Fractals 22.12.

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smallest n s.t. $< r_{\omega|n} \le r$

$r < r_{\omega|n-1}$

$r \cdot r_{min} < r_{\omega|n} \le r$

- $\boxed{\alpha < \frac{\mu(B(x,r))}{r^D} < \beta}$ $\rightarrow 0 < \mathcal{H}^D(K) < \infty$

$\downarrow$

$\mathcal{H}^D$

- (i) $\limsup\limits_{r\to 0+} \frac{\mu(B(x,r))}{r^s} < C$ $\forall x \in F$ $\implies$ $\boxed{\mathcal{H}^s(F) > \frac{\mu(F)}{C}}$

  (ii) $\limsup\limits_{r\to 0+} \frac{\mu(B(y,r))}{r^s} > C$ $\forall x \in F \implies \mathcal{H}^s(F) \le \frac{\mu(\mathbb{R}^d)}{C} \cdot 2^d$

Pr (i) $\limsup\limits_{r\to 0+} \frac{\mu(B(x,r))}{r^s} < C \iff \exists \varepsilon > 0 \ \limsup < C - \varepsilon$

$C-\varepsilon$

$\delta > 0$

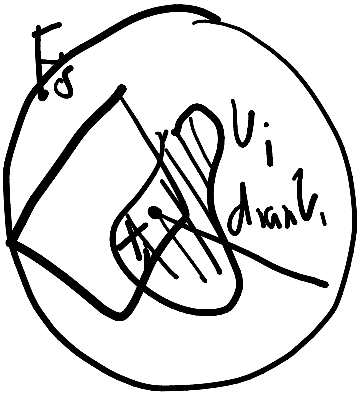
$F_\delta = \left\{ x \in F : \frac{\mu(B(x,r))}{r^s} < C - \varepsilon : \text{for some } \varepsilon > 0 \ \ldots\ 0 < r \le \delta \right\}$

$F_\delta = \Big\{ x \in F : \frac{\ \cdots\ }{r^s} < C - \varepsilon \cdot$ for some $\cdots$ and every $0 < r \le \delta \Big\}$

$\{U_i\}$ is a δ-covering of F.

$x_i \in F_\delta \cap U_i$

$$F_\delta \subseteq \bigcup_{U_i \cap F_\delta \neq 0} U_i \subseteq \bigcup B(x_i, \operatorname{diam} U_i)$$



$$\mu(F_\delta) \lesssim \sum_i \mu(B(x_i, \operatorname{diam} U_i)) \le C \cdot \sum_i (\operatorname{diam} U_i)^s$$

- $\mu(F_\delta) \le C \cdot \mathcal{H}^s_\delta(F) \le C \cdot \mathcal{H}^s(F)$

$\delta \to 0+$: $\mu(F_\delta) \to \mu(F)$

$$\boxed{\mathcal{H}^s(F) \ge \frac{\mu(F)}{C}}$$

$\frac{1}{|x-y|^s} = \infty$ $\quad x = y$

$$\phi_s(x, \mu) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^s} \, d\mu(y)$$ s-potential of μ

$$I_s(\mu) = \int \phi_s(x, \mu) \, d\mu(x)$$ s-energy of μ

$$I_s(\mu) = \int_{\mathbb{R}^d} \phi_s(x,\mu)\, d\mu(x) \qquad \text{s-energy of } \mu$$

$$\mathcal{M}(F) = \quad \operatorname{supp}\mu \subseteq F, \quad 0 < \mu(F) < \infty$$

(1) • If there is a measure $\mu \in \mathcal{M}(F)$ s.t. $I_s(F) < \infty$, then $\mathcal{H}^s(F) = \infty$

• $\mathcal{H}^s(F) > 0$ then there is $\mu \in \mathcal{M}(F)$ s.t. $I_t(F) < \infty$ for every $t < s$

HW: if $t < s$ and $I_s(\mu) < \infty \Rightarrow I_t(\mu) < \infty$

Pr (1) $\quad E := \left\{ x \in F : \limsup_{r \to 0+} \frac{\mu(B(x,r))}{r^s} > 0 \right\}$.

$\mu(E) = 0$

$x \in E \Rightarrow \quad \limsup_{r \to 0+} \frac{\mu(B(x,r))}{r^s} > \varepsilon$ for some $\varepsilon > 0$

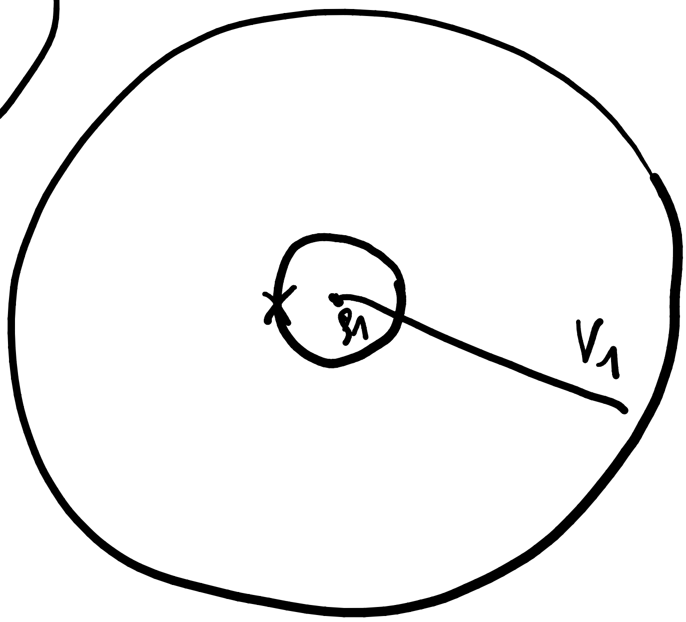
HW: $\mu(\{x\}) \neq 0 \quad \Rightarrow I_s(\mu) = \infty$

