**DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS**
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 119 – 127

# A class of noncompact homogeneous Einstein manifolds

Hiroshi Tamaru

Abstract. In this article we explain a construction of a class of solvmanifolds given by the author ([**14**]) and study their Einstein properties. We also give their complete list up to dimension 7 with their eigenvalue types.

## 1. Introduction

A Riemannian manifold is called a *solvmanifold* (resp. *nilmanifolds*) if it admits a transitive solvable (resp. nilpotent) group of isometries. Solvmanifolds and nilmanifolds provide many examples of Riemannian manifolds with good geometric structures and counterexamples for some conjectures. Damek-Ricci spaces are one of the most successive examples, which are certain one-dimensional solvable extensions of $H$-type groups, and provide examples of harmonic spaces (they are counterexamples of Lichnerowicz conjecture).

In this article, we study a class of solvmanifolds constructed by the author ([**14**]), which provides many examples of noncompact homogeneous Einstein manifolds. We will show that our solvmanifold is Einstein if the nilradical is two-step or three-step nilpotent.

Einstein solvmanifolds are important examples of noncompact homogeneous Einstein manifolds, actually no other examples have been known (so-called Alekseevkii's conjecture states that noncompact homogeneous Einstein manifolds are solvmanifolds). Let $(\mathfrak{s}, \langle , \rangle)$ be a metric solvable Lie algebra, which often identified with the corresponding solvmanifolds, the corresponding simply-connected Lie group with the induced left-invariant metric. A solvmanifold is called *standard* if $\mathfrak{a} = \mathfrak{n}^{\perp}$ is abelian, where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$. Standard Einstein solvmanifolds have been deeply studied by Heber ([**5**]), who obtained many fundamental results. One of his results states that there is distinguished element $H_0$ (called the mean curvature vector) such that $(\mathbb{R}H_0 + \mathfrak{n})$ is also Einstein. This procedure is called *rank one reduction*, by which the study of Einstein solvmanifolds can be reduced to the study of algebraic rank one Einstein solvmanifolds ($\dim \mathfrak{a}$ is called the *algebraic rank*).

Our solvmanifolds have algebraic rank one. Furthermore, the idea of rank one reduction is essential in our construction. We start from gradations of semisimple Lie algebras, construct nilmanifolds, and take certain rank one solvable extensions by adding "mean curvature vectors". It is remarkable that our class of solvmanifolds contains the class of rank one reduction of symmetric spaces of noncompact type.

---

2000 *Mathematics Subject Classification.* 53C30, 22E25, 53C25, 53C35,
This paper is in final form and no version of it will be published elswhere.

Other fundamental idea of Heber ([**5**]) is the notation of eigenvalue types. For standard Einstein solvmanifolds, the *eigenvalue types* are

$$(\mu_1 < \cdots < \mu_r; d_1, \ldots, d_r),$$

where $\mu_i$ are the eigenvalues of $\mathrm{ad}_{H_0}$ on $\mathfrak{n}$, and $d_i$ are the corresponding multiplicities. Heber studied the moduli spaces of standard Einstein solvmanifolds of fixed eigenvalue types, and showed that such moduli spaces have large dimensions in general.

In this article we make a complete list of our solvmanifolds up to dimension 7, and determine their eigenvalue types. One will see that eigenvalue types can be calculated in terms of root systems. We have to remark that this list is just an example of our solvmanifold. Our class of solvmanifolds contains many large dimensional ones, the nilradical of them may have large nilpotency, and the eigenvalue types can be "unusual".

## 2. Graded Lie algebras

In this section we briefly recall semisimple graded Lie algebras. We refer to [**8**] and [**13**] for details. A decomposition of a semisimple Lie algebra $\mathfrak{g}$ into subspaces with indices,

$$\mathfrak{g} = \sum_{k \in \mathbb{Z}} \mathfrak{g}_k$$

is called a *gradation* if the bracket product is compatible with indices, that is, $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$ holds for every $k$ and $l$. A gradation is said to be of *$\nu$-th kind* if $\mathfrak{g}_k = 0$ for every $|k| > \nu$. Two gradations on $\mathfrak{g}$ is called *isomorphic* if there exists an automorphism of $\mathfrak{g}$ preserving gradations. An easy example is given by the following second kind gradation on $\mathrm{sl}(3, \mathbb{R})$,

$$\mathfrak{g}_{-2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{bmatrix}, \quad \mathfrak{g}_{-1} = \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & * & 0 \end{bmatrix},$$

$$\mathfrak{g}_0 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \mathfrak{g}_1 = \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{g}_2 = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly, one can also construct gradations on $\mathrm{sl}(n, \mathbb{R})$ by block decompositions of matrices. It is easy to see that $\mathrm{sl}(n, \mathbb{R})$ has $\nu$-th kind gradation for every $\nu = 1, \ldots, n-1$.

