DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 165 – 177

Stable constant mean curvature hypersurfaces inside convex domains

César Rosales

Abstract. This paper is a survey about ***stable free boundary hypersurfaces***, i.e., second order minima of area under a volume constraint, inside smooth Euclidean domains. We review analytic, geometric and topological properties of these hypersurfaces, and above all, we gather the most relevant classification results of stable hypersurfaces in certain convex domains which are invariant under a large group of isometries. Then, we give a brief description of our main result in [**RRo**], where we add to the literature the characterization of stable hypersurfaces inside ***solid convex cones***. We finish by giving applications of stability to the ***partitioning problem*** inside a convex domain.

Introduction

In this report we review some facts about ***stable hypersurfaces*** inside a smooth Euclidean domain: they are defined as minima, up to second order, of the area functional associated to any variation inside the domain leaving invariant the volume ***separated*** by the hypersurface.

Stability is usually studied by using methods of the Calculus of Variations and geometric properties of the ambient domain. In Section 1 we recall the first and second variation of area and volume, in order to give a geometric and analytical description of stable hypersurfaces. In particular, we deduce the well-known result that any stable hypersurface has ***constant mean curvature*** and meets the boundary of the ambient domain ***orthogonally***. Moreover, such a hypersurface has associated a quadratic form, called the ***index form*** (see (3)), with at most index one over smooth functions defined on the hypersurface.

The fact that the boundary term appearing in the index form is non-negative when the domain has locally convex boundary suggests that the notion of stability is more restrictive for convex domains. In fact, by inserting suitable test functions in the index form, we can deduce interesting geometrical and topological information from the stability condition. With this idea, we recall in Section 1 some known restrictions on the topology of a stable hypersurface within a convex domain.

The classification of stable hypersurfaces inside a given domain is a difficult and interesting global problem in Riemannian Geometry. In Section 2 we gather the most relevant results related to this question, obtained for certain domains which are

2000 *Mathematics Subject Classification.* 53C20, 49Q20.
Key words and phrases. Stable hypersurfaces, isoperimetric hypersurfaces, constant mean curvature.
This paper is in final form and no version of it will be published elswhere.

invariant under a large group of Euclidean isometries, such as half-spaces, slabs and round balls. We also give, in Section 3, a scheme of the proof of a result established jointly with Ritoré [**RRo**] in which we characterize stable hypersurfaces inside a ***solid convex cone***: they are round spheres contained in the closure of the cone, or half-spheres centered and lying over a flat piece of the boundary of the cone, or spherical caps centered at the vertex. The method we used to prove this result was introduced by Barbosa and do Carmo [**BdC**] to show that round spheres are the only immersed, compact stable hypersurfaces in $\mathbb{R}^{n+1}$, and was adapted later by Morgan and Ritoré [**MR**] to identify bounded stable hypersurfaces with a small singular set contained in certain cones over compact submanifolds of the unit sphere.

Section 4 of the paper is devoted to the ***partitioning problem*** inside a smooth convex domain. It consists of finding those hypersurfaces included in the closure of the domain that ***globally minimize*** the area under the restriction of the volume they enclose. The first questions taken into consideration associated to this problem were related to the existence and regularity of the solutions, called ***isoperimetric hypersurfaces***. In the light of standard results in Geometric Measure Theory (see [**M**], [**GMT**] and [**Gr**]), inside a smooth bounded domain, isoperimetric hypersurfaces exist and they are smooth, up to a closed set of singularities with high Hausdorff codimension. Moreover, when the ambient domain is invariant under a large group of isometries, the solutions are smooth and bounded. We must remark that, in spite of the last advances, the complete description of isoperimetric hypersurfaces in a given domain has been achieved only in a few number of cases.

It is clear that isoperimetric hypersurfaces are also stable. This relation allows us to give, after a comparison among the areas of the different candidates, the complete solution to the partitioning problem in half-spaces, slabs, round balls and solid convex cones. The case of a convex cone requires some additional arguments to prove the existence of isoperimetric hypersurfaces and to deal with the possible non-empty set of singularities.

The partitioning problem and the study of stable hypersurfaces are object of an intensive study. A beautiful report on these topics containing recent progress and open questions is the one by Ros [**R2**], see also [**RR**].

1. The stability condition. Geometric and topological consequences

From now on, we denote by D a domain (connected, open set) of $\mathbb{R}^{n+1}$ with C^∞ boundary ∂D. Let $\Sigma \subset \overline{D}$ be a smooth, compact hypersurface with interior $\mathrm{int}(\Sigma)$ and boundary $\partial\Sigma$. We assume that the following conditions hold

(a) $\mathrm{int}(\Sigma) \subset D$,
(b) $\partial\Sigma \subset \partial D$,
(c) $\mathrm{int}(\Sigma)$ induces a partition of D into two open sets Ω_1 and Ω_2 such that $\overline{\partial\Omega_i \cap D} = \Sigma$. We suppose that one of these sets is bounded and we call it Ω.

