

# DGM for convection-diffusion equation

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Lecture 4

- model scalar convection-diffusion equation,
- Let  $\Omega \subset \mathbb{R}^d$ ,  $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ ,  $\partial\Omega_D \cap \partial\Omega_N = \emptyset$ ,  $Q_T \equiv \Omega \times (0, T)$ , we seek  $u : Q_T \rightarrow \mathbb{R}$  such that

## Scalar convection-diffusion problem

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) - \varepsilon \Delta u = g \quad \text{in } Q_T, \quad (1)$$

$$u = u_D \quad \text{on } \partial\Omega_D, \quad t \in (0, T),$$

$$\nabla(u) \cdot \mathbf{n} = g_N \quad \text{on } \partial\Omega_N, \quad t \in (0, T),$$

$$u(x, 0) = u^0(x) \quad x \in \Omega,$$

where:  $\vec{f} = (f_1, \dots, f_d)$ ,  $f_s \in C^1(\mathbb{R})$ ,  $s = 1, \dots, d$ ,  $\varepsilon > 0$ .

Suitable assumptions on  $\vec{f}$ ,  $g$ ,  $u_D$ ,  $g_N$  and  $u^0$  guarantee the existence and uniqueness of the weak solution.

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) - \varepsilon \Delta u = g \quad (2)$$

- let  $u$  be a strong (regular) solution,
- we multiply (2) by  $v \in H^2(\Omega, \mathcal{T}_h)$ ,
- integrate over each  $K \in \mathcal{T}_h$ ,
- apply Green's theorem,
- sum over all  $K \in \mathcal{T}_h$ ,
- we include additional terms vanishing for regular solution,
- we obtain the identity

$$\left( \frac{\partial u}{\partial t}(t), v \right) + \varepsilon a_h(u(t), v) + b_h(u(t), v) + \varepsilon J_h^\sigma(u(t), v) = \ell_h(v)(t) \\ \forall v \in H^2(\Omega, \mathcal{T}_h) \quad \forall t \in (0, T), \quad (3)$$

Convective term (“finite volume approach”):

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \vec{f}(u) v \, dx \\ &= - \sum_{K \in \mathcal{T}_h} \int_K \vec{f}(u) \cdot \nabla v \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \vec{f}(u) \cdot \mathbf{n} v \, dS \end{aligned}$$

Numerical flux

$$\vec{f}(u) \cdot \mathbf{n}|_{\Gamma} \approx H \left( u|_{\Gamma}^{(L)}, u|_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma} \right), \quad \Gamma \in \mathcal{F}_h,$$

$$\begin{aligned} b_h(u, v) &= - \sum_{K \in \mathcal{T}_h} \int_K \vec{f}(u) \cdot \nabla v \, dx \\ &+ \sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} H \left( u|_{\Gamma}^{(L)}, u|_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma} \right) [v]_{\Gamma} \, dS \end{aligned}$$

## Diffusive form

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx - \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_{\Gamma} \langle \nabla u \rangle \cdot \mathbf{n} [v] \, dS \\ + \theta \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_{\Gamma} \langle \nabla v \rangle \cdot \mathbf{n} [u] \, dS,$$

- $\theta = -1$  SIPG formulation,
- $\theta = 1$  NIPG formulation,
- $\theta = 0$  IIPG formulation.

## Interior and boundary penalty

$$J_h^\sigma(u, v) = \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_{\Gamma} \sigma[u][v] \, dS, \quad \sigma_{\Gamma} = \frac{C_W}{h_{\Gamma}},$$

## Right-hand-side

$$\begin{aligned} \ell_h(v)(t) = & \int_{\Omega} g(t) v \, dx + \sum_{\Gamma \in \mathcal{F}_h^N} \int_{\Gamma} g_N(t) v \, dS \\ & + \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} (\theta \nabla v \cdot \mathbf{n} u_D(t) + \sigma u_D(t) v) \, dS \end{aligned}$$