Here we explain general construction of gradations on semisimple Lie algebras $\mathfrak{g}$. Let $\sigma$ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the corresponding Cartan decomposition. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$. As usual, we call $\alpha \in \mathfrak{a}^* - \{0\}$ a *root* if

$$0 \neq \mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for every } H \in \mathfrak{a}\ \}.$$

We call $\mathfrak{g}_\alpha$ the *root spaces*. Denote by $\Delta$ the set of roots and consider the root lattice

$$\{H \in \mathfrak{a} \mid \alpha(H) \in \mathbb{Z} \text{ for every } \alpha \in \Delta\ \}.$$

Now take an element $Z$ in the root lattice. Then the eigenspace decomposition of $\mathfrak{g}$ with respect to $\mathrm{ad}_Z$ gives a gradation. Note that every eigenvalue of $\mathrm{ad}_Z$ is an integer by the definition of the root lattice.

In fact, every gradation can be constructed in this way. Let $\mathfrak{g} = \sum \mathfrak{g}_k$ be a semisimple graded Lie algebra. It is known that there exists the unique element $Z$ such that $\mathfrak{g}_k$ coincides with the eigenspace of $\mathrm{ad}_Z$ with eigenvalue $k$. Such element $Z$ is called the *characteristic element*. Furthermore, it is also known that there exists a grade-reversing Cartan involution $\sigma$, that is, $\sigma(\mathfrak{g}_k) = \mathfrak{g}_{-k}$ for every $k$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the corresponding Catan decomposition and $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{m}$ containing $Z$. Then, $Z$ is contained in the root lattice of the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$.

For further description of the characteristic element, we take a simple root system $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta$. Let $\{H^1, \ldots, H^r\}$ be the dual basis of $\Pi$, that is, $\alpha_i(H^j) = \delta_{ij}$. By considering the action of the Weyl group, without loss of generality, one can assume that the characteristic element is contained in

$$\begin{aligned} C \ &:= \ \{H \in \mathfrak{a} \mid \alpha(H) \in \mathbb{Z}_{\geqslant 0} \text{ for every } \alpha \in \Delta\ \} \\ &= \ \{c_1 H^1 + \cdots + c_r H^r \mid c_1, \ldots, c_r \in \mathbb{Z}_{\geqslant 0}\}. \end{aligned}$$

If a gradation $\mathfrak{g} = \sum \mathfrak{g}_k$ has the characteristic element $Z \in C$, one has

$$\mathfrak{g}_k \ = \ \{X \in \mathfrak{g} \mid [Z, X] = kX\} \ = \ \sum_{\alpha(Z)=k} \mathfrak{g}_\alpha.$$

Therefore the structures of graded Lie algebras can be determined in terms of roots.

Finally we define "effectivity" of gradations. A gradation is said to be of *type* $\alpha_0$ if $\mathfrak{g}_1$ generates $\sum_{k>0} \mathfrak{g}_k$. For our construction of nilmanifolds and solvmanifolds, it is enough to consider gradations of type $\alpha_0$.

## 3. Construction of the solvmanifolds

In this section we describe a construction of solvmanifolds from semisimple graded Lie algebras $\mathfrak{g} = \sum \mathfrak{g}_k$. As in the previous section, let $Z$ be the characteristic element, $\sigma$ a grade-reversing Cartan involution, $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the corresponding Cartan decomposition, and $\mathfrak{a}$ a maximal abelian subspace in $\mathfrak{m}$ containing $Z$.