In the situation above, we denote by $\mathrm{vol}(\Omega)$ the Lebesgue measure of Ω in $\mathbb{R}^{n+1}$, and by $A(\Sigma)$ the Riemannian measure of Σ induced by the Euclidean metric of $\mathbb{R}^{n+1}$. It is well-known that $A(\Sigma)$ coincides with $\mathcal{H}^n(\Sigma) = n$-dimensional Hausdorff measure of Σ in $\mathbb{R}^{n+1}$, so that we must think of $A(\Sigma)$ as the "area" of Σ.

In this paper we are interested in those hypersurfaces that ***locally minimize*** the area under the restriction of the volume they enclose. In order to precise this idea we introduce the next definition

Definition 1. *A* ***volume preserving variation*** *of Σ in D is a smooth map $\varphi : \Sigma \times (-\varepsilon, \varepsilon) \to \overline{D}$, such that*

(i) *$\Sigma_t = \varphi(\Sigma \times \{t\})$ is a smooth hypersurface satisfying (a), (b) and (c),*
(ii) *$\Sigma_0 = \Sigma$,*
(iii) *$vol(\Omega_t) = vol(\Omega)$ for any $t \in (-\varepsilon, \varepsilon)$, where $\{\Omega_t\}_t$ is a family of bounded open sets in D such that $\overline{\partial\Omega_t \cap D} = \Sigma_t$ and $\{\chi_{\Omega_t}\} \to \chi_\Omega$ in $L^1(D)$ as $t \to 0$.*

Geometrically, a volume preserving variation is a local deformation of Σ in such a way that the volume enclosed by the hypersurfaces of the deformation remains constant. Note that we do not impose that the variation leaves invariant the boundary $\partial\Sigma$.

Now, we can give the definition of stable hypersurface as was introduced by Barbosa and do Carmo [**BdC**], and by Ros and Vergasta [**RV**].

Definition 2. *A hypersurface Σ is* ***stable*** *in D if the area functional $A(t) = A(\Sigma_t)$ associated to any volume preserving variation of Σ in D satisfies*

$$A'(0) = 0 \qquad \text{and} \qquad A''(0) \geqslant 0.$$

Remark 1. *The terminology* ***stable hypersurface*** *is not used consistently in the literature. In the classical theory of* ***minimal surfaces****, a stable hypersurface is a second order minimum of area for variations* ***preserving the boundary*** *of the hypersurface. As we do not impose volume preserving variations to fix $\partial\Sigma$, our notion of stability in Definition 2 is usually referred to as* ***free boundary stability****. In these notes we will only consider this notion of stability.*

Now, we shall use the first and second variation formula for area and volume in order to give an analytical characterization of stability.

Let $\varphi : \Sigma \times (-\varepsilon, \varepsilon) \to \overline{D}$ be any variation of Σ in D. We denote by X the vector field on Σ defined by $X_p = \frac{d}{dt}\big|_{t=0} \varphi(p, t)$. Call N to the unit normal vector to Σ pointing into Ω, and η to the inner normal vector to $\partial\Sigma$ in Σ. Consider the functions $A(t) = A(\Sigma_t)$ and $V(t) = \mathrm{vol}(\Omega_t)$. It is well-known ([**RV**]) that

$$A'(0) = -n \int_\Sigma H u \, d\mathcal{H}^n - \int_{\partial\Sigma} \langle X, \eta \rangle \, d\mathcal{H}^{n-1}, \tag{1}$$

$$V'(0) = -\int_\Sigma u \, d\mathcal{H}^n, \tag{2}$$

where $u = \langle X, N \rangle$ and H is the mean curvature of Σ with respect to N (defined as the arithmetic mean of the principal curvatures). If $\partial\Sigma = \emptyset$ we adopt the convention that the integrals over this set are all equal to 0. By using (1) and (2) with appropriate variations of Σ, one can show that if Σ is stable in D, then H is constant and Σ meets ∂D orthogonally.

When the variation φ preserves volume, the derivative $A''(0)$ is given by ([**RS**])

$$A''(0) = Q(u, u) = \int_\Sigma \{|\nabla_\Sigma u|^2 - |\sigma|^2 u^2\} \, d\mathcal{H}^n - \int_{\partial\Sigma} \mathrm{II}(N, N) \, u^2 \, d\mathcal{H}^{n-1}, \tag{3}$$

where $\nabla_\Sigma u$ is the gradient of u relative to Σ, $|\sigma|^2$ is the squared norm of the second fundamental form of Σ with respect to N, and II is the second fundamental form of ∂D with respect to the inner normal vector. The expression (3) defines a quadratic form, called the ***index form*** of Σ. We see that it involves not only analytical quantities but also other terms related to the geometry of Σ and ∂D.