- Let  $u(t, x) \in C^1(0, T; H^2(\Omega))$  be the weak solution (1), then

$$\begin{aligned} \left( \frac{\partial u}{\partial t}(t), v \right) + \varepsilon a_h(u(t), v) + b_h(u(t), v) + \varepsilon J_h^\sigma(u(t), v) \\ = \ell_h(v)(t), \quad v \in H^2(\Omega, \mathcal{T}_h), \quad t \in (0, T), \quad (4) \end{aligned}$$

- (4) makes sense also for  $u(t) \in H^2(\Omega, \mathcal{T}_h)$ ,  $t \in (0, T)$ .
- since  $S_{hp} \subset H^2(\Omega, \mathcal{T}_h)$ , identity (4) makes sense for  $u, v \in S_{hp}$

## Definition

We say that  $u_h$  is a DGFE solution iff

- a)  $u_h \in C^1(0, T; S_{hp})$ ,
- b) 
$$\left( \frac{\partial u_h(t)}{\partial t}, v_h \right) + b_h(u_h(t), v_h) + \varepsilon a_h(u_h(t), v_h) \quad (5)$$
$$+ \varepsilon J_h^\sigma(u_h(t), v_h) = \ell_h(v_h)(t) \quad \forall v_h \in S_{hp}, t \in (0, T)$$
- c)  $u_h(0) = \Pi_{hp} u^0$ ,

where  $\Pi_{hp} u^0$  is a projection of IC in  $S_{hp}$

- system of ODEs,
- number of equations =  $\dim S_{hp}$
- (semi)-implicit ODE solver advantageous,



## Main theorem

Let

- $\partial u / \partial t \in L^2(0, T; H^s(\Omega))$  be the exact regular weak solution,
- $u_h \in C^1(0, T; S_{hp})$  be the approximate solution given by (5)
- assumptions on meshes, numerical flux, problem data,
- $e_h \equiv u_h - u$ ,

Then

$$\begin{aligned} \max_{t \in [0, T]} \|e_h(t)\|_{L^2(\Omega)}^2 + \varepsilon \int_0^T \|e_h(\vartheta)\|_{L^2(\Omega)}^2 d\vartheta & \quad (6) \\ & \leq C_2 h^{2\mu-2} \left( |u|_{L^2(\cdot, H^\mu(\Omega))}^2 + |\partial u / \partial t|_{L^2(\cdot, H^\mu(\Omega))}^2 \right), \end{aligned}$$

$$\mu = \min(p + 1, s).$$

# Auxiliary results $\Gamma_N = \emptyset$ !! (No Neumann BC)

$$|b_h(u, v) - b_h(\bar{u}, v)| \leq C \|v\| \left( \|u - \bar{u}\|_{L^2(\Omega)}^2 + \sum_{K \in \mathcal{T}_h} h_K \|u - \bar{u}\|_{L^2(\partial K)}^2 \right)^{1/2},$$
$$u, \bar{u} \in H^1(\Omega, \mathcal{T}_h) \cap L^\infty(\Omega), v \in H^1(\Omega, \mathcal{T}_h), h \in (0, \bar{h}) \quad (7)$$

$$|b_h(u_h, v_h) - b_h(\bar{u}_h, v_h)| \leq C \|v_h\| \|u_h - \bar{u}_h\|_{L^2(\Omega)}, \quad u_h, \bar{u}_h, v_h \in S_{hp},$$

Let  $\Pi_{hp} : H^s(\Omega) \rightarrow S_{hp}$  and  $\eta = u - \Pi_{hp}u$ , then

$$|b_h(u, v_h) - b_h(\Pi_{hp}u, v_h)| \leq CR_b(\eta) \|v_h\|, \quad v_h \in S_{hp}, h \in (0, \bar{h}), \quad (8)$$

Let  $\xi = u_h - \Pi_{hp}u$ , then under the above assumptions,

$$|b_h(u, v_h) - b_h(u_h, v_h)| \leq C \|v_h\| (R_b(\eta) + \|\xi\|_{L^2(\Omega)}), \quad v_h \in S_{hp}. \quad (9)$$

where  $R_b(\eta) = \left( \sum_{K \in \mathcal{T}_h} (\|\eta\|_{L^2(K)}^2 + h_K^2 |\eta|_{H^1(K)}^2) \right)^{1/2}$ .

