Adaptive numerical solution of time-dependent PDEs

V. Dolejší

Numerical Software

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho,\ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0,\ 0,\ -\rho g,\ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81\mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0, \ 0, \ -\rho g, \ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81 \mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}} \theta + \rho |\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho,\ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0,\ 0,\ -\rho g,\ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81\mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{f}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{f} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0, \ 0, \ -\rho g, \ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81 \mathrm{m/s^2}$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho,\ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0,\ 0,\ -\rho g,\ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81\mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0, \ 0, \ -\rho g, \ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81\mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,

  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho,\ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0,\ 0,\ -\rho g,\ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81\mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0, \ 0, \ -\rho g, \ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81 \mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho,\ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0,\ 0,\ -\rho g,\ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81\mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho, \ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0, \, 0, \, -\rho g, \, -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81 \mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,
  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho,\ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0,\ 0,\ -\rho g,\ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81\mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,

  (Θ is constant is atmospheric steady state)

Governing equations

Compressible Navier-Stokes equations

$$\partial_t w + \nabla \cdot \vec{\boldsymbol{f}}(w) - \nabla \cdot \vec{R}(w, \nabla w) = \boldsymbol{g}(w) \quad \text{in } Q_T := \Omega \times (0, T)$$

- $w = (\rho,\ \rho v_1, \rho v_2, e)^{\mathrm{T}}$ – state vector
- $\vec{\boldsymbol{f}} = (f_1, f_2)$ – inviscid fluxes
- $\vec{R} = (R_1, R_2)$ – viscous fluxes
- $\boldsymbol{g}(w) = (0,\ 0,\ -\rho g,\ -\rho g v_2)^{\mathrm{T}}$ – gravity forces, $g = 9.81\mathrm{m/s}^2$

Constitutive relations

- state equation $\mathrm{p} = R\rho\theta$, energy $e = \rho c_{\mathrm{v}}\theta + \rho|\boldsymbol{v}|^2/2$,
- Exner pressure $\mathrm{P} = (\mathrm{p}/\mathrm{p}_0)^{(\kappa-1)/\kappa}$, κ – Poisson constant
- potential temperature $\Theta = \theta/\mathrm{P}$,

  (Θ is constant is atmospheric steady state)

Density current flow [Straka et all IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
  - mesh
  - time steps
  - polynomial degrees
  - preconditioners

Density current flow [Straka et all IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
  - mesh
  - time steps
  - polynomial degrees
  - preconditioners

Density current flow [Straka et all IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
  - mesh
  - time steps
  - polynomial degrees
  - preconditioners

Density current flow [Straka et all IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
  - mesh
  - time steps
  - polynomial degrees
  - preconditioners

Density current flow [Straka et all IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
  - mesh
  - time steps
  - polynomial degrees
  - preconditioners

Density current flow [Straka et all IJNMF 93], [Giraldo Rostelli, JCP 08]

Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
  - mesh
  - time steps
  - polynomial degrees
  - preconditioners

Density current flow [Straka et all IJNMF 93], [Giraldo Rostelli, JCP 08]

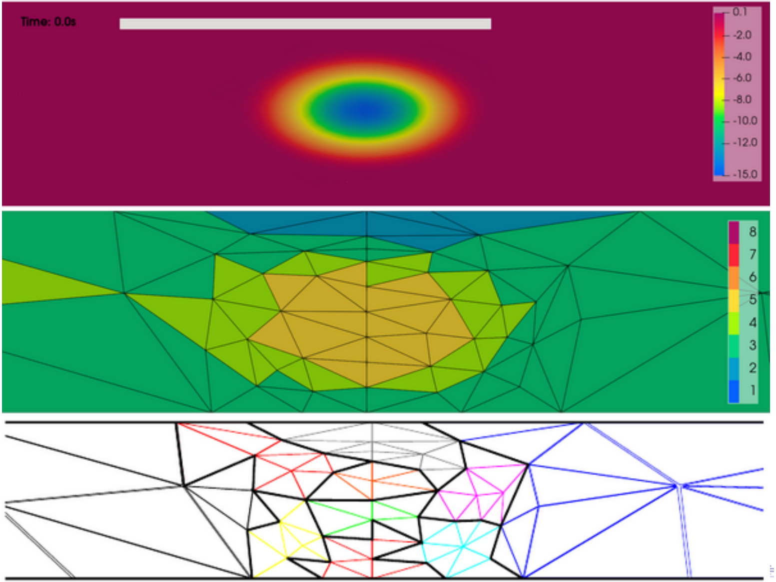
Problem setting

- an atmosphere in equilibrium in rectangular domain
- we add a cold bubble, it sinks to the impermeable ground
- Kelvin–Helmholtz vortices are formed

Numerical approach

- discontinuous FE space-time discretization
- adaptation of
    - mesh
    - time steps
    - polynomial degrees
    - preconditioners

Density current flow [Straka et all IJNMF 93], [Giraldo Rostelli, JCP 08]



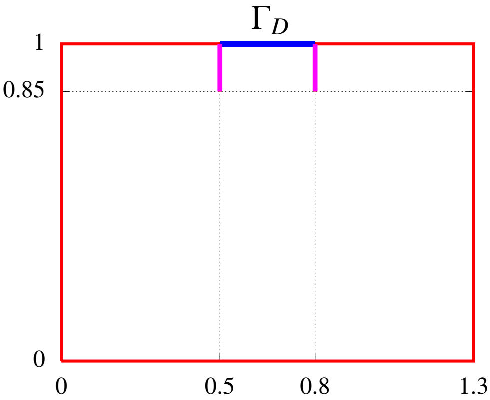
$\Delta\Theta$

hp

DD

Simulation of the single ring infiltration

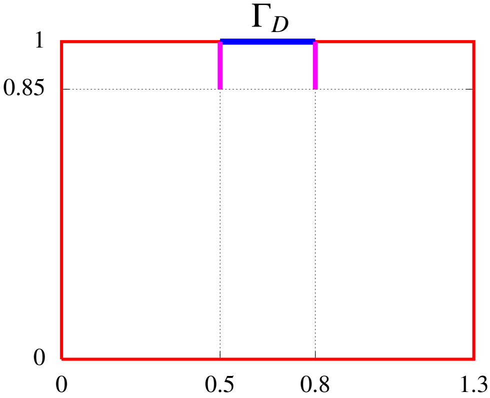
- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D: $\Psi = 1$
- homogeneous Neumann BC otherwise