Let u be a smooth mean zero function over Σ. By using the arguments in [**BdC**, Lemma 2.4] we can construct a volume preserving variation of Σ in D with associated vector field $X = uN$. Thus, if Σ is stable, then we get $Q(u,u) \geqslant 0$. We have obtained the following

PROPOSITION 1. *Let $\Sigma \subset \overline{D}$ be a smooth compact hypersurface. Then, Σ is stable in D if and only if*

(i) *Σ has constant mean curvature with respect to the normal pointing into Ω.*
(ii) *Σ meets ∂D orthogonally.*
(iii) *The index form defined in (3) satisfies $Q(u,u) \geqslant 0$ for any smooth mean zero function over Σ.*

The result above establishes a relation between stable hypersurfaces and the geometric theory of ***constant mean curvature hypersurfaces***; in fact, the previous arguments show that constant mean curvature hypersurfaces appear as critical points of area for volume preserving variations. It follows by the well-known Alexandrov uniqueness theorem that any stable, embedded hypersurface with empty boundary coincides with a round sphere in $\mathbb{R}^{n+1}$.

Inequality $Q(u,u) \geqslant 0$ provides interesting geometrical and topological information when a suitable function u is inserted. As we shall see, the best test functions are those involving geometric information about the ambient domain.

Note that the boundary term in the index form (3) is non-negative when D is ***convex***. This indicates us that the notion of stability is more restrictive for convex domains. As a matter of fact, some topological consequences of stability are deduced in this setting. For example, we have

LEMMA 1. *Any stable hypersurface Σ inside a convex domain D is connected or flat. Moreover, if D is strictly convex then Σ is connected.*

The proof of Lemma 1 is easy. In fact, if there were two different components of Σ, then we could consider a locally constant nowhere vanishing function u with mean zero over Σ and such that $Q(u,u) < 0$ unless $|\sigma|^2$ and $\mathrm{II}(N,N)$ vanish. Geometrically, this deformation corresponds to contract one component and expand the other one, so that the volume enclosed is preserved while the area decreases.

The next example illustrates that it is possible to find disconnected stable hypersurfaces inside a convex domain.

EXAMPLE. *Let D be a solid right cylinder in $\mathbb{R}^{n+1}$. It is clear that the hypersurface $\Sigma \subset \overline{D}$ consisting of two parallel n-dimensional discs meeting ∂D orthogonally is stable since $|\sigma|^2 = 0$ and $II(N,N) = 0$.*

The following result, due to Sternberg and Zumbrun [**SZ**], shows that the example above is essentially the unique situation in which disconnected stable hypersurfaces appear.

THEOREM 1 ([**SZ**, Theorem 3.1]). *If Σ is a disconnected stable hypersurface inside a convex domain D, then the part of D lying between any two components of Σ is a right cylinder.*

In general, it is difficult to obtain more precise information about the topology of a stable hypersurface. However, Ros and Vergasta [**RV**] obtained some restrictions on the genus and the number of boundary components of a stable surface inside a convex domain of $\mathbb{R}^3$. Their arguments use the existence of conformal spherical maps on certain compact Riemann surfaces in order to construct a suitable test function.

Theorem 2 ([**RV**, Theorem 5]). *Let Σ be a connected, stable surface with non-empty boundary inside a convex domain of $\mathbb{R}^3$. Then, the genus g and the number r of boundary components of Σ satisfy $g \leqslant 3$ and $r \leqslant 3$. More precisely, the only possible values for g and r are*

(i) $g \in \{0, 1\}$ *and* $r \in \{1, 2, 3\}$.
(ii) $g \in \{2, 3\}$ *and* $r = 1$.

In the next section we shall see that under additional assumptions on the convex domain D we can exactly determine the topology of any stable $\Sigma \subset \overline{D}$.

2. Classification of stable hypersurfaces inside smooth convex domains

The complete description of stable hypersurfaces inside a given convex domain is a difficult and interesting problem in Riemannian Geometry. It has been solved only in some specific situations, and even the apparently simple case of a round ball in $\mathbb{R}^{n+1}$ remains open. In this section we gather the most relevant results in relation to this question.