# Sketch of the proof of (7)

We assume Lipschitz continuity of  $\vec{f}$  and  $H$ :

$$|\vec{f}(u) - \vec{f}(\bar{u})| \leq L_f |u - \bar{u}| \quad \forall u, \bar{u} \in \mathbb{R},$$

$$|H(u_1, u_2, \mathbf{n}) - H(v_1, v_2, \mathbf{n})| \leq L_H (|u_1 - v_1| + |u_2 - v_2|) \quad \forall u_1, u_2, v_1, v_2 \in \mathbb{R}$$

$$\begin{aligned} & b_h(u, v) - b_h(\bar{u}, v) && (10) \\ = & - \sum_{K \in \mathcal{T}_h} \int_K (\vec{f}(u) - \vec{f}(\bar{u})) \cdot \nabla v \, dx && =: \sigma_1 \\ & + \sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} \left( H(u_{\Gamma}^{(L)}, u_{\Gamma}^{(R)}, \mathbf{n}) - H(\bar{u}_{\Gamma}^{(L)}, \bar{u}_{\Gamma}^{(R)}, \mathbf{n}) \right) [v] \, dS && =: \sigma_2. \end{aligned}$$

$$|\sigma_1| \leq L_f \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d |u - \bar{u}| \left| \frac{\partial v}{\partial x_s} \right| dx \leq \sqrt{d} L_f \|u - \bar{u}\|_{L^2(\Omega)} \|v\|_{H^1(\Omega, \mathcal{T}_h)}.$$

$$|\sigma_2| \leq L_H \sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} \left( |u_{\Gamma}^{(L)} - \bar{u}_{\Gamma}^{(L)}| + |u_{\Gamma}^{(R)} - \bar{u}_{\Gamma}^{(R)}| \right) |[v]| \, dS \dots$$

# Sketch of the proof of error estimates (1)

## Approximate solution

$$\left( \frac{\partial u_h(t)}{\partial t}, v_h \right) + b_h(u_h(t), v_h) + \varepsilon a_h(u_h(t), v_h) + \varepsilon J_h^\sigma(u_h(t), v_h) = \ell_h(v_h)$$

## Consistency

$$\left( \frac{\partial u(t)}{\partial t}, v_h \right) + b_h(u(t), v_h) + \varepsilon a_h(u(t), v_h) + \varepsilon J_h^\sigma(u(t), v_h) = \ell_h(v_h)$$

$$\text{Error } e_h = u_h - u = \xi + \eta$$

$$\xi = u_h - \Pi_{hp} u \in S_{hp},$$

$$\eta = \Pi_{hp} u - u \in H^s(\Omega, \mathcal{T}_h): \text{ available info from } u \Rightarrow \text{aim: } \|\xi\| \leq \|\eta\|$$

Putting  $A_h := a_h + J_h^\sigma$ , subtracting both above relations and  $v_h := \xi$  gives

$$\left( \frac{\partial \xi}{\partial t}, \xi \right) + \varepsilon A_h(\xi, \xi) = b_h(u, \xi) - b_h(u_h, \xi) - \left( \frac{\partial \eta}{\partial t}, \xi \right) - \varepsilon A_h(\eta, \xi).$$

$$\frac{1}{2} \frac{d}{dt} \|\xi\|_{L^2(\Omega)}^2 + \varepsilon C_C \|\xi\|^2 \leq b_h(u, \xi) - b_h(u_h, \xi) - \left( \frac{\partial \eta}{\partial t}, \xi \right) - \varepsilon A_h(\eta, \xi).$$

# Sketch of the proof of error estimates (2)

$$\frac{1}{2} \frac{d}{dt} \|\xi\|_{L^2(\Omega)}^2 + \varepsilon C_C \|\xi\|^2 \leq b_h(u, \xi) - b_h(u_h, \xi) - \left( \frac{\partial \eta}{\partial t}, \xi \right) - \varepsilon A_h(\eta, \xi).$$