First of all, define

$$\mathfrak{n} := \sum_{k>0} \mathfrak{g}_k.$$

This is obviously a nilpotent subalgebra. Note that one can take "negative part" instead of "positive part", since there exists a grade-reversing Cartan involution. Define an inner product $B_\sigma$ on $\mathfrak{g}$ by

$$B_\sigma(X, Y) := -B(X, \sigma(Y)),$$

where $B$ denotes the Killing from of $\mathfrak{g}$. Consider the metric solvable Lie algebra $(\mathfrak{a} + \mathfrak{n}, B_\sigma)$. Let $H_0$ be the mean curvature vector of this solvmanifold, that is,

$$B_\sigma(H_0, H) = \mathrm{tr}(\mathrm{ad}_H |_{\mathfrak{n}})$$

for every $H \in \mathfrak{a}$.

DEFINITION 1. The metric nilpotent Lie algebra $(\mathfrak{n}, B_\sigma |_{\mathfrak{n}\times\mathfrak{n}})$ is called the *nilmanifold attached to a graded Lie algebra*. The solvable Lie algebra $\mathfrak{s} := \mathbb{R}H_0 + \mathfrak{n}$ endowed with the inner product $\langle , \rangle := 2B_\sigma |_{\mathbb{R}H_0 \times \mathbb{R}H_0} + B_\sigma |_{\mathfrak{n}\times\mathfrak{n}}$ is called the *solvmanifold attached to a graded Lie algebra*.

Here we describe the mean curvature vector. Let $\{E_i^{(k)}\}$ be an orthonormal basis of $\mathfrak{g}_k$, and we call $Z_k := \sum[\sigma(E_i^{(k)}), E_i^{(k)}]$ the *$k$-th mean curvature vector*. It is easy to see that, for a $\nu$-th kind gradation, the mean curvature vector satisfies $H_0 = Z_1 + \cdots + Z_\nu$. Furthermore, these mean curvature vectors can be calculated in terms of the roots. We call $H_\alpha$ the *root vector* if $B_\sigma(H_\alpha, H) = \alpha(H)$ holds for every $H \in \mathfrak{a}$. Let $\Delta_k := \{\alpha \mid \alpha(H) = k\}$, where $Z$ is the characteristic element (note that $\mathfrak{g}_k = \sum_{\alpha\in\Delta_k} \mathfrak{g}_\alpha$ holds).

LEMMA 1. *The $k$-th mean curvature vectors satisfy* $Z_k = \displaystyle\sum_{\alpha\in\Delta_k} (\dim \mathfrak{g}_\alpha) \cdot H_\alpha$.

This expression is quite useful to determine the mean curvature vectors and calculate the Ricci curvatures of our solvmanifolds.

## 4. Curvature properties of our solvmanifolds

Denote by $\mathrm{Ric}^{\mathfrak{n}}$ the Ricci curvatures of the nilmanifolds $(\mathfrak{n}, B_\sigma)$ (of course we consider the corresponding simply-connected Lie group endowed with the induced left-invariant metric), and by Ric the Ricci curvatures of the attached solvmanifolds $(\mathfrak{s}, \langle,\rangle)$. These curvatures can be calculated as follows. Define $Z_k = 0$ for $k > \nu$.

THEOREM 1 ([**14**]). (i) *For every $U_l \in \mathfrak{g}_l$, one has*

$$\mathrm{Ric}^{\mathfrak{n}}(U_l) = -(1/4)\sum_{k<l} A_k(U_l) + (1/2)\sum_{k>l} A_k(U_l),$$

*where the operator $A_k : \mathfrak{n} \to \mathfrak{n}$ is defined by*

$$A_k(U_l) := -[Z_k, U_l] - [Z_{k+l}, U_l] - [Z_{k+2l}, U_l] - \cdots .$$

(ii) *For the Ricci curvatures* Ric *of the solvmanifolds* $(\mathfrak{s}, \langle,\rangle)$*, one has*

(1) $\mathrm{Ric}(H_0) = -(1/4)H_0$,
(2) $\mathrm{Ric}(U) = \mathrm{Ric}^{\mathfrak{n}}(U) - (1/2)[H_0, U]$ *for every* $U \in \mathfrak{n}$.