Simulation of the single ring infiltration

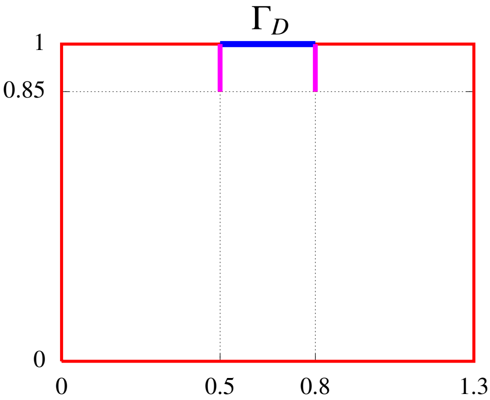
- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D: $\Psi = 1$
- homogeneous Neumann BC otherwise

Simulation of the single ring infiltration

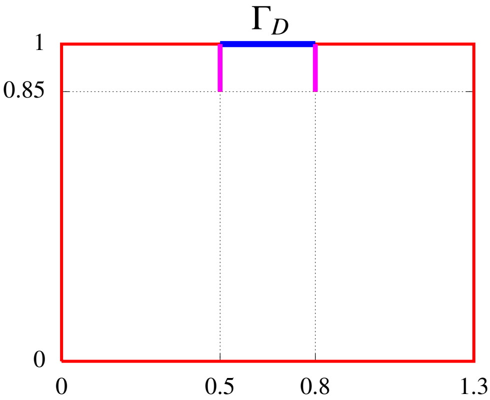
- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D: $\Psi = 1$
- homogeneous Neumann BC otherwise

Simulation of the single ring infiltration

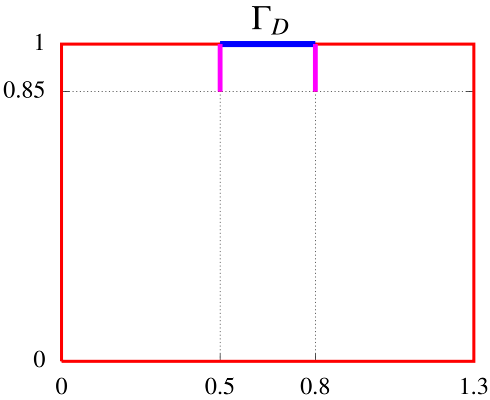
- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D: $\Psi = 1$
- homogeneous Neumann BC otherwise

Simulation of the single ring infiltration

- flow through variably saturated media
- ψ – pressure head [m], $\Psi = \psi + z$ – hydraulic head [m]



- initial BC $\Psi = -1$
- Dirichlet BC on Γ_D: $\Psi = 1$
- homogeneous Neumann BC otherwise

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot \left(K(\psi)\nabla(\psi + z)\right) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \ldots$ van Genuchten formula
- $K(\psi) = \ldots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot \big(K(\psi)\nabla(\psi + z)\big) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \dots$ van Genuchten formula
- $K(\psi) = \dots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot \big(K(\psi)\nabla(\psi + z)\big) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \ldots$ van Genuchten formula
- $K(\psi) = \ldots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot \big(K(\psi)\nabla(\psi + z)\big) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \ldots$ van Genuchten formula
- $K(\psi) = \ldots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

$$\partial_t \vartheta(\psi) - \nabla \cdot \big(K(\psi)\nabla(\psi + z)\big) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \dots$ van Genuchten formula
- $K(\psi) = \dots$ Mualem formula
- nonlinear functions depending on material parameters

Governing equations

Richards equation

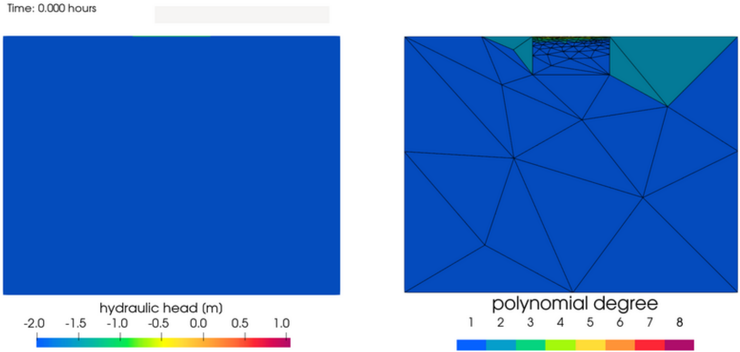
$$\partial_t \vartheta(\psi) - \nabla \cdot \big(K(\psi)\nabla(\psi + z)\big) = 0 \quad \text{in } Q_T := \Omega \times (0, T)$$

- ψ – pressure head
- $\vartheta(\psi)$ – water content
- $K(\psi)$ – hydraulic conductivity

Constitutive relations

- $\vartheta(\psi) = \ldots$ van Genuchten formula
- $K(\psi) = \ldots$ Mualem formula
- nonlinear functions depending on material parameters

Single ring infiltration – mesh adaptation



Single ring infiltration – domain decomposition