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A moving frames technique and the invariant theory of Killing tensors

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Abstract. The inductive construction introduced recently to the theory of moving frames by Irina Kogan is employed to solve the problems of the determination of fundamental sets of invariants and covariants under the action of the isometry group for a vector space of Killing tensors defined in the Minkowski plane.

1. Introduction

Our ultimate goal is to employ the methods of the invariant theory of Killing tensors (ITKT) in the study of the so-called *Drach integrable potentials* [**2**]. Recall that in 1935 Drach listed ten completely integrable Hamiltonian systems defined in the Minkowski plane $\mathbb{R}^2_1$. Their complete integrability is afforded by the existence of additional first integrals which are cubic in the momenta. In recent years the ten integrable systems of Drach have received much attention from the viewpoint of the theory of superintegrable systems [**7, 11**] (see also the references therein). Thus, it has been shown [**18, 21**] that seven of the ten Drach potentials are, in fact, superintegrable, admitting in addition first integrals which are quadratic in the momenta. It is still unclear, however, *whether or not the Drach list is exhaustive.* We note that each of the ten integrable Hamiltonian systems is specified by its first integral cubic in the momenta. Moreover, the leading term of each such first integral is determined by the corresponding Killing tensor of valence three defined in the Minkowski plane. In order to answer the above question we propose to employ invariants (covariants) of the isometry group $I(\mathbb{R}^2_1)$ acting on the vector space $\mathcal{K}^3_0(\mathbb{R}^2_1)$ of the Killing tensors of valence three defined on $\mathbb{R}^2_1$ to classify the elements of $\mathcal{K}^3_0(\mathbb{R}^2_1)$, *including those that determine the first integrals cubic in the momenta of the Drach integrable potentials.* Note, the isometry group action preserves the dynamics of the ten integrable Hamiltonian systems.

The main aim of this paper is to make the first step in solving the more general problem described above, that is to determine the fundamental sets of $I(\mathbb{R}^2_1)$-invariants and covariants of the vector space $\mathcal{K}^3_0(\mathbb{R}^2_1)$, with the property that any other invariant or covariant is an analytic function of the elements of the corresponding fundamental set, which are algebraic polynomials. To solve this problem we make use of the construction of moving frames introduced recently by Kogan [**12**]

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to the theory of moving frames [**5, 6**] (see also the references therein), which has been successfully applied to the theory of invariant differential forms on jet bundles. The application of the algorithm due to Kogan for our purposes in ITKT is especailly simple, effective and greatly simplifies the procedure.

2. Invariant theory of Killing tensors

In this section we briefly review the main features of the invariant theory of Killing tensors (ITKT). Let $(M, \boldsymbol{g})$ be an m-dimensional pseudo-Riemannian manifold of constant curvture.

Definition 1. *A symmetric contravariant tensor $\boldsymbol{K}$ of valence p defined on $(M, \boldsymbol{g})$ is said to be a* generalized Killing tensor (GKT) of order n *iff*

$$[[\dots[\boldsymbol{K}, \boldsymbol{g}], \boldsymbol{g}], \dots, \boldsymbol{g}] = 0 \quad (n+1 \textit{ brackets}), \tag{1}$$

where $[\ ,\ ]$ denotes the Schouten bracket.

It follows immediately from the $\mathbb{R}$-bilinear properties of the Schouten bracket that GKTs of the same valence and order constitute a *vector space*. We observe next that such a vector space specified by the system of overdetermined PDEs (1) includes the subspace of the standard Killing tensors of valence p defined on $(M, \boldsymbol{g})$ by the system of overdetermined PDEs $[\boldsymbol{K}, \boldsymbol{g}] = 0$. Thus, the GKTs of order zero are the standard Killing tensors. The concept of a generalized Killing tensor defined on the Minkowski space $\mathbb{R}^m_1$ was introduced by Nikitin and Prilipko [**13**] (see also Appendix A of the monograph [**8**]) as generalized symmetries of the Klein-Gordon-Fock equation. The authors also derived the formulas for the dimensions of the vector spaces that GKTs constitute. More recently, these differential geometric objects were independently re-introduced in a more general setting by Eastwood [**3, 4**] within the framework of the study of overdetermined systems of PDEs by the methods of representation theory. More specifically, it has been shown that the vector space of the solutions of the overdetermined system of PDEs (1) is preserved by the action induced by $SL(m+1, \mathbb{R})$. In this view the author defines such a vector space as an irreducible representation of $\mathfrak{sl}(m+1, \mathbb{R})$ (see also [**14**]), and then derives the formula for the dimension of the vector space $\mathcal{K}^p_n(M)$ of the generalized Killing tensors of valence p and order n defined on $(M, \boldsymbol{g})$. Thus, the dimension d of the vector space $\mathcal{K}^p_n(M), n \geqslant 0, p \geqslant 1$ is determined by the *Nikitin-Prilipko-Eastwood (NPE) formula* given by:

