

Adaptive numerical solution of time-dependent PDEs

V. Dolejší

Numerical Software

Numerical solution of PDE

Possible applications

- modelling in atmosphere
- compressible/incompressible flows in industrial devices
- flows in porous media

Source of errors

- spatial discretization
 - temporal discretization
 - solution of algebraic systems
-
- higher DoF $\Rightarrow$ smaller error $\Rightarrow$ higher computational costs
 - adaptive techniques are necessary
 - balance between accuracy & efficiency

Numerical solution of PDE

Possible applications

- modelling in atmosphere
- compressible/incompressible flows in industrial devices
- flows in porous media

Source of errors

- spatial discretization
 - temporal discretization
 - solution of algebraic systems
-
- higher DoF $\Rightarrow$ smaller error $\Rightarrow$ higher computational costs
 - adaptive techniques are necessary
 - balance between accuracy & efficiency

Numerical solution of PDE

Possible applications

- modelling in atmosphere
- compressible/incompressible flows in industrial devices
- flows in porous media

Source of errors

- spatial discretization
 - temporal discretization
 - solution of algebraic systems
-
- higher DoF $\Rightarrow$ smaller error $\Rightarrow$ higher computational costs
 - adaptive techniques are necessary
 - balance between **accuracy** & **efficiency**

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{ m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{ m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t \mathbf{w} + \nabla \cdot \vec{\mathbf{f}}(\mathbf{w}) - \nabla \cdot \vec{\mathbf{R}}(\mathbf{w}, \nabla \mathbf{w}) = \mathbf{g}(\mathbf{w}) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $\mathbf{w} = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{\mathbf{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{\mathbf{R}} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(\mathbf{w}) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{ m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{ m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{ m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{ m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{ m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{ m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,

(Θ is constant in atmospheric steady state)

Atmospheric modelling

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \mathbf{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \rho v_1, \rho v_2, e)^T$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\mathbf{g}(w) = (0, 0, -\rho g, -\rho g v_2)^T$ – gravity forces, $g = 9.81 \text{m/s}^2$

Constitutive relations

- state equation $p = R\rho\theta$, energy $e = \rho c_v \theta + \rho |\mathbf{v}|^2/2$,
- Exner pressure $P = (p/p_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/P$,
(Θ is constant in atmospheric steady state)

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
 - mesh
 - time steps
 - polynomial degrees
 - preconditioners

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an **atmosphere in equilibrium** in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
 - mesh
 - time steps
 - polynomial degrees
 - preconditioners

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a **cold bubble**, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
 - mesh
 - time steps
 - polynomial degrees
 - preconditioners

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it **sinks to the impermeable ground**
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
 - mesh
 - time steps
 - polynomial degrees
 - preconditioners

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- **Kelvin–Helmholtz vortices** are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
 - mesh
 - time steps
 - polynomial degrees
 - preconditioners

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
 - mesh
 - time steps
 - polynomial degrees
 - preconditioners

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

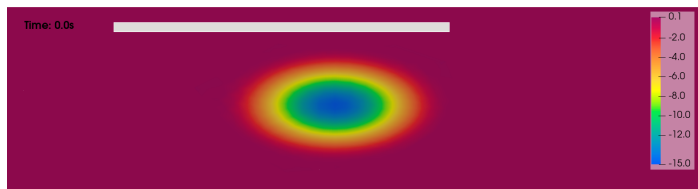
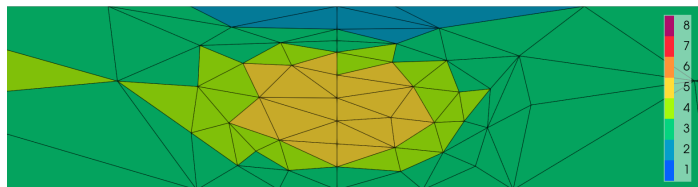
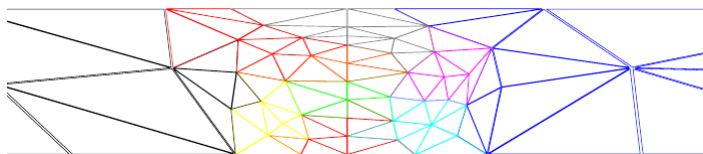
Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