2.1. The whole space and half-spaces. In 1984, Barbosa and do Carmo obtained the first classification result of stable hypersurfaces. They proved the following

Theorem 3 ([**BdC**, Theorem 1.3]). *Let Σ be an immersed, compact hypersurface in $\mathbb{R}^{n+1}$. If Σ is stable, then Σ is a round sphere.*

As any stable hypersurface in $\mathbb{R}^{n+1}$ has constant mean curvature, the result above trivially follows from Alexandrov uniqueness theorem when Σ is embedded. In the general case we cannot appeal to this theorem and stable immersed tori could appear, see [**W**]. The argument used by Barbosa and do Carmo consists of inserting in the index form (3) the test function $u = 1 + H\,\langle X, N\rangle$, where H is the mean curvature of Σ with respect to a unit normal vector field N, and X is the position vector field in $\mathbb{R}^{n+1}$ given by $X(p) = p$. It was shown by Wente [**W2**] that the function u appears when one considers first a contraction of Σ by parallel hypersurfaces and then applies a dilation to restore the enclosed volume. As u is a smooth mean zero function on Σ, the stability condition in Proposition 1 (iii) implies $Q(u, u) \geqslant 0$. An explicit calculation of $Q(u, u)$ shows that inequality $Q(u, u) \geqslant 0$ is equivalent to $|\sigma|^2 = nH^2$. This means that any stable Σ is a totally umbilical hypersurface of $\mathbb{R}^{n+1}$. On the other hand, as Σ is compact with empty boundary, we get by using Lemma 1 that Σ is connected. We conclude that Σ coincides with a round sphere of $\mathbb{R}^{n+1}$.

Now, let us consider a half-space $\mathbb{H}$ of $\mathbb{R}^{n+1}$ and a stable hypersurface Σ inside $\mathbb{H}$. If $\partial\Sigma = \emptyset$, then Σ must be a round sphere by Theorem 3. On the other hand, if $\partial\Sigma \neq \emptyset$, then we can use again the test function given by Barbosa and do Carmo, or a reflection argument with respect to $\partial\mathbb{H}$, to deduce

Theorem 4. *Let Σ be an immersed, compact hypersurface in a half-space $\mathbb{H}$. If Σ is stable, then Σ is a round sphere contained in $\mathbb{H}$ or a half-sphere centered at $\partial\mathbb{H}$.*

FIGURE 1. Stable hypersurfaces in a half-space: spheres and half-spheres.

2.2. Stable hypersurfaces inside a slab. Another interesting situation to consider is the case of a domain $S \subset \mathbb{R}^{n+1}$ bounded by two parallel hyperplanes. It was first studied in dimension three by Athanassenas [**At**], who showed that the only stable surfaces in S are round spheres, half-spheres centered at one of the planes contained in ∂S, and some circular cylinders meeting ∂S orthogonally. In higher dimension, the problem was treated by Pedrosa and Ritoré [**PR**]. Their first result, obtained as an application of Alexandrov reflection principle, is the following

LEMMA 2. *Let Σ be a compact, embedded, constant mean curvature hypersurface in a slab $S \subset \mathbb{R}^{n+1}$ such that Σ meets ∂S orthogonally. Then, Σ is rotationally symmetric with respect to a line perpendicular to ∂S.*

This lemma implies that any stable $\Sigma \subset S$ belongs to the well-known family of constant mean curvature hypersurfaces of revolution in $\mathbb{R}^{n+1}$. The complete description of such hypersurfaces was given by Delaunay [**D**] in 1841 (they are depicted in Figure 2). It follows by the orthogonality condition that Σ must be a half-sphere, or a cylinder about a line perpendicular to ∂S, or a certain closed piece of an unduloid.

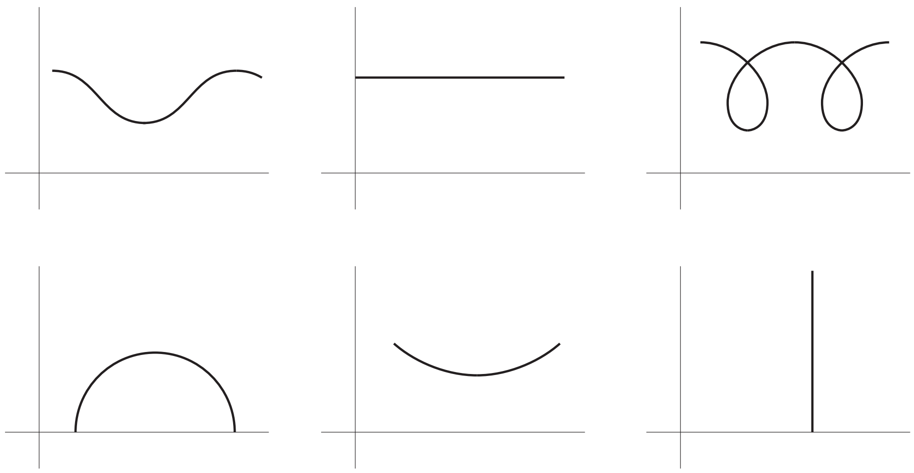

FIGURE 2. The different possibilities for the generating curve of a hypersurface of revolution with constant mean curvature in $\mathbb{R}^{n+1}$: unduloid, cylinder, nodoid, sphere, catenoid and hyperplane.