Therefore our solvmanifolds are Einstein if and only if $\mathrm{Ric}(U) = -(1/4)U$ for every $U \in \mathfrak{n}$. This condition can be written in terms of the mean curvature vectors $Z_1, \ldots, Z_\nu$. For example, one has

LEMMA 2. *For the Ricci curvatures* Ric *of the solvmanifolds* $(\mathfrak{s}, \langle,\rangle)$*, one has*

(1) $\mathrm{Ric}(U_1) = -(1/2)[\sum kZ_k, U_1]$ *for every* $U_1 \in \mathfrak{g}_1$.
(2) $\mathrm{Ric}(U_2) = -(1/4)[\sum kZ_k, U_2]$ *for every* $U_2 \in \mathfrak{g}_2$.

Since we assumed that our gradation is of type $\alpha_0$, Einstein condition implies

$$2\sum kZ_k = Z$$

(it follows from Lemma 2 (1) and the uniqueness of the characteristic element). Conversely, if the above equation holds, then one has $\mathrm{Ric} = -(1/4)\cdot \mathrm{id}$ on $\mathbb{R}H_0 + \mathfrak{g}_1 + \mathfrak{g}_2$.

THEOREM 2 ([**14**]). *The solvmanifolds attached to graded Lie algebras of second kind are Einstein.*

For the proof of this theorem, we have only to check $Z = 2(Z_1 + 2Z_2)$ for every second kind gradation.

## 5. Third kind case

In this section we give a sketch of the proof of the following theorem.

THEOREM 3. *The solvmanifolds attached to graded Lie algebras of third kind are Einstein.*

Because of Lemma 2, the solvmanifold attached to a graded Lie algebra of third kind is Einstein if and only if $Z = 2(Z_1 + 2Z_2 + 3Z_3)$ and $\mathrm{Ric}(U_3) = -(1/4)U_3$ for every $U_3 \in \mathfrak{g}_3$. By definition, the operator $A_k$ satisfies

$$A_k(U_3) = -[Z_k, U_3] \quad \text{for } k = 1, 2, 3.$$

Therefore Theorem 1 (i) leads that

$$\mathrm{Ric}^{\mathfrak{n}}(U_3) = -(1/4)(A_1(U_3) + A_2(U_3)) = (1/4)[Z_1 + Z_2, U_3].$$

Combining with Theorem 1 (ii), one has

$$\begin{aligned}\mathrm{Ric}(U_3) &= (1/4)[Z_1 + Z_2, U_3] - (1/2)[Z_1 + Z_2 + Z_3, U_3] \\ &= -(1/6)[Z_1 + 2Z_2 + 3Z_3, U_3] - (1/12)[Z_1 - Z_2, U_3].\end{aligned}$$

This implies that

LEMMA 3. *The solvmanifold attached to a graded Lie algebra of third kind is Einstein if and only if* $Z = 2(Z_1 + 2Z_2 + 3Z_3)$ *and* $[Z_1 - Z_2, \mathfrak{g}_3] = 0$.

These conditions can be checked in terms of the root system, since all $Z_k$ can be written as the linear combinations of the root vectors. Theorem 3 can be proved by checking these conditions individually.

## 6. The attached nilmanifolds up to dimension 6

In this section we give a complete list of the nilmanifolds attached to graded Lie algebras up to dimension 6. The list contains 3-step and 5-step nilmanifolds. In the next section we will investigate their Einstein solvable extensions.

THEOREM 4. *The nilmanifolds attached to simple graded Lie algebras which are not abelian and up to dimension* 6 *are listed in Table 1.*

Here we mention notations in the table. The multiplicities $(m_1, \ldots, m_r)$ mean that the multiplicity of the simple root $\alpha_i$ is $m_i$. Since the multiplicities of roots are invariant under the actions of the Weyl groups, $(m_1, \ldots, m_r)$ determine the multiplicities of all roots. For the root system of $BC_1$-type, $((m, m'))$ means that the multiplicities of $\alpha_1$ and $2\alpha_1$ are $m$ and $m'$, respectively. For the numbering of the simple roots, we use the notations in the book of Bourbaki ([**2**]). Denote by $\{H^1, \ldots, H^r\}$ the dual basis of the simple root system $\Pi \subset \Delta$. For the structures of $\mathfrak{n}$, the notation $(3 + 2 + 1)$ means that $\dim \mathfrak{g}_1 = 3$, $\dim \mathfrak{g}_2 = 2$ and $\dim \mathfrak{g}_3 = 1$ (therefore $\mathfrak{n}$ is of 3-step nilpotent).

