Fractals 1.12.

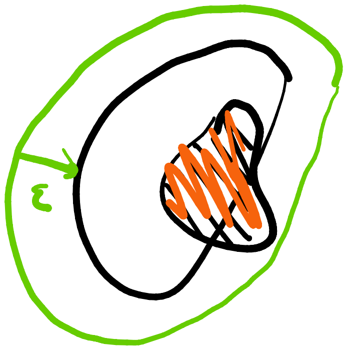
úterý 1. prosince 2020 12:00

$\mathcal{K}(\mathbb{R}^d)$ - K nonempt compact

- $\mathrm{dist}_{\mathcal{H}}(K,M) = \inf\{\varepsilon \geq 0 : K \subseteq M_\varepsilon, M \subseteq K_\varepsilon\}$

min ↑ compact

$M_0 = \overline{M} = M$



(1) $\mathrm{dist}_{\mathcal{H}}(K,M) = 0$

$\Updownarrow$

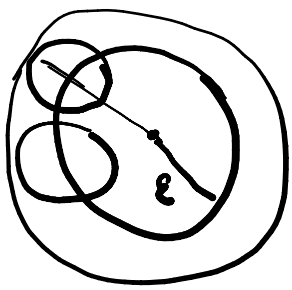
$K \subseteq M, M \subseteq K \Rightarrow K = M$

(2) $\mathrm{dist}_{\mathcal{H}}(K,M) = \mathrm{dist}_{\mathcal{H}}(M,K)$

(3) $\mathrm{dist}_{\mathcal{H}}(K,M) \leq \mathrm{dist}_{\mathcal{H}}(K,L) + \mathrm{dist}_{\mathcal{H}}(L,M)$ $\quad K,L,M \in \mathcal{K}(\mathbb{R}^d)$

$\mathrm{dist}_{\mathcal{H}}(K,L) = \varepsilon$ $\quad \mathrm{dist}_{\mathcal{H}}(L,M) = \delta$

- $K \subseteq L_\varepsilon$
- $L \subseteq K_\varepsilon$
- $L \subseteq M_\delta$
- $M \subseteq L_\delta$

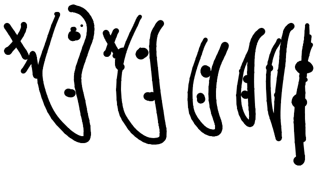


$$\left.\begin{array}{l} K \subseteq L_\varepsilon \subseteq (M_\delta)_\varepsilon = M_{\delta+\varepsilon} \\ M \subseteq L_\delta \subseteq (K_\varepsilon)_\delta = K_{\delta+\varepsilon} \end{array}\right\} \mathrm{dist}_{\mathcal{H}}(K,M) \leq \delta + \varepsilon$$

$M \subseteq L_\delta \subseteq (K_\varepsilon)_\delta = K_{\varepsilon+\delta}$ $\Big\}$ $\mathrm{dist}_{\mathcal{X}}(K,M) \le \delta+\varepsilon$

$\mathrm{dist}_{\mathcal{X}}$ is a metric on $\mathcal{K}$

$\{K_n\}$



- $(\mathcal{K}, \mathrm{dist}_{\mathcal{X}})$ is a complete metric space

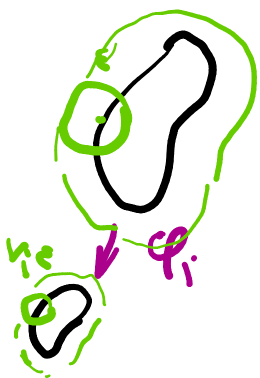
$\Phi: \underset{\mathcal{K}}{\underset{\in}{K}} \to \underset{\mathcal{K}}{\underset{\in}{\bigcup \varphi_i(K)}}$ – Hutchinson operator

$\Phi(K) = K$

r_i – contraction ratio of φ_i $\quad 0 < r_i < 1$

$r_{max} = \max_i r_i < 1$

- $\mathrm{dist}_{\mathcal{X}}(\Phi(K), \Phi(M)) \le r_{max}\, \mathrm{dist}_{\mathcal{X}}(K,M)$



$\varepsilon = \mathrm{dist}_{\mathcal{X}}(K,M) \rightarrow K \subseteq M_\varepsilon,\ M \subseteq K_\varepsilon$

$\Phi(K) = \bigcup \varphi_i(K) \subseteq \bigcup \varphi_i(M_\varepsilon) = \bigcup [\varphi_i(M)]_{r_i \varepsilon}$

$\subseteq \bigcup (\varphi_i(M))_{\ldots} = [\bigcup \varphi_i(M)] \quad = [\Phi(M)]$

$$\subseteq \bigcup \left(\varphi_i(M)\right)_{r_{max}\varepsilon} = \left[\bigcup \varphi_i(M)\right]_{r_{max}\varepsilon} = \left[\Phi(M)\right]_{r_{max}\varepsilon}$$

- $\Phi(K) \subseteq [\Phi(M)]_{r_{max}\varepsilon}$, $\Phi(M) \subseteq [\Phi(K)]_{r_{max}\varepsilon}$