Let Σ be a closed piece of an unduloid meeting ∂S orthogonally. For this hypersurface Pedrosa and Ritoré [**PR**, Proposition 3.2] consider a suitable mean zero function depending on the period of the unduloid, which gives instability of Σ in dimension $n \leqslant 7$ when it is inserted in the index form (3). As a consequence they obtain the next result, analogous to the one by Athanassenas.

THEOREM 5 ([**PR**, Corollary 5.4]). *Let Σ be a stable, embedded hypersurface in a slab S of $\mathbb{R}^{n+1}$, $n \leqslant 7$. Then, Σ is either*

(i) *A round sphere contained in S, or*
(ii) *A half-sphere attached to one of the hyperplanes contained in ∂S, or*
(iii) *The intersection with the slab of a solid cylinder of suitable radius, whose axis is perpendicular to ∂S.*

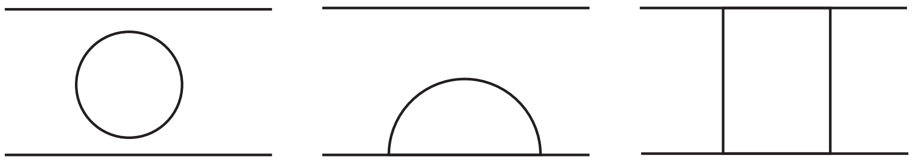

FIGURE 3. Stable hypersurfaces in a slab of $\mathbb{R}^{n+1}$, $n \leqslant 7$: spheres, half-spheres, and cylinders perpendicular to the boundary of the slab.

As it is pointed out in [**PR**] not all cylinders are stable. Stability depends on the distance between the two boundary hyperplanes and on the radius of the cylinder.

In higher dimension the previous theorem does not hold. In fact, in [**PR**, Proposition 5.3] it is shown that for any $n \geqslant 9$ some stable pieces of unduloid appear. The discussion about the stability of unduloids remains open in the case $n = 8$.

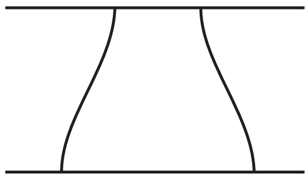

FIGURE 4. Stable pieces of unduloid inside a slab of $\mathbb{R}^{n+1}$ appear for any $n \geqslant 9$.

2.3. Stable hypersurfaces inside a round ball. This case was first treated by Nitsche [**N**], who showed that the only stable disc-type surfaces inside a ball of $\mathbb{R}^3$ are the totally geodesic discs and the spherical caps. The problem was also considered by Ros and Vergasta [**RV**] in dimension three. More recently, Sternberg and Zumbrun [**SZ**] have extended to arbitrary dimension some of the results in [**RV**].

The main stability result for round balls obtained by Ros and Vergasta is the following

THEOREM 6 ([**RV**, Theorem 11]). *Let Σ be a stable hypersurface inside a round ball B of the Euclidean three-space. Then Σ is either*

(i) *A round sphere contained in B, or*
(ii) *A spherical cap meeting ∂B orthogonally, or*
(iii) *A flat disc passing through the center of B, or*
(iv) *A surface of genus one and at most two boundary components.*

An example of a constant mean curvature surface $\Sigma \subset \overline{B}$ of genus one and two boundary components contained in ∂B is a piece of a catenoid, see Figure 2. In

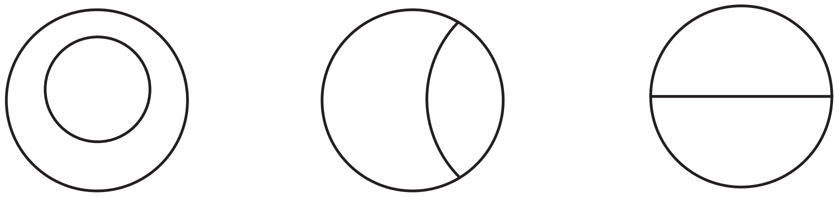

FIGURE 5. Some stable hypersurfaces inside a round ball: spheres, spherical caps, and flat discs. The complete classification is still an open question.

fact, for any $H > 0$ there is a piece of an unduloid of mean curvature H meeting ∂B orthogonally. However, Ros [**R**, Theorem 4] proved that neither catenoid nor unduloid pieces are stable. His argument uses the Courant's nodal domain theorem [**Ch**, p. 19] and a test function associated to the infinitesimal vector field of rotations about a line. As a consequence, we have

PROPOSITION 2. *Let Σ be a compact, embedded piece of a Delaunay hypersurface inside a round ball $B \subset \mathbb{R}^{n+1}$ meeting ∂B orthogonally. If Σ is stable, then Σ is either a flat disc or a spherical cap.*

At the present moment not much is known about a stable hypersurface $\Sigma \subset \overline{B}$ without the additional assumption of an axis of revolution. The following result provides information about the sign of the support function $u = \langle X, N \rangle$, where $X(p) = p - p_0$ is the position vector field with respect to the center p_0 of the ball, and N is a unit normal vector field along Σ. Recall that Σ is said to be a ***graph over*** ∂B if u does not change sign on Σ.