$$d = \dim \mathcal{K}^p_n(M) = \frac{n+1}{m} \binom{p+m-1}{m-1} \binom{p+n+m}{m-1}, \tag{2}$$

where $m = \dim(M, \boldsymbol{g})$. We immediately recognize that for $n = 0$ the formula (2) reduces to the *Delong-Takeuchi-Thompson (DTT) formula* (see [**14**] and [**15**] as well as the relevant references therein for more details). In this view the elements of $\mathcal{K}^p_n(M)$ are specified by d arbitrary parameters $a_1, \dots, a_d$, where d is given by (2). The (continuous) isometry group $I(M)$ of $(M, \boldsymbol{g})$ acts on $\mathcal{K}^p_n(M)$ as an automorphism inducing at the same time the corresponding action on the parameter space Σ^p_n determined by $a_1, \dots, a_d$. The same applies to the extended space $\mathcal{K}^p_n(M) \times M$ and the corresponding extended parameter space $\Sigma^p_n \times M$ defined by $a_1, \dots, a_d, x^1, \dots, x^m$, where $x^1, \dots, x^m$ are local coordinates on M. More specifically, the isometry group action on $\Sigma_n^p \times M$ is represented by the transformation laws

$$
\begin{aligned}
\tilde{a}_1 &= \varphi^1(a_1, \ldots, a_d, g^1, \ldots, g^r), \\
&\vdots \\
\tilde{a}_d &= \varphi^d(a_1, \ldots, a_d, g^1, \ldots, g^r), \\
\tilde{x}^1 &= \psi^1(x^1, \ldots, x^m, g^1, \ldots, g^r), \\
&\vdots \\
\tilde{x}^m &= \psi^m(x^1, \ldots, x^m, g^1, \ldots, g^r),
\end{aligned} \tag{3}
$$

where $g^1, \ldots, g^r$ are local coordinates that parametrize the group $I(M)$. Note that $r = \frac{1}{2}m(m+1)$. The formulas (3) can be derived explicitly by employing the standard tensor transformation laws (see, for example, [**15**]).We observe that $d = \dim \mathcal{K}_n^p(M) > r = \dim I(M), p \neq 1$. Therefore the action of the isometry group $I(M)$ on the vector space $\mathcal{K}_n^p(M)$ is *not* transitive. Hence, it is natural to proceed with the following definitions.

DEFINITION 2. *An $I(M)$*-invariant $\mathcal{I}$ of $\mathcal{K}_n^p(M)$ *is an analytic function $F : \Sigma_n^p \to \mathbb{R}$ satisfying the condition*

$$
\mathcal{I} = F(a_1, \ldots, a_d) = F(\tilde{a}_1, \ldots, \tilde{a}_d), \tag{4}
$$

while an $I(M)$-covariant *of $\mathcal{K}_n^p(M)$, $p \geqslant 1$ is an analytic function $G : \Sigma_n^p \times M \to \mathbb{R}$ satisfying the condition*

$$
\mathcal{C} = G(a_1, \ldots, a_d, x^1, \ldots, x^m) = G(\tilde{a}_1, \ldots, \tilde{a}_d, \tilde{x}^1, \ldots, \tilde{x}^m) \tag{5}
$$

under the transformation laws (3) induced by the isometry group $I(M)$ acting on $\mathcal{K}_n^p(M) \times M$, $p \geqslant 1$, where $(x^1, \ldots, x^m)$ are local coordinates on M. The following Fundamental Theorem on Invariants of a regular Lie group action [**17**] is employed to determine the number of fundamental invariants (or covariants).