Particular error estimates

$$\begin{aligned} \frac{d}{dt} \|\xi\|_{L^2}^2 + 2\varepsilon C_C \|\xi\|^2 \\ \leq C ((R_b(\eta) + \|\xi\|_{L^2}) \|\xi\| + \varepsilon R_a(\eta) \|\xi\| + \|\partial_t \eta\|_{L^2} \|\xi\|_{L^2}) \end{aligned}$$

where  $R_b(\eta) = (\sum_{K \in \mathcal{T}_h} (\|\eta\|_{L^2(K)}^2 + h_K^2 |\eta|_{H^1(K)}^2))^{1/2} = O(h^\mu)$ ,

$R_a(\eta) = (\sum_{K \in \mathcal{T}_h} (|\eta|_{H^1(K)}^2 + h_K^2 |\eta|_{H^2(K)}^2 + h_K^{-2} \|\eta\|_{L^2(K)}^2))^{1/2} = O(h^{\mu-1})$

$$\begin{aligned} \frac{d}{dt} \|\xi\|_{L^2}^2 + 2\varepsilon C_C \|\xi\|^2 \\ \leq C (R_b(\eta) + \varepsilon R_a(\eta) + \|\xi\|_{L^2}) \|\xi\| + C \|\partial_t \eta\|_{L^2} \|\xi\|_{L^2} \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \|\xi\|_{L^2}^2 + 2\varepsilon C_C \|\xi\|^2 \\ \leq \frac{C}{\varepsilon} (R_b(\eta) + \varepsilon R_a(\eta) + \|\xi\|_{L^2})^2 + \varepsilon C_C \|\xi\|^2 + C \|\partial_t \eta\|_{L^2}^2 + C \|\xi\|_{L^2}^2 \end{aligned}$$

# Sketch of the proof of error estimates (3)

$$\begin{aligned} \frac{d}{dt} \|\xi\|_{L^2}^2 + \varepsilon \|\xi\|^2 &\leq \frac{C}{\varepsilon} (R_b(\eta) + \varepsilon R_a(\eta) + \|\xi\|_{L^2})^2 + C \|\partial_t \eta\|_{L^2}^2 + C \|\xi\|_{L^2}^2 \\ &\leq \frac{C}{\varepsilon} (R_b(\eta) + \varepsilon R_a(\eta))^2 + \frac{C}{\varepsilon} \|\xi\|_{L^2}^2 + C \|\partial_t \eta\|_{L^2}^2 + C \|\xi\|_{L^2}^2 \\ &\leq C \left(1 + \frac{1}{\varepsilon}\right) \|\xi\|_{L^2}^2 + CR_Q(\eta) \end{aligned}$$

integration over  $\int_0^t d\vartheta$  gives (we have  $\xi(0) = u_h(0) - \Pi_{hp} u^0 = 0$ )

$$\begin{aligned} \|\xi(t)\|_{L^2(\Omega)}^2 + \varepsilon \int_0^t \|\xi(\vartheta)\|^2 d\vartheta \\ \leq C \left(1 + \frac{1}{\varepsilon}\right) \int_0^t \|\xi(\vartheta)\|_{L^2(\Omega)}^2 d\vartheta + \int_0^t R_Q(\eta(\vartheta)) d\vartheta. \end{aligned}$$

Gronwall's Lemma

$$\|\xi(t)\|_{L^2(\Omega)}^2 + \varepsilon C_C \int_0^t \|\xi(\vartheta)\|^2 d\vartheta \leq R(\eta, \varepsilon) c_1(\exp(1/\varepsilon)), \quad (11)$$