THEOREM 7 ([**RV**, Theorem 8], [**SZ**, Theorem 3.5]). *Let Σ be a stable hypersurface with $\partial\Sigma \neq \emptyset$ inside a round ball $B \subset \mathbb{R}^{n+1}$. Suppose that the area A of Σ and the $(n-1)$-Hausdorff measure L of $\partial\Sigma$ satisfy the relation $L \geqslant nA$. Then, Σ is a graph over ∂B.*

The proof is based on this reasoning: if there were two domains Σ^+ and Σ^- of Σ where u is signed, then we could construct a non-trivial mean zero function v on Σ such that $Q(v, v) < 0$, a contradiction with the stability condition in Proposition 1 (iii).

3. Stable hypersurfaces inside solid convex cones

In this section we give a brief description of a result obtained jointly with Ritoré in [**RRo**], where we classify stable hypersurfaces in a solid convex cone. The presence of a singular point at the vertex of the cone leads us to consider hypersurfaces with a possible non-empty set of singularities. Let us precise the situation.

We denote by C a ***solid cone*** of $\mathbb{R}^{n+1}$, that is, a set of the form $\{t\nu : t > 0,\, \nu \in U\}$, for some domain U with smooth non-empty boundary of the sphere $\mathbb{S}^n$. The cone C is open, unbounded, and has a singularity at the vertex unless it coincides with a half-space. We remark that we allow a hypersurface $\Sigma \subset \overline{C}$ to contain the vertex of the cone. Under these conditions we have proved

THEOREM 8 ([**RRo**, Theorem 4.9]). *Let Σ be a stable hypersurface inside a solid convex cone C of $\mathbb{R}^{n+1}$ $(n \geqslant 2)$. Then Σ is either*

(i) *A round sphere contained in C, or*

(ii) *A half-sphere centered and lying over a flat piece of ∂C, or*
(iii) *A spherical cap centered at the vertex of the cone.*

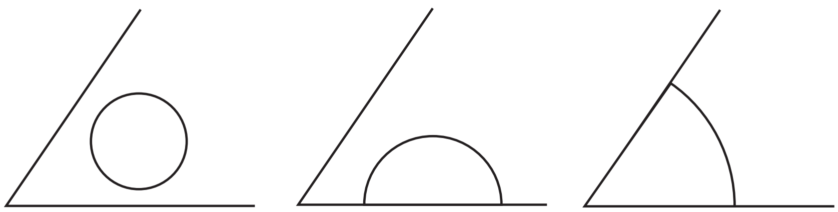

FIGURE 6. Stable hypersurfaces in a convex cone: spheres, half-spheres and spherical caps centered at the vertex.

PROOF. We will only give a sketch of the proof. Call N to the unit normal vector along Σ pointing into the enclosed domain Ω. In order to use the stability condition we must construct an appropriate volume preserving variation of Σ inside the cone. We follow the same idea employed by Barbosa and do Carmo in Theorem 3. For any t in a small neighborhood of the origin, we take the parallel to Σ at distance t in the direction of the normal vector N. Then, as the cone is invariant under dilations centered at the vertex, we can apply a dilation of ratio $s(t)$ to restore the enclosed volume without leaving the cone. This procedure gives a volume preserving variation $\{\Sigma_t\}_t$ of Σ inside C. The normal component of this variation coincides with the function $u = 1 + H\langle X, N\rangle$ introduced by Barbosa and do Carmo. When u is inserted in the index form (3), the stability condition reads $Q(u,u) \geqslant 0$, which implies, after some computations, that $|\sigma|^2 = nH^2$ on Σ and $\mathrm{II}(N,N) = 0$ on $\partial\Sigma \setminus \{0\}$. In particular, Σ is a totally umbilical hypersurface. On the other hand, by using the first Minkowski formula for cones, see [**RRo**, (4.8)], we deduce that $H > 0$ and so, each component of Σ is a piece of a sphere. Moreover, Σ is connected by Lemma 1. The proof finishes by invoking Lemma 3 below. □

LEMMA 3 ([**RRo**, Lemma 4.10]). *Let $C \subset \mathbb{R}^{n+1}$ be a convex cone and $\Sigma \subset \overline{C}$ a connected piece of a round sphere such that $\partial\Sigma \subset \partial C$ and Σ meets $\partial C \setminus \{0\}$ orthogonally. Then, Σ is either a round sphere contained in C, or a spherical cap centered at the vertex, or a half-sphere lying over a flat piece of ∂C.*