THEOREM 1. *Let G be a Lie group acting regularly on an m-dimensional manifold M with s-dimensional orbits. Then, in a neighbourhood N of each point $x_0 \in M$, there exist $m-s$ functionally independent G-invariants $\Delta_1, \ldots, \Delta_{m-s}$. Any other G-invariant $\mathcal{I}$ defined near x_0 can be locally uniquely expressed as an analytic function of the fundamental invariants through $\mathcal{I} = F(\Delta_1, \ldots, \Delta_{m-s})$.*

In what follows we present a complete solution to the problem of the determination of the $I(\mathbb{R}_1^2)$-invariants and covariants of the vector space $\mathcal{K}_0^3(\mathbb{R}_1^2)$, where $\mathbb{R}_1^2$ denotes the Minkowski plane.

3. The moving frames method and the isometry groups of flat spaces

The origins of the method of moving frames can be traced back to Cartan [**1**] (for more references, see [**19**]). It was defined originally as an equivariant map from the space of submanifolds to a bundle of frames. The recent generalization of the moving frames method due to Fels and Olver [**5, 6**] makes it possible to dispence with the bundle of frames and employ in this setting very general transformation groups, including infinite-dimensional Lie pseudo-groups. We review very briefly the underlying ideas of the generalized moving frames method (see Olver[**17**] for more details).

DEFINITION 3. *A* moving frame *is a smooth, G-equivariant map $\rho : M \to G$, where G is a r-dimensional group acting smoothly on an m-dimensional underlying manifold M.*

THEOREM 2. *A moving frame exists in a neighborhood of a point $\mathbf{x} \in M$ iff G acts freely and regularly near $\mathbf{x}$.* To construct a moving frame, one employs Cartan's *normalization method* [**1**].

THEOREM 3. *Let G act freely and regularly on M and let $K \subset M$ be a (local) cross-section to the group orbits. Given $\mathbf{x} \in M$, let $\mathbf{g} = \rho(\mathbf{x})$ be the unique group element that maps $\mathbf{x}$ to the cross-section: $\mathbf{g} \cdot \mathbf{x} = \rho(\mathbf{x}) \cdot \mathbf{x} \in K$. Then $\rho : M \to G$ is a right moving frame.* More specifically, let $\mathbf{x} = (x^1, \dots, x^m) \in M$ be local coordinates. Consider the explicit formulas for the coordinate transformations induced by the action of G: $\omega(\mathbf{g}, \mathbf{x}) = \mathbf{g} \cdot \mathbf{x}$. The right moving frame $\mathbf{g} = \rho(\mathbf{x})$ can be constructed by making use of a *coordinate cross-section*

$$K = \{x^1 = c_1, x^2 = c_2, \dots, x^r = c_r\},$$

where $c_i, i = 1, \dots, r$ are some constants and solving the corresponding *normalization equations*

$$\omega_1(\mathbf{g}, \mathbf{x}) = c_1, \quad \omega_2(\mathbf{g}, \mathbf{x}) = c_2, \quad \dots, \quad \omega_r(\mathbf{g}, \mathbf{x}) = c_r, \tag{6}$$

for the group G locally parametrized by $\mathbf{g} = (g_1, \dots, g_r)$ in terms of the local coordinates $(x^1, \dots, x^m)$. Substituting the resulting expressions for $g_1, \dots, g_r$ in terms of the local coordinates $(x^1, \dots x^m)$ into the remaining $m - r$ formulas for the transformation rules $\omega(\mathbf{g}, \mathbf{x}) = \mathbf{g} \cdot \mathbf{x}$ yields a complete set of fundamental invariants for the action of G on M.

THEOREM 4. *If $\mathbf{g} = \rho(\mathbf{x})$ is the moving frame solution to the normalization equations (6), then the functions*

$$\mathcal{I}_1(\mathbf{x}) = \omega_{r+1}(\rho(\mathbf{x}), \mathbf{x}), \dots, \mathcal{I}_{m-r}(\mathbf{x}) = \omega_m(\rho(\mathbf{x}), \mathbf{x}) \tag{7}$$

form a complete system of fundamental G-invariants. In Kogan [**12**] the method of moving frames is further developed. In particular, it is observed that if the group G (see above) factors as a product of its subgroups A and B: $G = AB$, such that $A \cap B$ is discrete, one can apply the method of moving frames in two steps. In many instances this approach drastically simplifies the procedure. Thus one first applies the moving frames method to find a set of fundamental invariants of the subroup A and then treats them as new parameters under the action of the subgroup B. To solve the problem of the determination of fundamental sets of invariants and covariants of the vector space $\mathcal{K}_0^3(\mathbb{R}_1^2)$, we employ the method developed in [**12**]. Indeed, in this case the isometry group $I(\mathbb{R}_1^2)$ is a product of two subgroups (i.e., "translations" and "rotations"). More generally, this approach appears to be very potent in applications arising in ITKT when one deals with isometry groups of flat spaces of arbitrary dimensions which have the above property.

