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Minkowski dimension (boxing d., ...)

- Counting approach

$\varepsilon > 0$ we define the ε-covering of a set $K \subseteq \mathbb{R}^d$ as a sytsem of balls $B(x_i, \varepsilon)$ satisfying $K \subseteq \bigcup B(x_i, \varepsilon)$.

closed ball

ε-packing will be system of balls $B(x_i, \varepsilon)$ if: (1) all x_i belong to K

(2) $B(x_i, \varepsilon) \cap B(x_j, \varepsilon) = \emptyset$ if $i \neq j$

For a bonded set $K \subseteq \mathbb{R}$, $K \neq \emptyset$.

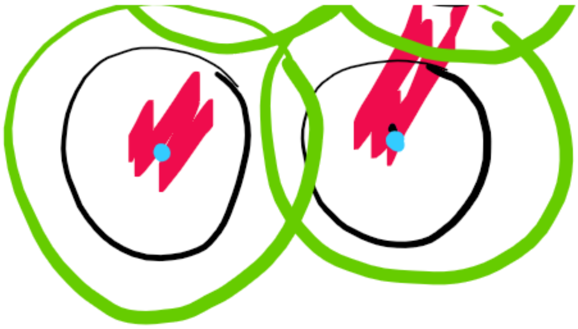
We define $C(K, \varepsilon)$ - minimal cardinality of ε-covering of K

$P(K, \varepsilon)$ - maximal cardinality of ε-packing of K

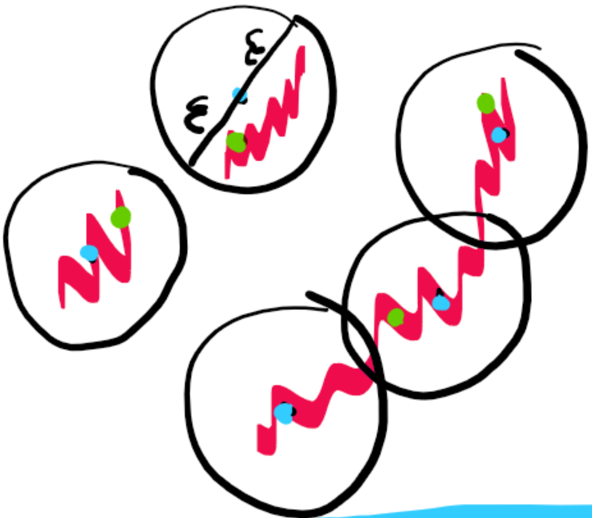
$C(K, \varepsilon)$ - ε-covering number

$P(K, \varepsilon)$ - ε packing number

$P(K, \varepsilon) \geq C(K, 2\varepsilon)$



$C(K,\varepsilon) \geq P(K,\varepsilon)$



$$\boxed{C(K,2\varepsilon) \leq P(K,\varepsilon) \leq C(K,\varepsilon)}$$

Def: For $K \subseteq \mathbb{R}^d$, $K \neq \emptyset$ we define

- $\overline{\dim}_M(K) = \limsup\limits_{\varepsilon \to 0+} \dfrac{\log C(K,\varepsilon)}{\log \frac{1}{\varepsilon}}$ – upper Minkowski dimension
- $\underline{\dim}_M(K) = \liminf\limits_{\varepsilon \to 0+} \dfrac{\log C(K,\varepsilon)}{\log \frac{1}{\varepsilon}}$ – lower — " —

If $\overline{\dim}_M(K) = \underline{\dim}_M(K)$ then we define $\dim_M(K)$ (Minkowski dimension) as their common value

$\log P(K,\varepsilon)$

L: $\overline{\dim}_\mu(K) = \limsup\limits_{\varepsilon\to 0+} \dfrac{\log P(K,\varepsilon)}{\log\frac{1}{\varepsilon}}$

$\underline{\dim}_\mu(K) = \liminf\limits_{\varepsilon\to 0+} \dfrac{\log P(K,\varepsilon)}{\log\frac{1}{\varepsilon}}$

P: • $C(K,2\varepsilon) \le P(K,\varepsilon) \le C(K,\varepsilon)$

$$\frac{\log C(K,2\varepsilon)}{\log\frac{1}{\varepsilon}} \le \frac{\log P(K,\varepsilon)}{\log\frac{1}{\varepsilon}} \le \frac{\log C(K,\varepsilon)}{\log\frac{1}{\varepsilon}}$$

• $\limsup\limits_{\varepsilon\to 0+} \dfrac{\log P(K,\varepsilon)}{\log\frac{1}{\varepsilon}} \le \limsup\limits_{\varepsilon\to 0+} \dfrac{\log C(K,\varepsilon)}{\log\frac{1}{\varepsilon}} = \overline{\dim}_\mu(K)$

$$\frac{\log P(K,\varepsilon)}{\log\frac{1}{\varepsilon}} \ge \frac{\log C(K,2\varepsilon)}{\log\frac{1}{2\varepsilon}+\log 2} = \underbrace{\frac{\log C(K,2\varepsilon)}{\log\frac{1}{2\varepsilon}}\cdot\boxed{\frac{\log\frac{1}{2\varepsilon}}{\log\frac{1}{2\varepsilon}+\log 2}}}_{\downarrow\ \overline{\dim}_\mu(K)}$$