REMARK 2. *By using an approximation argument* [**RRo**, Lemma 4.3] *we can show that Theorem 8 is also valid if we allow the presence in Σ of a closed singular set Σ_0 which is negligible, in the sense that $\mathcal{H}^{n-2}(\Sigma_0) = 0$ (here, $\mathcal{H}^{n-2}$ is the $(n-2)$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$). This argument also holds for the whole space $\mathbb{R}^{n+1}$ and for a half-space $\mathbb{H} \subset \mathbb{R}^{n+1}$; as a consequence, we can extend Theorem 3 and Theorem 4 to hypersurfaces with small singular sets.*

4. Applications to the partitioning problem

In this section we will show how to use the stability condition to study the ***partitioning problem*** inside a convex domain $D \subseteq \mathbb{R}^{n+1}$. This problem consists of finding those hypersurfaces separating a given amount of volume inside D with the least possible area. More precisely

DEFINITION 3. *Let $\Sigma \subset \overline{D}$ be a smooth hypersurface separating a bounded set $\Omega \subset D$. We say that Σ is an* ***isoperimetric hypersurface*** *enclosing volume $V < vol(D)$ if $vol(\Omega) = V$ and*

$$A(\Sigma) \leqslant A(\Sigma')$$

for all hypersurfaces $\Sigma' \subset \overline{D}$ such that $vol(\Omega') = V$.

It is clear that any isoperimetric hypersurface is also stable. Along this section we will give many examples showing that the reverse statement is not true in general. The relation between stability and isoperimetry implies that stable hypersurfaces are natural candidates to solve the partitioning problem.

Suppose that D is a smooth domain where the existence of isoperimetric hypersurfaces enclosing any given volumen is ensured. If stable hypersurfaces in D are classified, then we only have to compare the area of the different stable candidates for fixed volume in order to find the best ones. We shall apply this scheme to describe isoperimetric hypersurfaces for the cases treated in Section 2 and Section 3.

Usually, a difficult step in the scheme above is to prove the existence and the regularity of isoperimetric hypersurfaces in D. By standard results in Geometric Measure Theory [**M**], isoperimetric hypersurfaces exist in some situations: for example, when D is bounded or invariant under the action of a large group of Euclidean isometries. Moreover, the singular set $\Sigma_0 \subset \Sigma$ is closed in $\mathbb{R}^{n+1}$ and satisfies $\mathcal{H}^{n-2}(\Sigma_0) = 0$, where $\mathcal{H}^{n-2}$ is the $(n-2)$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. Sometimes it is possible to show that the singular set Σ_0 is empty. Otherwise, we can use approximation arguments to deal with the singularities, see [**SZ2**, Lemma 2.4] and [**RRo**, Lemma 4.3].

We begin by considering the case $D = \mathbb{R}^{n+1}$. The results alluded to above provide existence and regularity of isoperimetric hypersurfaces in $\mathbb{R}^{n+1}$ enclosing any given volume. Moreover, they are also bounded and connected. Therefore, by using Theorem 3 we obtain the classical isoperimetric property of spheres in $\mathbb{R}^{n+1}$

THEOREM 9. *Isoperimetric hypersurfaces in $\mathbb{R}^{n+1}$ are round spheres.*

We can use similar arguments and Theorem 4 to deduce that the only candidates to solve the partitioning problem in a half-space of $\mathbb{R}^{n+1}$ are round spheres and half-spheres centered at the boundary of the half-space. An easy comparison of areas shows

THEOREM 10. *Isoperimetric hypersurfaces in a Euclidean half-space are half-spheres centered at the boundary of the half-space.*

Now, we consider a slab S in $\mathbb{R}^{n+1}$. By taking into account that S is invariant under translations parallel to ∂S and rotations about any line orthogonal to ∂S, we can establish existence and regularity of isoperimetric hypersurfaces for any given volume. In this case, the family of stable candidates is bigger, see Theorem 5. However, after a comparison between the areas, it is obtained

THEOREM 11 ([**PR**, Thm. 3.5]). *Isoperimetric hypersurfaces in a slab of $\mathbb{R}^{n+1}$, $n \leqslant 7$ are*

(i) *Half-spheres centered at one of the boundary hyperplanes, or*
(ii) *Certain tubes around a line orthogonal to the boundary of the slab.*

This result illustrates that the topology of isoperimetric hypersurfaces changes for certain values of volume. For any $n \geqslant 9$ it was shown in [**PR**, Proposition 3.4] that there are isoperimetric hypersurfaces in S of unduloid type. The argument is the following: for $n \geqslant 9$ a half-sphere centered at one component of ∂S and meeting tangentially the other one cannot be isoperimetric by regularity but has strictly less area than a tube enclosing the same volume. Hence, there is an isoperimetric hypersurface which is neither a half-sphere nor a tube. The only possibility is that the hypersurface is of unduloid type. The solution to the partitioning problem for $n = 8$ is still an open question.