4. Fundamental invariants and covariants of the vector space $\mathcal{K}_0^3(\mathbb{R}_1^2)$

We begin our investigation by solving the corresponding system of overdetermined PDEs (1) for $p = 3$, $n = 0$ and $M = \mathbb{R}_1^2$ with respect to the metric $\boldsymbol{g}$ of $\mathbb{R}_1^2$

given by

$$\boldsymbol{g} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{8}$$

We note that the metric of the Minkowski plane $\mathbb{R}^2_1$ of the form (8) was used in [**2**] to define the Hamiltonian functions in question. The general solution to (1) in this case takes the form

$$\begin{aligned} K^{111} &= a_1 - 3a_5 x - 3a_7 x^2 - a_{10} x^3, \\ K^{112} &= (a_3 + a_9 x + a_8 x^2) + y(a_5 + 2a_7 x + a_{10} x^2), \\ K^{122} &= (a_4 - a_9 y - a_7 y^2) + x(a_6 - 2a_8 y - a_{10} y^2), \\ K^{222} &= a_2 - 3a_6 y + 3a_8 y^2 + a_{10} y^3, \end{aligned} \tag{9}$$

where $K^{111}, K^{112}, K^{122}$ and K^{222} are the general formulas for the components of the elements of the vector space $\mathcal{K}^3_0(\mathbb{R}^2_1)$. The ten parameters (constants of integration) $a_i, i = 1, \ldots, 10$ represent the dimension $d = \dim \mathcal{K}^3_0(\mathbb{R}^2_1) = 10$, which agrees with the NPE formula (2). The infinitesimal action of the isomtetry group $I(\mathbb{R}^2_1)$ on the parameter space Σ^3_0 determined by the parameters $a_i, i = 1, \ldots, 10$ is found to be determined by the vector fields

$$\begin{aligned} \boldsymbol{U}_1 &= -3a_5\partial_{a_1} + a_9\partial_{a_3} + a_6\partial_{a_4} + 2a_7\partial_{a_5} + 2a_8\partial_{a_9} + a_{10}\partial_{a_7}, \\ \boldsymbol{U}_2 &= -3a_6\partial_{a_2} - a_9\partial_{a_4} + a_5\partial_{a_3} - 2a_8\partial_{a_6} + 2a_7\partial_{a_9} + a_{10}\partial_{a_8}, \\ \boldsymbol{V} &= -3a_1\partial_{a_1} + 3a_2\partial_{a_2} - a_3\partial_{a_3} + a_4\partial_{a_4} - 2a_5\partial_{a_5} + 2a_6\partial_{a_6} \\ &\quad -a_7\partial_{a_7} + a_8\partial_{a_8}, \end{aligned} \tag{10}$$

where $\partial_{a_i} = \partial/\partial a_i$, $i = 1, \ldots, 10$. The three vector fields given by (10) represent the infinitesimal action of the group $I(\mathbb{R}^2_1)$ on the parameter space Σ^3_0 generated by $a_i, i = 1, \ldots, 10$. Indeed, it is easy to verify that they enjoy the same commutator relations as the generators (Killing vectors)

$$\boldsymbol{X} = \partial_x, \quad \boldsymbol{Y} = \partial_y, \quad \boldsymbol{H} = x\partial_x - y\partial_y \tag{11}$$

of the Lie algebra $i(\mathbb{R}^2_1)$ of the isometry group with respect to the metric given by (8). The vector fields (10) have been derived by employing the *MST-procedure* (see [**16, 22**] for more details). This procedure utilizes the technique of Lie derivative deformations and allows one to determine the infinitesimal action of the isometry group on a vector space of Killing tensors based on the infinitesimal action of the group, *without* knowing how the group itself acts on the vector space in question. The MST-procedure can be used with much success, for example, when the parameter space in question is too large, thus precluding the determination of the group action on it explicitly (see [**22**]).