We will give a sketch of this theorem. The following lemma gives a restriction for the dimension of the attached nilmanifold.

LEMMA 4. *Let* $\mathfrak{g} = \sum \mathfrak{g}_k$ *and* $\mathfrak{g} = \sum \mathfrak{g}'_k$ *be the semisimple graded Lie algebras with the characteristic elements* $H^i$ *and* $H^i + H^j$, *and denote by* $\mathfrak{n}$ *and* $\mathfrak{n}'$ *the attached nilpotent Lie algebras, respectively. Then one has* $\dim \mathfrak{n} < \dim \mathfrak{n}'$.

TABLE 1. The attached nilmanifolds up to dimension 6

| $\dim \mathfrak{n}$ | $\mathfrak{g}$ | $\Delta$ | multiplicities | $Z$ | $\mathfrak{n}$ | |
|---|---|---|---|---|---|---|
| 3 | $\mathrm{sl}(3,\mathbb{R})$ | $A_2$ | $(1,1,1)$ | $H^1+H^2$ | $(2+1)$ | Heisenberg |
| | $\mathrm{so}(2,3)$ | $B_2$ | $(1,1)$ | $H^2$ | $(2+1)$ | Heisenberg |
| | $\mathrm{su}(1,2)$ | $BC_1$ | $((2,1))$ | $H^1$ | $(2+1)$ | Heisenberg |
| 4 | $\mathrm{so}(2,3)$ | $B_2$ | $(1,1)$ | $H^1+H^2$ | $(2+1+1)$ | |
| 5 | $\mathrm{sl}(4,\mathbb{R})$ | $A_3$ | $(1,1,1)$ | $H^1+H^2$ | $(3+1+1)$ | |
| | $\mathrm{sl}(4,\mathbb{R})$ | $A_3$ | $(1,1,1)$ | $H^1+H^3$ | $(4+1)$ | Heisenberg |
| | $\mathrm{so}(2,4)$ | $B_2$ | $(1,2)$ | $H^2$ | $(4+1)$ | Heisenberg |
| | $\mathrm{sp}(3,\mathbb{R})$ | $C_3$ | $(1,1,1)$ | $H^1$ | $(4+1)$ | Heisenberg |
| | $\mathrm{su}(1,3)$ | $BC_1$ | $((4,1))$ | $H^1$ | $(4+1)$ | Heisenberg |
| | $\mathfrak{g}_{2(2)}$ | $G_2$ | $(1,1)$ | $H^1$ | $(2+1+2)$ | |
| | $\mathfrak{g}_{2(2)}$ | $G_2$ | $(1,1)$ | $H^2$ | $(4+1)$ | Heisenberg |
| 6 | $\mathrm{sl}(4,\mathbb{R})$ | $A_3$ | $(1,1,1)$ | $H^1+H^2+H^3$ | $(3+2+1)$ | |
| | $\mathrm{so}(3,4)$ | $B_3$ | $(1,1,1)$ | $H^3$ | $(3+3)$ | |
| | $\mathrm{so}(2,4)$ | $B_2$ | $(1,2)$ | $H^1+H^2$ | $(3+2+1)$ | |
| | $\mathrm{so}(3,\mathbb{C})$ | $B_3$ | $(2,2)$ | $H^2$ | $(4+2)$ | $H$-type |
| | $\mathfrak{g}_{2(2)}$ | $G_2$ | $(1,1)$ | $H^1+H^2$ | $(2+1+1+1+1)$ | |

PROOF. Recall that, for the gradation with the characteristic element $Z$, one has

$$\mathfrak{n} = \sum_{\alpha(Z)>0} \mathfrak{g}_\alpha .$$

It is easy to see that $\alpha(H^i)>0$ implies $\alpha(H^i+H^j)>0$. The equality can not hold since $\alpha_j(H^i)=0$ but $\alpha_j(H^j)\neq 0$. □

Now we start to make the list. Note that we omit the case $\mathfrak{n}$ is abelian, in other words, the gradations are of first kind. We just prove the case $\mathfrak{g}=\mathrm{sl}(n,\mathbb{R})$, since other cases are quite similar.