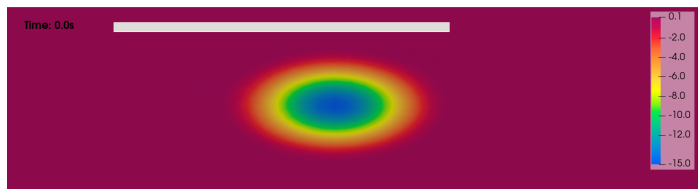
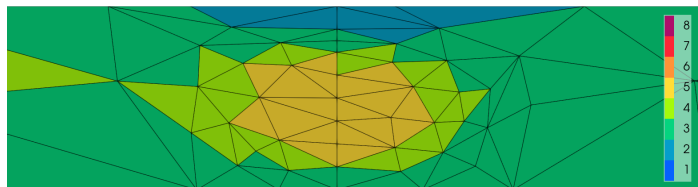
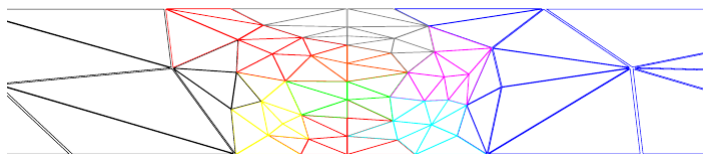
- discontinuous FE space-time discretization
- adaptation of
 - mesh
 - time steps
 - polynomial degrees
 - preconditioners

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

 $\Delta\Theta$  h_p 

DD

Density current flow [Straka et al IJNMF 93], [Giraldo Rostelli, JCP 08]

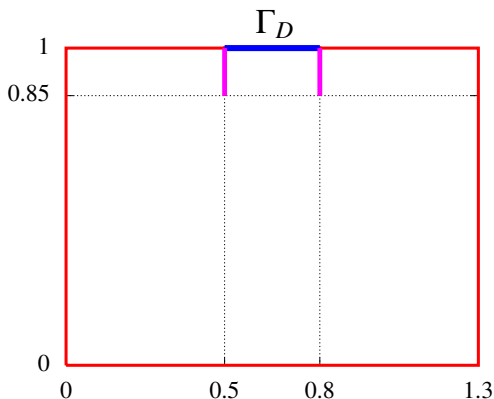
 $\Delta\Theta$  h_p 

DD

Simulation of the single ring infiltration

- flow through variably saturated media

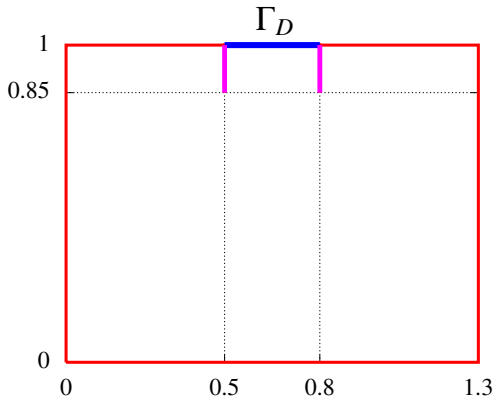
• ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D :
 $\Psi = 1$
- homogeneous
Neumann BC
otherwise

Simulation of the single ring infiltration

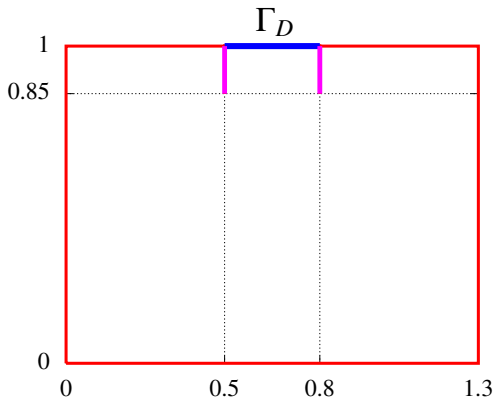
- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D :
 $\Psi = 1$
- homogeneous
Neumann BC
otherwise