To solve the problem of the determination of fundamental sets of $I(\mathbb{R}^2_1)$-invariants and covariants of the vector space $\mathcal{K}^3_0(\mathbb{R}^2_1)$ one can attempt to solve the system of PDEs generated by the vector fields (10). Unfortuntely, the method of characteristics fails in this case (see [**9**] for more details). Employing the method of moving frames for the *whole group* $I(\mathbb{R}^2_1)$ leads to equally frustrating results (see [**20**] for more details) due to unsurmountable computational difficulties. To overcome the difficulties and actually solve the problem we employ in what follows the method developed by Kogan [**12**] within the theory of moving frames.

We first observe that the rank of the system of linear PDEs (10) is three almost everywhere. Therefore there is an open subspace of Σ_0^3 where the group acts with 3-dimensional orbits. The latter means that in view of Theorem 1 we have to derive 10 (dimenension of Σ_0^3) - 3 (dimension of the orbits) $=$ 7 fundamental invariants in the subspace of Σ_0^3 where the group acts freely and regularly with 3-dimensional orbits. Observe first that the isometry group $I(\mathbb{R}_1^2)$ is a product $I(\mathbb{R}_1^2) = TH$ of the subgroups T of translations and H of hyperbolic rotations. Moreover, $T \cap H$ is discrete. The two-dimensional subgroup T acts in $\mathbb{R}_1^2$ as follows: $\tilde{x} = x + b, \tilde{y} = y + c,\ b, c \in \mathbb{R}$. The action induces the corresponding transformation laws on the parameters $a_i, i = 1, \ldots, 10$ which we derive via the standard tensor transformation laws

$$\tilde{K}^{ijk}(\tilde{y^1}, \tilde{y^2}, \tilde{a}_1, \ldots, \tilde{a}_{10}) = K^{\alpha\beta\gamma}(y^1, y^2, a_1, \ldots, a_{10}) \frac{\partial \tilde{y}^i}{\partial y^\alpha} \frac{\partial \tilde{y}^j}{\partial y^\beta} \frac{\partial \tilde{y}^k}{\partial y^\gamma}, \tag{12}$$

$i, j, k, \alpha, \beta, \gamma = 1, 2$, where the tensor components K^{ijk} are given by (9), $y^1 = x, y^2 = y$ and the formulas (9):

$$\begin{aligned}
\tilde{a}_1 &= a_1 + 3ba_5 - 3b^2a_7 + b^3a_{10}, \\
\tilde{a}_2 &= a_2 + 3ca_6 + 3c^2a_8 - c^3a_{10}, \\
\tilde{a}_3 &= a_3 - ba_9 - ca_5 + b^2a_8 + 2bca_7 - b^2ca_{10}, \\
\tilde{a}_4 &= a_4 + ca_9 - ba_6 - c^2a_7 - 2bca_8 + bc^2a_{10}, \\
\tilde{a}_5 &= a_5 - 2ba_7 + b^2a_{10}, \\
\tilde{a}_6 &= a_6 + 2ca_8 - c^2a_{10}, \\
\tilde{a}_7 &= a_7 - ba_{10}, \\
\tilde{a}_8 &= a_8 - ca_{10}, \\
\tilde{a}_9 &= a_9 - 2ba_8 - 2ca_7 + 2bca_{10}, \\
\tilde{a}_{10} &= a_{10}.
\end{aligned} \tag{13}$$

Using (13), we first find a set of fundamental T-invariants of $\mathcal{K}_0^3(\mathbb{R}_1^2)$. Indeed, consider the cross-section $K = \{\tilde{a}_7 = \tilde{a}_8 = 0\}$. It leads to the normalization equations $b = a_7/a_{10}$ and $c = a_8/a_{10}$. Using the moving frames method, we substitute these expressions for the subgroup T parameters into the remaining eight equations (13) and thus arrive at the following fundamental T-invariants of $\mathcal{K}_0^3(\mathbb{R}_1^2)$:

$$
\begin{aligned}
I_1 &= a_1a_{10}^2 + 3a_5a_7a_{10} - 2a_7^3,\\
I_2 &= a_2a_{10}^2 + 3a_6a_8a_{10} + 2a_8^3,\\
I_3 &= a_3a_{10}^2 - a_7a_9a_{10} - a_5a_8a_{10} + 2a_7^2a_8,\\
I_4 &= a_4a_{10}^2 + a_8a_9a_{10} - a_6a_7a_{10} - 2a_7a_8^2,\\
I_5 &= a_5a_{10} - a_7^2,\\
I_6 &= a_6a_{10} + a_8^2,\\
I_7 &= a_9a_{10} - 2a_7a_8,\\
I_8 &= a_{10}.
\end{aligned}
\tag{14}
$$

