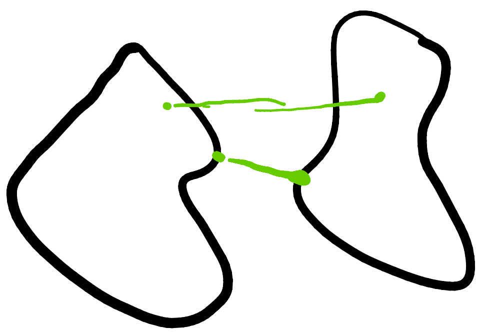
s-dimensional Hausdorff outer measure.

Metric outer measure

- $\operatorname{dist}(K, M) > 0 \implies \mathcal{H}^s(K \cup M) = \mathcal{H}^s(K) + \mathcal{H}^s(M)$

ε "

$\inf\{|x-y| : x \in K, y \in M\}$



$0 < \delta < \varepsilon$

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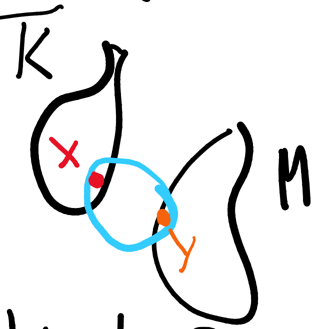
$\bigcup H_i \supseteq K \cup M \qquad \underline{\operatorname{diam} H_i \leq \delta} \quad (<\varepsilon)$



if $\quad H_i \cap K \neq \emptyset \Rightarrow H_i \cap M = \emptyset$

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$|x-y| \leq \delta < \varepsilon$

$I_K \rightarrow \quad K \subseteq \bigcup_{i \in I_K} H_i$

$I_M \qquad M \subseteq \bigcup_{i \in I_M} H_i$

$\underline{I_0} \qquad K \cap H_i = M \cap H_i = \emptyset \quad i \in I_0$

I_K, I_M, I_0 are pairwise disjoint

$$\sum \operatorname{diam}^s H_i \geq \underbrace{\sum_{i \in I_M} \operatorname{diam}^s H_i} + \underbrace{\sum_{i \in I_K} \operatorname{diam}^s H_i}$$

$$\sum \operatorname{diam}^s H_i \geq \underline{\mathcal{H}^s_\delta(M)} + \underline{\mathcal{H}^s_\delta(K)}$$

$\underline{0 < \delta < \varepsilon}$ • $\mathcal{H}^s_\delta(M \cup K) \geq \mathcal{H}^s_\delta(M) + \mathcal{H}^s_\delta(K)$

• $\bigcup K_i \supseteq K \qquad \operatorname{diam} K_i \leq \delta$

- $\bigcup K_i \supseteq K \qquad \operatorname{diam} K_i \leq \delta$
- $\bigcup M_i \supset M \qquad \operatorname{diam} M_i \leq \delta$

then $\bigcup K_i \cup \bigcup M_i \supseteq K \cup M$

$$\left[\ \mathcal{H}^s_\delta(K\cup M) \leq \mathcal{H}^s_\delta(K) + \mathcal{H}^s_\delta(M)\right.$$

$0<\delta<\varepsilon$ then

$$\mathcal{H}^s_\delta(K\cup M) = \mathcal{H}^s_\delta(K) + \mathcal{H}^s_\delta(M)$$

$B \in \mathcal{B} \Rightarrow$ then B is measurable (Caratheodory)

$(\mathcal{B}, \mathcal{H}^s)$ – the s-dimensional Hausdorff measure.

$$\dim_{\mathcal{H}}(K) = \sup\{s : \underset{=\infty}{\mathcal{H}^s(K) > 0}\}$$

$$= \inf\{s : \underset{=0}{\mathcal{H}^s(K) < \infty}\}$$

Observation

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if $0 \le r < s$ $\quad \mathcal{H}^s(K) > 0 \Rightarrow \mathcal{H}^r(K) = \infty$

HW1: $\quad \mathcal{H}^r(K) < \infty \Rightarrow \mathcal{H}^s(K) = 0$

φ similarity with scaling ratio r

HW2: $|\varphi(x) - \varphi(y)| = r \cdot |x - y|$

then $\mathcal{H}^s(\varphi(K)) = r^s \mathcal{H}^s(K)$.

HW3: $\dim_{\mathcal{H}}\left(\bigcup K_i\right) = \sup_i \dim_{\mathcal{H}}(K_i)$

$\dim_{\mathcal{H}}\left(\{0\} \cup \{\frac{1}{n}, n \in \mathbb{N}\}\right) = \sup 0 = 0$