# Sketch of the proof of error estimates (4)

$$\|\xi(t)\|_{L^2(\Omega)}^2 + \varepsilon C_C \int_0^t \|\xi(\vartheta)\|_{L^2(\Omega)}^2 d\vartheta \leq R(\eta, \varepsilon) c_1(\exp(1/\varepsilon)),$$

since  $e_h = \xi + \eta$

$$\|e_h(t)\|_{L^2(\Omega)}^2 + \varepsilon C_C \int_0^t \|e_h(\vartheta)\|_{L^2(\Omega)}^2 d\vartheta \leq \tilde{R}(\eta, \varepsilon) c_1(\exp(1/\varepsilon)),$$

## Approximation properties

- $\mu = \min(p + 1, s)$ ,  $u \in C([0, T], H^s(\Omega))$ ,  $u_h \in S_{hp}$
- $R_a(\eta) = O(h^{\mu-1})$ ,  $R_b(\eta) = O(h^\mu) \Rightarrow \tilde{R}(\eta, \varepsilon) = O(h^{\mu-1})$

Finally,

$$\|e_h(t)\|_{L^2(\Omega)}^2 + \varepsilon C_C \int_0^t \|e_h(\vartheta)\|_{L^2(\Omega)}^2 d\vartheta \leq c_1(\exp(1/\varepsilon)) O(h^{\mu-1})$$

## SIPG

- optimal order of convergence – duality arguments,
- $C_W$  chosen carefully.

## NIPG

- sub-optimal order of convergence,
- $C_W$  arbitrary.

## IIPG

- sub-optimal order of convergence,
- $C_W$  chosen carefully,
- simpler implementation.



## Linear convection-diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial x_1} - \varepsilon \Delta u &= 1 \quad \text{in } \Omega = (0, 1) \times (0, 1), & (12) \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

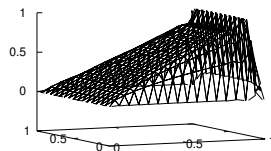
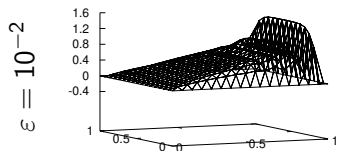
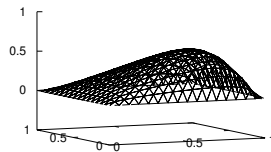
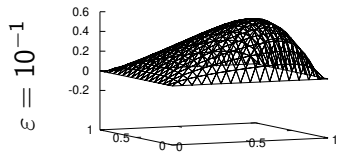
where  $\varepsilon > 0$

steep boundary layer

## Numerical methods

- conforming FEM
- IIPG variant of DGM
- $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$  and  $10^{-6}$

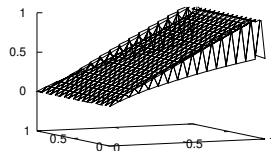
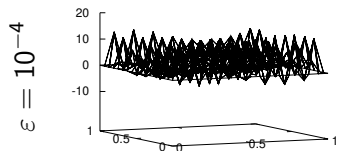
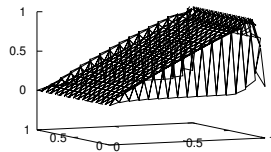
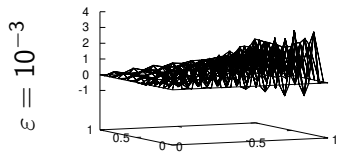
# Numerical results: cases $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-2}$



FEM

DGM

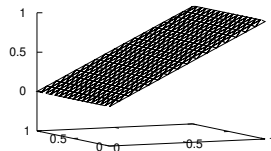
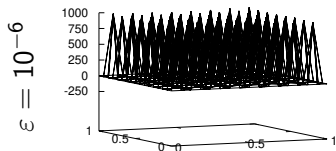
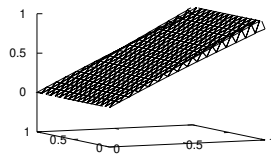
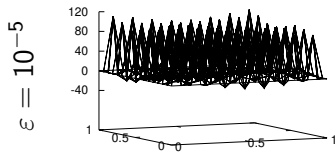
# Numerical results: cases $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$



FEM

DGM

# Numerical results: cases $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-6}$



FEM

DGM