EXAMPLE. *Assume a gradation of* $\mathrm{sl}(r+1,\mathbb{R})$ *is not of first kind. Then the attached nilmanifold* $(\mathfrak{n},\langle,\rangle)$ *satisfies* $\dim\mathfrak{n}\leqslant 6$ *if and only if*

(1) $r=3$ *and* $Z=H^1+H^2$; $\dim\mathfrak{n}=5$,
(2) $r=3$ *and* $Z=H^1+H^3$; $\dim\mathfrak{n}=5$,
(3) $r=3$ *and* $Z=H^1+H^2+H^3$; $\dim\mathfrak{n}=6$,
(4) $r=2$ *and* $Z=H^1+H^2$; $\dim\mathfrak{n}=3$.

PROOF. The root system of $\mathrm{sl}(r+1,\mathbb{R})$ is of $A_r$-type and every multiplicity is one. Consider the gradation with the characteristic element $H^i$. In this case the gradation is of first kind, which will be excluded from our list. A direct calculation shows that $\dim\mathfrak{n}=i(r+1-i)$. Because of Lemma 4, we are just interested in the case $\dim\mathfrak{n}<6$, so necessarily $r<6$.

If $r=5$, by counting the roots, one can get the following.

| $Z$ | $H^1$ | $H^2$ | $H^3$ | $H^4$ | $H^5$ |
|---|---|---|---|---|---|
| $\dim\mathfrak{n}$ | 5 | 8 | 9 | 8 | 5 |

Therefore the only possible combination is $Z=H^1+H^5$, but in this case we have $\dim\mathfrak{n}=9$, which is out of our condition.

If $r = 4$, by the similar way, one has

| $Z$ | $H^1$ | $H^2$ | $H^3$ | $H^4$ |
|---|---|---|---|---|
| $\dim \mathfrak{n}$ | 4 | 6 | 6 | 4 |

Again there is only one possibility, $Z = H^1 + H^4$. It is denied by $\dim \mathfrak{n} = 7$.

If $r = 3$, all solvmanifolds attached to gradations satisfy $\dim \mathfrak{n} \leqslant 6$. The structures of $\mathfrak{n}$ can be described as follows.

| $Z$ | $\mathfrak{n}$ | $\dim \mathfrak{n}$ |
|---|---|---|
| $H^1 + H^2$ | $(\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_2+\alpha_3}) + (\mathfrak{g}_{\alpha_1+\alpha_2} + \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3})$ | 5 |
| $H^1 + H^3$ | $(\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_3} + \mathfrak{g}_{\alpha_1+\alpha_2} + \mathfrak{g}_{\alpha_2+\alpha_3}) + (\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3})$ | 5 |
| $H^1 + H^2 + H^3$ | $(\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_3}) + (\mathfrak{g}_{\alpha_1+\alpha_2} + \mathfrak{g}_{\alpha_2+\alpha_3}) + (\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3})$ | 6 |

In case $Z = H^1 + H^2$, $\mathfrak{n}$ is of two-step nilpotent. Note that the root spaces satisfy $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. Therefore, $\mathfrak{g}_{\alpha_1+\alpha_2} + \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3}$ is the center and $\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_2+\alpha_3}$ is its orthogonal complement. In case $Z = H^2 + H^3$, $\mathfrak{n}$ is also a two-step nilpotent Lie algebra, furthermore, it is a Heisenberg Lie algebra. In case $Z = H^1 + H^2 + H^3$, $\mathfrak{n}$ is of 3-step nilpotent. In this case $\mathfrak{n}$ coincides with the nilpotent part of the Iwasawa decomposition of $\mathrm{sl}(4, \mathbb{R})$. Note that we omit the case $Z = H^2 + H^3$, since it is equivalent to the case $Z = H^1 + H^2$ by a Dynkin diagram automorphism.