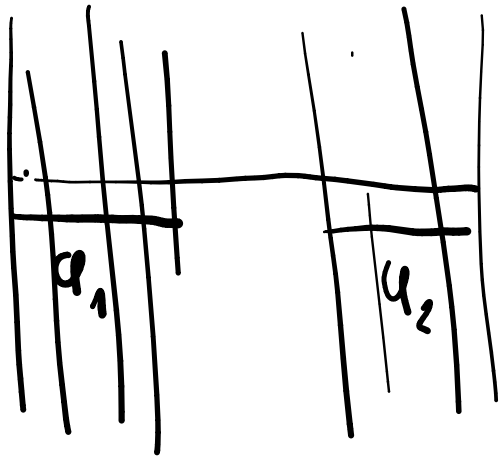
$$\mathrm{dist}(\Phi(K), \Phi(M)) \leq r_{max}\,\varepsilon = r_{max}\,\mathrm{dist}_{\mathcal{K}}(K, M)$$

Φ is a contraction on a complete (and nonempty) metric space $(\mathcal{K}, \mathrm{dist}_{\mathcal{K}})$.

$$\Big\downarrow \text{ Banach fixed point theorem}$$

$$\exists!\, K \in \mathcal{K} : \Phi(K) = K$$

$\Phi(\mathbb{R}^d) = \mathbb{R}^d$



self similar set generated by $(\varphi_1, \dots, \varphi_n)$.

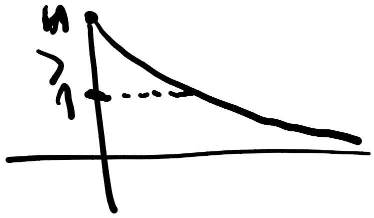
Similarity dimension ______ of $\varphi_1, \dots, \varphi_n$

Similarity dimension of $\varphi_1, \dots, \varphi_n$

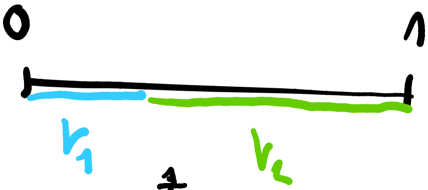
$$s : \underset{[0,\infty)}{D} \to \sum r_i^D \qquad s(D) = 1$$

$n \geq 2$

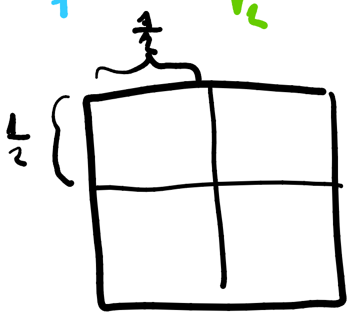
$0 < r_i < 1$



dim = 1



$r_1^1 + r_2^1 = 1$



$4 \cdot \left(\frac{1}{2}\right)^D = 1 \to D = 2$

$2 \cdot \left(\frac{1}{3}\right)^D = 1 \qquad D = \frac{\log 2}{\log 3}$

2

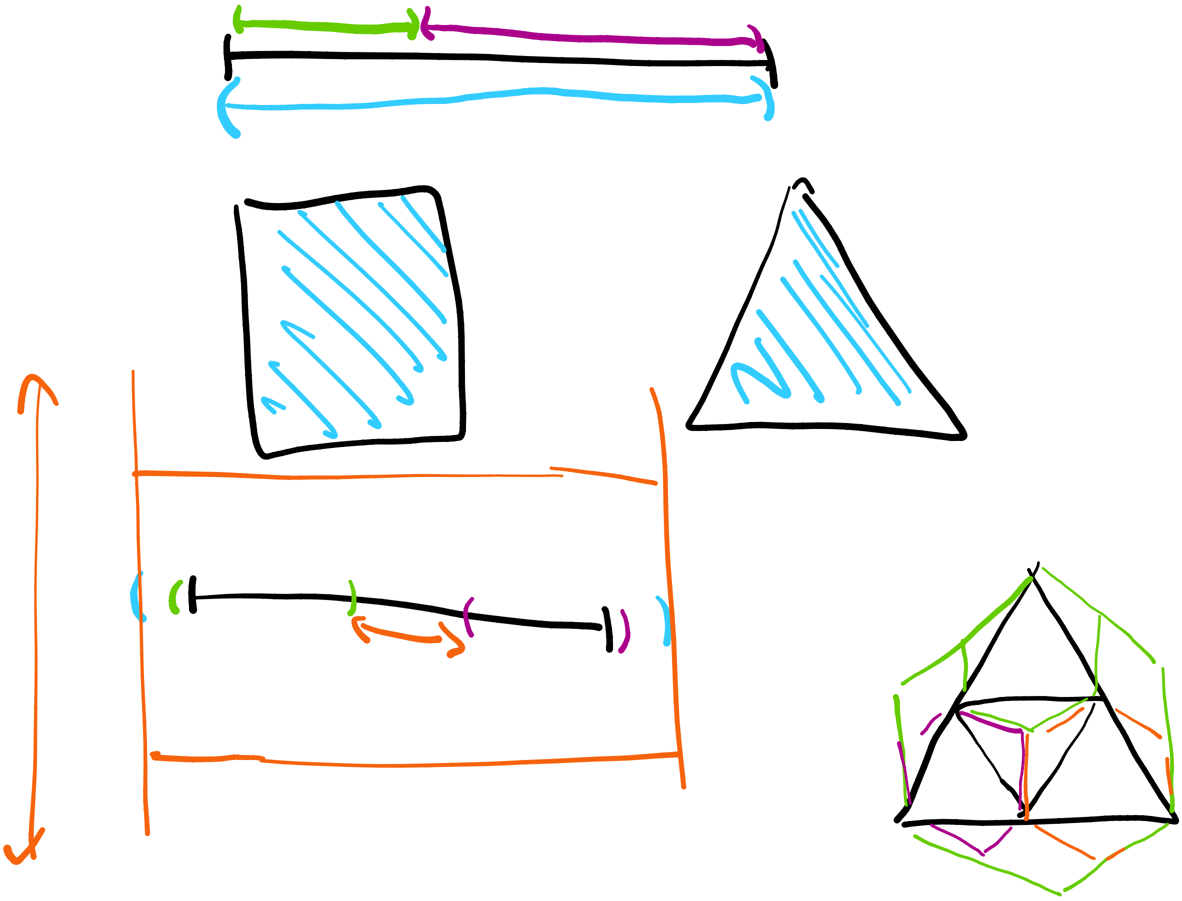
$\to \quad 3\left(\frac{1}{2}\right)^D = 1 \qquad D \neq 1$