[illegible]

assum $\mu(\{x\}) = 0$

$B(x,\rho) \to \{x\}$

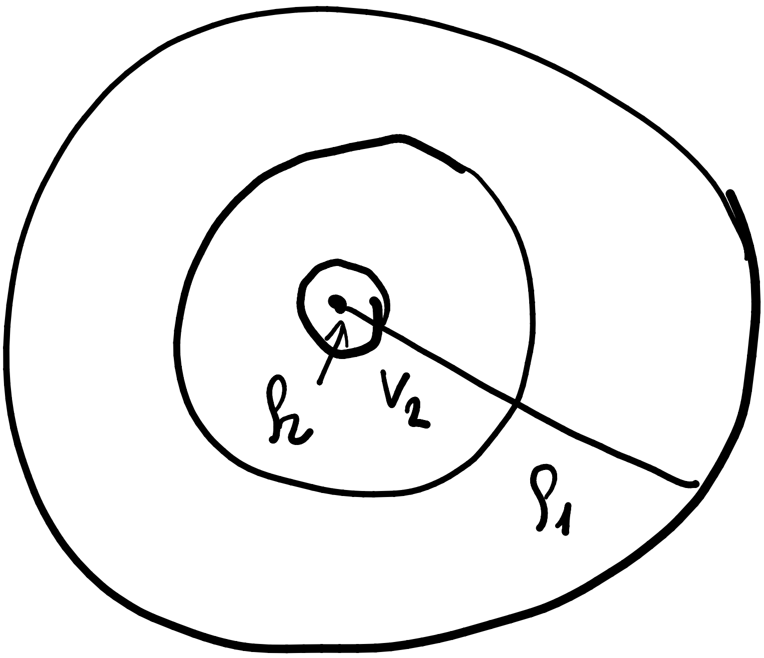
$\rho \to 0$



$$\frac{\mu(B(x,r_1))}{r_1^s} > \varepsilon$$

$$\frac{\mu(B(x,r_1) \setminus B(x,\rho_1))}{r_1^s} > \varepsilon$$

$$r_2 < \rho_1 < r_1$$



$$\frac{\mu(B(x,r_2))}{r_2^s} > \varepsilon$$

$$\frac{\mu(B(x,r_2) \setminus B(x,\rho_2))}{r_2^s} > \varepsilon$$

$\vdots$

$$r_1 > \rho_1 > r_2 > \rho_2 > \dots$$

$$A_i = B(x,r_i) \setminus B(x,\rho_i)$$

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- $\mu(A_i) > \varepsilon \cdot r_i^s$
- A_i are pairwise disjoint
- $y \in A_i \Rightarrow |x-y| < r_i \Rightarrow \frac{1}{r_i^s} < \frac{1}{|x-y|^s}$

$$\int\limits_{\bigcup A_i} \frac{1}{|x-y|^s} \, d\mu(y) = \sum_i \int\limits_{A_i} \frac{1}{|x-y|^s} \, d\mu(y)$$

$$\geq \sum_i \int\limits_{A_i} \frac{1}{r_i^s} \, d\mu(y)$$

$$= \sum_i \frac{1}{r_i^s} \mu(A_i)$$

$$\geq \sum_i \frac{1}{r_i^s} \cdot \varepsilon \cdot r_i^s = \sum_i \varepsilon = \infty$$

$$x \in E \Rightarrow \phi_s(x, \mu) = \infty$$

$$\infty > I(\mu) \geq \int \phi_s(x, \mu) \, d\mu(x) \Rightarrow \mu(E) = 0$$

$$\infty > I_s(\mu) \geq \int_E \phi_s(x,\mu)\,d\mu(x) \Rightarrow \mu(E)=0$$

$$F\setminus E = \left\{x\in F : \limsup_{\varepsilon\to 0+} \frac{\mu(B(x,r))}{r^s} = 0\right\}$$

$\uparrow$

$\mu(F\setminus E)=\mu(F)$ $\qquad\qquad\qquad\qquad$ $<C$

$$\forall C>0 \qquad \mathcal{H}^s(F) \geq \frac{\mu(F)}{C} \Rightarrow \mathcal{H}^s(F)=\infty \qquad \blacksquare$$

$$\Rightarrow \dim_{\mathcal{H}} F \geq s$$

$$C_s(F) := \sup\left\{\frac{1}{I_s(\mu)} : \mu\in\mathcal{M}(F)\right\}$$

Hk. $\quad \dim_{\mathcal{H}}(F) = \sup\{s : C_s(F)>0\}$