If $r = 2$, all solvmanifolds attached to gradations satisfy $\dim \mathfrak{n} \leqslant 6$. The structures of $\mathfrak{n}$ can be described as follows.

| $Z$ | $\mathfrak{n}$ | $\dim \mathfrak{n}$ |
|---|---|---|
| $H^1 + H^2$ | $(\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_2}) + (\mathfrak{g}_{\alpha_1+\alpha_2})$ | 3 |

It is easy to see that $\mathfrak{n}$ is a Heisenberg Lie algebra. On the other hand, $\mathfrak{n}$ also coincides with the nilpotent part of the Iwasawa decomposition of $\mathrm{sl}(3, \mathbb{R})$. □

## 7. The attached solvmanifolds up to dimension 7

In this section we show that the solvmanifolds attached to graded Lie algebras up to dimension 7 are Einstein, and determine their eigenvalue types. 7-dimensional Einstein solvmanifolds of algebraic rank one has been classified by Will [**15**], so the list presented here contains no new examples. But it seems to be good to know which of them are coming from semisimple gradations. Furthermore, 7-dimensional solvmanifolds are of particular interest because of the study of special geometric structures, such as $G_2$-structures.

THEOREM 5. *The solvmanifolds attached to simple graded Lie algebras up to dimension 7 are Einstein. Such Einstein solvmanifolds which are not Damek-Ricci spaces are listed in Table 2.*

Let $(\mathfrak{s}, \langle , \rangle)$ be the solvmanifolds attached to a simple graded Lie algebra and assume $\dim \mathfrak{s} \leqslant 7$. We have to show that it is Einstein. It has been showed that if the gradation is of second or third kind, then the attached solvmanifolds are Einstein. By looking at Table 1, the only remaining case is the fifth kind gradation of $\mathfrak{g}_{2(2)}$ with the characteristic element $Z = H^1 + H^2$. This is obviously Einstein, since $\mathfrak{n}$ coincides with the nilpotent part of the Iwasawa decomposition of $\mathfrak{g}_{2(2)}$ and hence $(\mathfrak{s}, \langle , \rangle)$ is the rank one reduction of the symmetric space $\mathrm{G}_{2(2)}/\mathrm{SO}(4)$.

The eigenvalue types of the attached solvmanifolds can be calculated in terms of the roots. We just mention the case $\mathfrak{g}_{2(2)}$.

TABLE 2. The attached solvmanifolds up to dimension 7

| $\dim \mathfrak{s}$ | $\mathfrak{g}$ | $\Delta$ | multiplicities | $Z$ | eigenvalue types |
|---|---|---|---|---|---|
| 5 | $\mathrm{so}(2,3)$ | $B_2$ | $(1,1)$ | $H^1+H^2$ | $(1<2<3<4;1,1,1,1)$ |
| 6 | $\mathrm{sl}(4,\mathbb{R})$ | $A_3$ | $(1,1,1)$ | $H^1+H^2$ | $(2<3<5;1,2,2)$ |
| | $\mathfrak{g}_{2(2)}$ | $G_2$ | $(1,1)$ | $H^1$ | $(1<2<3;2,1,2)$ |
| 7 | $\mathrm{sl}(4,\mathbb{R})$ | $A_3$ | $(1,1,1)$ | $H^1+H^2+H^3$ | $(1<2<3;3,2,1)$ |
| | $\mathrm{so}(3,4)$ | $B_3$ | $(1,1,1)$ | $H^3$ | $(1<2;3,3)$ |
| | $\mathrm{so}(2,4)$ | $B_2$ | $(1,2)$ | $H^1+H^2$ | $(1<2<3;3,2,1)$ |
| | $\mathfrak{g}_{2(2)}$ | $G_2$ | $(1,1)$ | $H^1+H^2$ | $(1<2<3<5<6<9;1,\dots,1)$ |

EXAMPLE. *Let us consider $\mathfrak{g}_{2(2)}$. The eigenvalue type of the solvmanifold attached to the gradation with $Z = H^1$ is $(1 < 2 < 3; 2, 1, 2)$. The eigenvalue type of the solvmanifold attached to the gradation with $Z = H^1 + H^2$ is $(1 < 2 < 3 < 5 < 6 < 9; 1, 1, 1, 1, 1, 1)$.*

PROOF. For the roots of $G_2$-type root system, we use the expression in [**2**], that is, $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

If $Z = H^1$, the attached nilmanifold is

$$\mathfrak{n} = (\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_1+\alpha_2}) + (\mathfrak{g}_{2\alpha_1+\alpha_2}) + (\mathfrak{g}_{3\alpha_1+\alpha_2} + \mathfrak{g}_{3\alpha_1+2\alpha_2}).$$