Open set condition (OSC)

There is $U \subseteq \mathbb{R}^d$ open nonempty st.:

(1) $\varphi_i(U) \subseteq U$ ← feasible open set

(2) $\varphi_i(U) \cap \varphi_j(U) = \emptyset$



$U \not\supset K$ always feasible

HW4: $K \subseteq \overline{U}$, there is always a bounded feasible open set

we can always find a feasible open set s.t. $U \cap K \neq \emptyset$
↑
(SOSC)

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$\Sigma_n = \{1, \dots, n\}^{\mathbb{N}}$ — infinite

$\Sigma_n^* = \{1, \dots, n\}^{<\mathbb{N}}$ — finite words on alphabet $\{1,..,n\}$

$\Sigma_n^N = \{1, \dots, n\}^N$

$N = |I|$

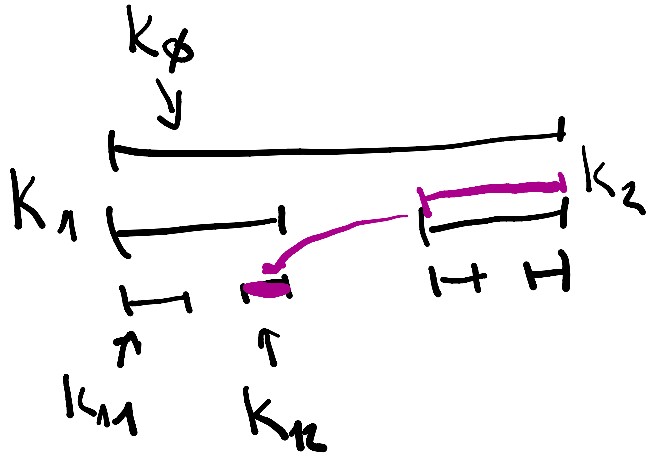
$I \in \Sigma_n^* \rightarrow \varphi_I = \varphi_{I_1} \circ \varphi_{I_2} \circ \dots \circ \varphi_{I_N}$

- $K_I = \varphi_I(K) \rightarrow K = \bigcup_i \varphi_i(K) = \bigcup_{i,j} \varphi_{ij}(K)$

$\bigcup_j \varphi_j(K) = \bigcup_{|I|=2} \varphi_I(K)$

$K = \bigcup_{|I|=N} K_I$

$\varphi_J \quad K_J = \bigcup_{|I|=N} K_{JI}$



$\omega \in \Sigma_N$

$\gamma : K_{\omega|_n} \longrightarrow \{x_\omega\} \in K$

$$\mathrm{diam}(K_I) \le r_{max}^{|I|} \cdot \mathrm{diam}\, K \to 0 \quad \text{as } |I| \to \infty$$

- $\mathcal{J} : \Sigma_N \xrightarrow{\text{onto}} K$

$$U_I = \varphi_I(U) \qquad \varphi_I(U) \cap \varphi_J(U) = \emptyset$$

$I \neq J$

$$r_I = r_{I_1} \cdot r_{I_2} \cdot \dots \cdot r_{I_N}$$

$|I| = N$

$$\mu : [1] \longrightarrow r_1^D$$

$I \in \Sigma_N^*$

$\downarrow$

$$\mu([I]) = r_I^D$$

$\uparrow$

sequences from Σ_N starting wit I

$$\mathcal{J}_*(\mu)$$