Next, following the approach developed in [**12**], we determine the action of the subgroup H of (hyperbolic) rotations on the space of the eight T-invariants (14). Indeed, observing that the subgroup H acts on $\mathbb{R}_1^2$ by $\tilde{x} = xe^t$, $\tilde{y} = ye^{-t}$, where $t \in \mathbb{R}$ is the parameter of H, we use the same technique as above to determine the transformation formulas induced by the action of H on the T-invariants $I_1, \dots, I_8$. Thus, we derive that the action of H on Σ_0^3 is as follows:

$$
\begin{aligned}
\tilde{a}_1 &= a_1e^{3t}, & \tilde{a}_2 &= a_2e^{-3t},\\
\tilde{a}_3 &= a_3e^{t}, & \tilde{a}_4 &= a_4e^{-t},\\
\tilde{a}_5 &= a_5e^{2t}, & \tilde{a}_6 &= a_6e^{-2t},\\
\tilde{a}_7 &= a_7e^{t}, & \tilde{a}_8 &= a_8e^{-t},\\
\tilde{a}_9 &= a_9, & \tilde{a}_{10} &= a_{10}.
\end{aligned}
\tag{15}
$$

Therefore, taking into account (14) and (15), we get

$$
\begin{aligned}
\tilde{I}_1 &= e^{3t}I_1, & \tilde{I}_2 &= e^{-3t}I_2,\\
\tilde{I}_3 &= e^{t}I_3, & \tilde{I}_4 &= e^{-t}I_4,\\
\tilde{I}_5 &= e^{2t}I_5, & \tilde{I}_6 &= e^{-2t}I_6\\
\tilde{I}_7 &= I_7, & \tilde{I}_8 &= I_8.
\end{aligned}
\tag{16}
$$

We immediately observe that I_7 and I_8 are fundamental $I(\mathbb{R}_1^2)$-invariants of $\mathcal{K}_0^3(\mathbb{R}_1^2)$, while $I_1, \dots, I_6$ are *conformal* $I(\mathbb{R}_1^2)$-invariants of $\mathcal{K}_0^3(\mathbb{R}_1^2)$ (see [**15**] for more details). Moreover, it easily follows that

Proposition 1. *The functions*

$$Z_i := \operatorname{sign} I_i, \quad i = 1, \dots, 6$$

are discrete $I(\mathbb{R}_1^2)$*-invariants of* $\mathcal{K}_0^3(\mathbb{R}_1^2)$*, where* $I_1, \dots, I_6$ *are given by (14).* To determine the remaining five continuous invariants, we employ the method of moving frames again, taking into account the action (16) of the subgroup H on $I_1, \dots, I_8$. Indeed, we note that no cross-section of the form $K = \{\tilde{I}_i = 0\}, i = 1, \dots, 6$ intersects the orbits transversally. Hence, we choose the cross-section $K = \{\tilde{I}_3 = c\}$, $c =$

const, $c \neq 0$, leading to the moving frames map $e^t = c/I_3$. Substituting the latter expression into the remaining five formulas (16), after some algebra we arrive at the following result.

THEOREM 5. *Any algebraic $I(\mathbb{R}^2_1)$-invariant $\mathcal{I}$ of $\mathcal{K}^3_0(\mathbb{R}^2_1)$ $\mathcal{I}$ defined over the open submanifold of Σ^3_0 where the isometry group $I(\mathbb{R}^2_1)$ acts freely and regularly with three-dimensional orbits can be locally uniquely expressed as an analytic function*

$$\mathcal{I} = F(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7),$$

where the fundamental $I(\mathbb{R}^2_1)$-invariants Δ_j, $j = 1, \ldots, 7$ are given by

$$\begin{array}{lllllllll} \Delta_1 & = & I_8, & \Delta_2 & = & I_7, & \Delta_3 & = & I_5 I_6, \\ \Delta_4 & = & I_1 I_2, & \Delta_5 & = & I_3 I_4, & \Delta_6 & = & I_1 I_4^3, \\ \Delta_7 & = & I_2 I_3^3, & & & & & & \end{array} \tag{17}$$

where $I_1, \ldots, I_8$ are given by (14).