The mean curvature vector $H_0$ can be calculated in terms of root vectors (Lemma 1), one has

$$\begin{aligned} H_0 &= H_{\alpha_1} + H_{\alpha_1+\alpha_2} + H_{\alpha_1+\alpha_2} + H_{\alpha_1+\alpha_2} + H_{\alpha_1+\alpha_2} \\ &= 5H_{-\varepsilon_2+\varepsilon_3}. \end{aligned}$$

For $X_\alpha \in \mathfrak{g}_\alpha$, by definition of the root spaces and root vectors, one has

$$\mathrm{ad}_{H_0}(X_\alpha) = \alpha(H_0)X_\alpha = 5\langle -\varepsilon_2 + \varepsilon_3, \alpha\rangle X_\alpha.$$

Therefore the eigenvalues of $\mathrm{ad}_{H_0}$ can be calculated.

If $Z = H^1 + H^2$, the attached nilmanifold is

$$\mathfrak{n} = (\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_2}) + (\mathfrak{g}_{\alpha_1+\alpha_2}) + (\mathfrak{g}_{2\alpha_1+\alpha_2}) + (\mathfrak{g}_{3\alpha_1+\alpha_2}) + (\mathfrak{g}_{3\alpha_1+2\alpha_2}).$$

Similarly, one has $H_0 = 2H_{-\varepsilon_1-2\varepsilon_2+3\varepsilon_3}$ and thus one can calculate the eigenvalues of $\mathrm{ad}_{H_0}$. □

**Acknowledgements.** This work was partially supported by Grant-in-Aid for Young Scientists (B) 14740049, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

## References

[1] J. Berndt, F. Tricerri, L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lect. Notes in Math. **1598** (1995), Springer-Verlag Berlin Heidelberg.

[2] N. Bourbaki, *Groupes et algèbres de Lie, Chapitre IV – VI*, Éléments de Mathématique, Masson, Paris, 1981.

[3] V. Cortés, Alekseevskian spaces, *Diff. Geom. Appl.* **6** (1996), 129–168.

[4] E. Damek, F. Ricci, Harmonic analysis on solvable extensions of $H$-type groups. *J. Geom. Anal.* **2** (1992), 213–248.

[5] J. Heber, Noncompact homogeneous Einstein spaces, *Invent. Math.* **133** (1998), 279-352.

[6] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics **34**, American Mathematical Society, Providence, RI, 2001.
[7] S. Kaneyuki, On the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_{\mathrm{ev}}$ of semisimple graded Lie algebras, *J. Math. Soc. Japan* **45** (1993), 1–19.
[8] S. Kaneyuki, H. Asano, Graded Lie algebras and generalized Jordan triple systems, *Nagoya Math. J.* **112** (1988), 81–115.
[9] J. Lauret, Finding Einstein solvmanifolds by a variational method, *Math. Z.* **241** (2002), 83–99.
[10] K. Mori, Einstein metrics on Boggino-Damek-Ricci-type solvable Lie groups, *Osaka J. Math.* **39** (2002), 345–362.
[11] S. Nishikawa, Harmonic maps and negatively curved homogeneous spaces, *Geometry and topology of submanifolds X* (Eds. W. H. Chen et. al.), World Sci. Publishing, Singapore, 2000, 200–215.
[12] H. Tamaru, The local orbit types of symmetric spaces under the actions of the isotropy subgroups, *Diff. Geom. Appl.* **11** (1999), 29–38.
[13] H. Tamaru, On certain subalgebras of graded Lie algebras, *Yokohama Math. J.* **46** (1999), 127–138.
[14] H. Tamaru, Noncompact homogeneous Einstein manifolds attached to graded Lie algebras, preprint (2004).
[15] C. Will, Rank-one Einstein solvmanifolds of dimension 7, *Diff. Geom. Appl.* **19** (2003), 307–318.
[16] T. H. Wolter, Einstein metrics on solvable Lie groups, *Math. Z.* **206** (1991), 457–471.

Department of Mathematics, Hiroshima University,
1-3-1 Kagamiyama, Higashi-Hiroshima 739-8526, Japan

e-mail: tamaru@math.sci.hiroshima-u.ac.jp