Simulation of the single ring infiltration

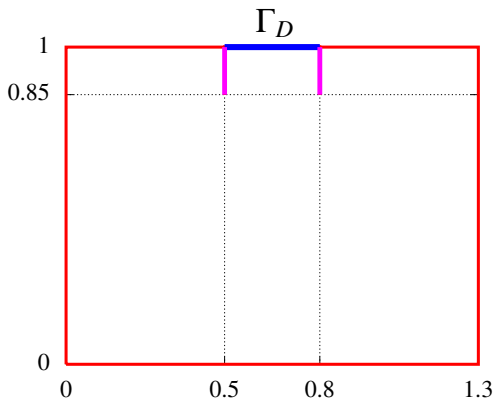
- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D :
 $\Psi = 1$
- homogeneous
Neumann BC
otherwise

Simulation of the single ring infiltration

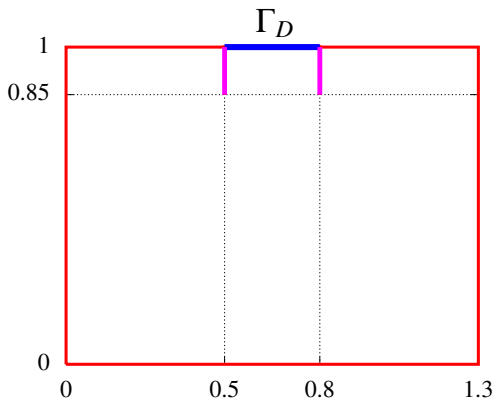
- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D :
 $\Psi = 1$
- homogeneous
Neumann BC
otherwise

Simulation of the single ring infiltration

- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D :
 $\Psi = 1$
- homogeneous Neumann BC otherwise

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot (K(\psi) \nabla (\psi + z)) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \dots$ van Genuchten formula
- $K(\psi) = \dots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot (K(\psi) \nabla (\psi + z)) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \dots$ van Genuchten formula
- $K(\psi) = \dots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot (K(\psi) \nabla (\psi + z)) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \dots$ van Genuchten formula
- $K(\psi) = \dots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot (K(\psi) \nabla (\psi + z)) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \dots$ van Genuchten formula
- $K(\psi) = \dots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot (K(\psi) \nabla(\psi + z)) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \dots$ van Genuchten formula
- $K(\psi) = \dots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot (K(\psi) \nabla (\psi + z)) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

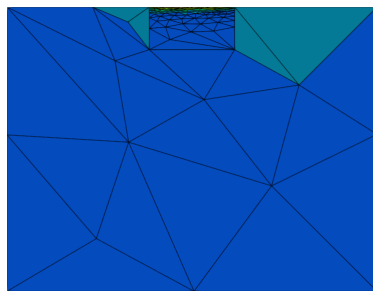
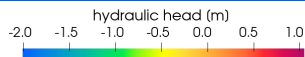
- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \dots$ van Genuchten formula
- $K(\psi) = \dots$ Mualem formula
- nonlinear functions depending on material parameters

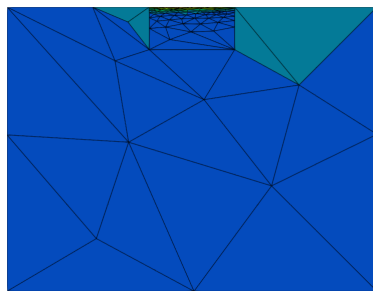
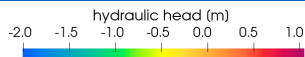
Single ring infiltration – mesh adaptation

Time: 0.000 hours



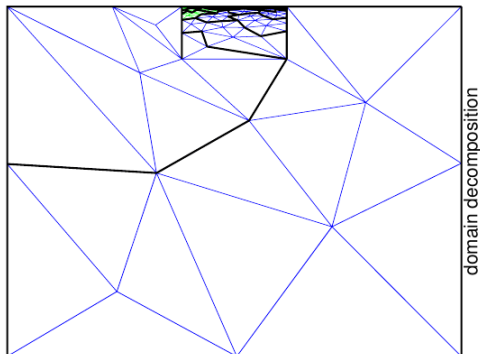
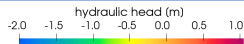
Single ring infiltration – mesh adaptation

Time: 0.000 hours



Single ring infiltration – domain decomposition

Time: 0.000 hours



Single ring infiltration – domain decomposition

Time: 0.000 hours