We can employ the same technique to determine the action of $I(\mathbb{R}^2_1)$ on the extended vector space $\mathcal{K}^3_0(\mathbb{R}^2_1) \times \mathbb{R}^2_1$, thus finding a set of *fundamental $I(\mathbb{R}^2_1)$-covariants*. This leads to the following result.

THEOREM 6. *Let $\mathcal{K}^3_0(\mathbb{R}^2_1) \times \mathbb{R}^2_1$ be the extended vector space of $\mathcal{K}^3_0(\mathbb{R}^2_1)$. Any algebraic $I(\mathbb{R}^2_1)$-covariant $\mathcal{C}$ defined over the open submanifold of $\Sigma^3_0 \times \mathbb{R}^2_1$ where the isometry group $I(\mathbb{R}^2_1)$ acts freely and regularly with three-dimensional orbits can be locally uniquely expressed as an analytic function*

$$\mathcal{C} = F(\Delta^c_1, \Delta^c_2, \Delta^c_3, \Delta^c_4, \Delta^c_5, \Delta^c_6, \Delta^c_7, \Delta^c_8, \Delta^c_9),$$

where the fundamental $I(\mathbb{R}^2_1)$-covariants Δ^c_i, $i = 1, \ldots, 9$ are given by

$$\begin{array}{lll} \Delta^c_1 & = & a_9 + 2(a_8 x + a_7 y + a_{10} xy), \\ \Delta^c_2 & = & (a_1 - 3a_5 x - 3a_7 x^2 - a_{10} x^3)(a_2 - 3a_6 y + 3a_8 y^2 + a_{10} y^3), \\ \Delta^c_3 & = & (a_3 + a_9 x + a_5 y + a_8 x^2 + 2a_7 xy + a_{10} x^2 y) \\ & & \times (a_4 - a_9 y + a_6 x - a_7 y^2 - 2a_8 xy - a_{10} xy^2), \\ \Delta^c_4 & = & (a_5 + 2a_7 x + a_{10} x^2)(a_6 - 2a_8 y - a_{10} y^2), \\ \Delta^c_5 & = & (a_7 + a_{10} x)(a_8 + a_{10} y), \\ \Delta^c_6 & = & (a_1 - 3a_5 x - 3a_7 x^2 - a_{10} x^3) \\ & & \times (-a_4 + a_9 y - a_6 x + a_7 y^2 + 2a_8 xy + a_{10} xy^2)^3, \\ \Delta^c_7 & = & (a_2 - 3a_6 y + 3a_8 y^2 + a_{10} y^3) \\ & & \times (-a_3 - a_9 x - a_5 y - a_8 x^2 - 2a_7 xy - a_{10} x^2 y)^3, \\ \Delta^c_8 & = & (a_5 + 2a_7 x + a_{10} x^2)(a_8 + a_{10} y)^2, \\ \Delta^c_9 & = & (a_6 - 2a_8 y - a_{10} y^2)(a_7 + a_{10} x)^2. \end{array} \tag{18}$$

and (x, y) are the local coordinates on $\mathbb{R}^2_1$ with respect to the metric (8). We remark that the fundamental $I(\mathbb{R}^2_1)$-invariants of $\mathcal{K}^3_0(\mathbb{R}^2_1)$ (17) are *syzygies* for the fundamental $I(\mathbb{R}_1^2)$-covariants (18). Indeed, it is easy to check that, for example, $\Delta_2 = \Delta_1\Delta_1^c - 2\Delta_5^c$.

The results of Theorems 5 and 6 constitute an analogue of the corresponding result presented in [**10**] for the vector space $\mathcal{K}_0^3(\mathbb{R}^2)$, where $\mathbb{R}^2$ denotes the Euclidean plane. We note, however, that the latter result was obtained by solving a system of PDEs analogous to (10) by the method of undetermined coefficients with the aid of MAPLE. In this work, based on the moving frames technique introduced in [**12**] we have solved a very similar problem in what appears to be a much more efficient way, in particular, - without the aid of a computer algebra system.