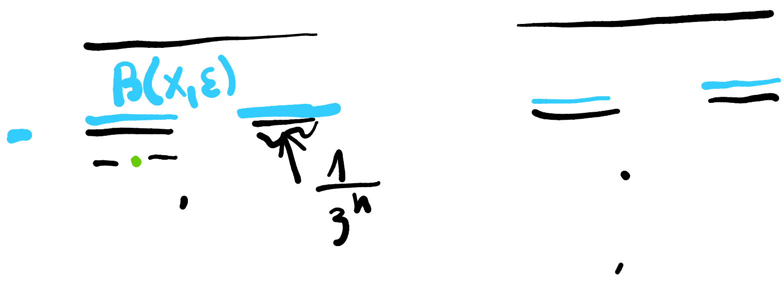
$\varepsilon_n \searrow 0$

$$\limsup_{\varepsilon\to 0+} \frac{P(K,\varepsilon)}{\log\frac{1}{\varepsilon}} \ge \overline{\dim}_\mu(K)$$

$$\overline{\dim}_\mu(K) = \limsup_{\varepsilon\to 0+} \frac{P(K,\varepsilon)}{\log\frac{1}{\varepsilon}}$$

$$\overline{\dim}_M(K) = \limsup_{\varepsilon \to 0^+} \frac{\ldots}{\log\frac{1}{\varepsilon}}$$

Analogously for $\underline{\dim}_M(K)$.



$$2^n \geq C(K,\varepsilon)$$

$$\frac{1}{3^{n-1}} > 2\varepsilon \geq \frac{1}{3^n}$$

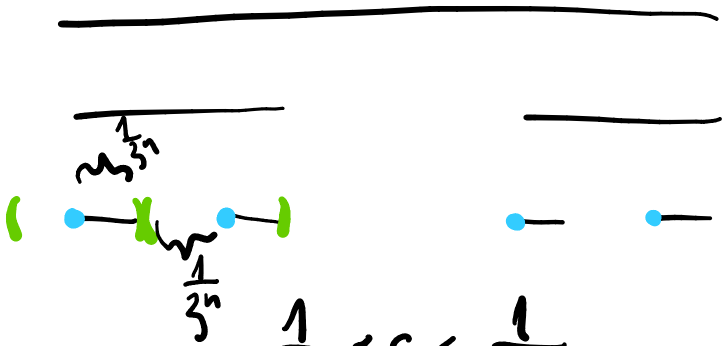
$$2\cdot 3^{n-1} < \frac{1}{\varepsilon} \leq 2\cdot 3^n$$

$$\frac{\log C(K,\varepsilon)}{\log\frac{1}{\varepsilon}} \leq \frac{\log 2^n}{\log(2\cdot 3^{n-1})} \to \frac{\log 2}{\log 3}$$

$$\overline{\dim}_M(K) \leq \frac{\log 2}{\log 3}$$

points

$$2^n \leq P(K,\varepsilon)$$



$$3^n < \frac{1}{\varepsilon} \leq 3^{n+1}$$

$$\frac{1}{3^{n+1}} \leq \varepsilon < \frac{1}{3^n}$$

$$\left[\frac{\log P(K,\varepsilon)}{\log\frac{1}{\varepsilon}} \geq \frac{\log 2^n}{\log 3^{n+1}} \to \frac{\log 2}{\log 3}\right]$$

$$\frac{\log 2}{\log 3} \geq \overline{\dim}_M(K) \geq \underline{\dim}_M(K) \geq \frac{\log 2}{\log 3} \Rightarrow \boxed{\dim_M(K) = \frac{\log 2}{\log 3}}$$

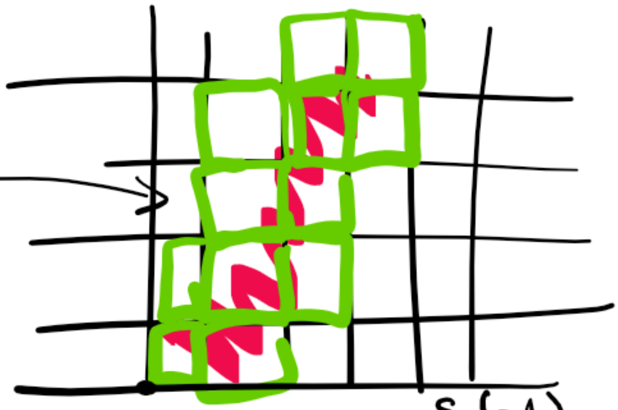
log

Ex. $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$

$\dim_M(K) = ?$

Boxing number

$B(K, \varepsilon)$



Can you define $\dim_M(K)$ in terms of $B(K, \varepsilon)$?