Now, let us consider a round ball B of $\mathbb{R}^{n+1}$. As B is bounded, the existence of isoperimetric solutions is guaranteed. The natural candidates to solve the partitioning problem provided by Theorem 6 are round spheres included in B, flat n-dimensional discs passing through the center of B, and spherical caps meeting ∂B orthogonally. At first, some other candidates could appear since the classification of stable hypersurfaces in a ball is not complete. However, we can discard them by using ***spherical symmetrization*** [**BZ**, p. 78] and Proposition 2.

Spherical symmetrization about a line R assigns to any set $A \subset \mathbb{R}^{n+1}$ another set $S(A)$ in this way: for any S in a family of concentric spheres centered at R, we replace the intersection $A \cap S$ by the spherical cap centered at one point in $S \cap R$ of the same area. This construction has the property of preserving the volume of A while the boundary area strictly decreases unless A were of revolution about a line parallel to R, and the sections $A \cap S$ were all connected. By applying this procedure we see that any isoperimetric hypersurface Σ inside a ball is rotationally symmetric with respect to a line passing through the center of the ball. By invoking Proposition 2 we deduce that Σ coincides with one of the natural candidates mentioned above. An easy comparison of areas finally gives

THEOREM 12 ([**BS**], [**A**]). *Isoperimetric hypersurfaces in a ball $B \subset \mathbb{R}^{n+1}$ are*

(i) *Flat n-dimensional discs containing the center of the ball, or*
(ii) *Spherical caps meeting ∂B orthogonally.*

Finally we treat the case of a cone $C \subset \mathbb{R}^{n+1}$ different from a half-space. For solid cones, we cannot apply general existence results in Geometric Measure Theory since they are neither bounded nor invariant under a large group of isometries. This leads us to study the question of existence in more detail. By using that the dilations centered at the vertex are diffeomorphisms of the cone, we proved in [**RRo**] some criteria ensuring existence of isoperimetric hypersurfaces.

THEOREM 13 ([**RRo**, Propositions 3.5 and 3.6]). *Let $C \subset \mathbb{R}^{n+1}$ be a solid cone over a smooth domain $U \subset \mathbb{S}^n$. Suppose that C satisfies one of the following conditions*

(i) *C admits a local support hyperplane at a point $p \in \partial C \setminus \{0\}$.*
(ii) *$\mathcal{H}^n(U) \leqslant \mathcal{H}^n(\mathbb{S}^n)/2$.*

Then, there are bounded isoperimetric hypersurfaces in C for any given volume.

Admit that the cone C is convex. Let Σ be an isoperimetric hypersurface in C (which exists by Theorem 13). At first, a singular set $\Sigma_0 \subset \Sigma$ consisting of isolated points or satisfying $\mathcal{H}^{n-2}(\Sigma_0) = 0$ could appear. However, as we pointed out in Remark 2, our classification of stable hypersurfaces in Theorem 8 is also valid in this situation. It turns out that Σ is either a round sphere inside C, or a half-sphere lying over a flat piece of ∂C, or a spherical cap centered at the vertex of C. A comparison between the areas of the different candidates finally gives us

Theorem 14 ([**LP**], [**RRo**]). *Isoperimetric hypersurfaces in a solid convex cone are spherical caps centered at the vertex of the cone.*

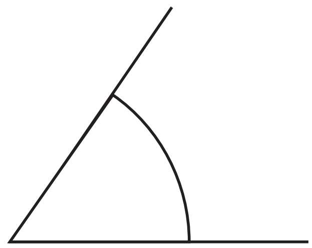

Figure 7. Isoperimetric hypersurfaces in a solid convex cone.

Theorem 14 was previously proved by Lions and Pacella [**LP**] by using the Brunn-Minkowski inequality in $\mathbb{R}^{n+1}$. The complete solution to the partitioning problem inside a convex cone over a non-smooth spherical domain remains open. Another interesting question to study is the next one

Problem 1. *Consider the Clifford torus $T \subset \mathbb{S}^3$*

$$T = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + t^2 = 1/2\}.$$

$\mathbb{S}^3 \setminus T$ is the union of two domains U_k, which are isometric via the antipodal map and satisfy $\mathcal{H}^3(U_1) = \mathcal{H}^3(U_2) = \mathcal{H}^3(\mathbb{S}^3)/2$. Hence, if C is the cone over U_1 we know by Theorem 13 that there exist isoperimetric hypersurfaces in C for any volume. However, in this case we cannot apply Theorem 8 since the cone C is non-convex (in fact, at any point of $\partial C \setminus \{0\}$ there always are two principal curvatures with opposite values).

Acknowledgements. The author has been partially supported by MCyT-Feder research project MTM2004-01387

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Departamento de Geometría y Topología
Universidad de Granada
E–18071 Granada España

e-mail: `crosales@ugr.es`