Following the idea exhibited in [**10**], we observe that the elements of $\mathcal{K}_0^3(\mathbb{R}_1^2)$ can be expressed by the general formula

$$\begin{aligned}\boldsymbol{K}_0^3 &= a^{ijk}\boldsymbol{X}_i \odot \boldsymbol{X}_j \odot \boldsymbol{X}_k + 3b^{ij}\boldsymbol{X}_i \odot \boldsymbol{X}_j \odot \boldsymbol{H} + 3c^i\boldsymbol{X}_i \odot \boldsymbol{H} \odot \boldsymbol{H} \\ &\quad + e\boldsymbol{H} \odot \boldsymbol{H} \odot \boldsymbol{H} \\ &= a^{ijk}\boldsymbol{X}_i \odot \boldsymbol{X}_j \odot \boldsymbol{X}_k + (3b^{ij}\boldsymbol{X}_i \odot \boldsymbol{X}_j + \boldsymbol{K}_0^1 \odot \boldsymbol{H}) \odot \boldsymbol{H} \\ &= a^{ijk}\boldsymbol{X}_i \odot \boldsymbol{X}_j \odot \boldsymbol{X}_k + \boldsymbol{K}_0^2 \odot \boldsymbol{H}, \quad i,j,k = 1,2,\end{aligned} \tag{19}$$

where $\boldsymbol{X}_1 = \boldsymbol{X}, \boldsymbol{X}_2 = \boldsymbol{Y}$ and $\boldsymbol{H}$ are given by (11), while $\boldsymbol{K}_0^1$ and $\boldsymbol{K}_0^2$ represent the elements of the vector spaces $\mathcal{K}_0^1(\mathbb{R}_1^2)$ and $\mathcal{K}_0^2(\mathbb{R}_1^2)$ respectively. Furthermore, the components of the tensors a^{ijk}, b^{ij}, c^i and e are given by $a^{111} = a_1$, $a^{112} = a_3$, $a^{221} = a_4$, $a^{222} = a_2$, $b^{11} = -a_5$, $b^{12} = b^{21} = a_9$, $b^{22} = -a_6$, $c^1 = -a_7$, $c^2 = a_8$ and $e = -a_{10}$ in terms of the parameters $a_i, i = 1, \ldots, 10$ that appear in (9). Taking into account the formulas (17), (18) and (19), we arrive at the following two results.

COROLLARY 1. *Any algebraic $I(\mathbb{R}_1^2)$-invariant $\mathcal{I}$ of the vector space $\mathcal{K}_0^2(\mathbb{R}_1^2)$ defined over the open submanifold of the corresponding parameter space Σ_0^2 where the isometry group $I(\mathbb{R}_1^2)$ acts freely and regularly with three-dimensional orbits can be locally uniquely expressed as an analytic function $\mathcal{I} = F(\Delta_1, \Delta_2, \Delta_3)$, where the fundamental $I(\mathbb{R}_1^2)$-invariants Δ_i, $i = 1, \ldots, 3$ are given by the formulas (17).*

COROLLARY 2. *Let $\mathcal{K}_0^2(\mathbb{R}_1^2) \times \mathbb{R}_1^2$ be the extended vector space of $\mathcal{K}_0^2(\mathbb{R}_1^2)$. Any algebraic $I(\mathbb{R}_1^2)$-covariant $\mathcal{C}$ of $\mathcal{K}_0^3(\mathbb{R}_1^2)$ defined over the open submanifold of $\Sigma_0^2 \times \mathbb{R}_1^2$ where the isometry group $I(\mathbb{R}_1^2)$ acts freely and regularly with three-dimensional orbits can be locally uniquely expressed as an analytic function $\mathcal{C} = F(\Delta_1^c, \Delta_4^c, \Delta_5^c, \Delta_8^c, \Delta_9^c)$, where the fundamental $I(\mathbb{R}_1^2)$-covariants $\Delta_1^c, \Delta_4^c, \Delta_5^c, \Delta_8^c$ and Δ_9^c are given by the formulas (18).* We note finally that the action of the isometry group $I(\mathbb{R}_1^2)$ in Σ_0^3 is not free globally due to the existence of isotropy subgroups (see (19)).

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