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Ricci generalized pseudo-parallel immersions

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Abstract. In the present study we consider the isometric immersions $f : M^n \longrightarrow N^{n+d}(c)$ satisfying the curvature condition $\bar{R}(X,Y)\cdot h = L_h Q(S,h)$. We call these type of immersions Ricci generalized pseudo-parallel. We have shown that, if f is Ricci pseudo-parallel with $H(p) = 0$ and $L_h(p)\tau(p) - nc \geqslant 0$, then the point p is a geodesic point, i.e. the second fundamental form of M^n vanishes at p.

1. Introduction

Let (M, g) be a semi-Riemannian manifold of class C^∞ and dimension $n \geqslant 3$. We denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M, g), respectively. A manifold (M, g) is said to be semisymmetric [**23**] if $R(X,Y)\cdot R = 0$ holds on M. Every locally symmetric manifold ($\nabla R = 0$) and every 2-dimensional Riemannian manifold are semisymmetric. The converse statement is not true. The first notes about semisymmetric Riemannian manifolds and some new examples appear in the works by E. Cartan, A. Lichnerowicz, R. S. Couty and N. S. Sinjukov. In [**21**] K. Nomizu asked the question if there exist complete, irreducible and simply connected Riemannian manifolds of dimension $n \geqslant 3$ satisfying $R(X,Y)\cdot R = 0$ and not locally symmetric. The first such example was constructed by H. Takagi in [**26**].

A systematic study of Riemannian semisymmetric manifolds was started first by Z.I. Szabó ([**23**] , [**24**]) and O. Kowalski [**12**], later by E. Boeckx, L. Vanhecke and others. See [**3**] for a survey. In the semi-Riemannian case this topic was studied, among others, by A. Derdzinski and W. Roter [**9**].

A semi-Riemannian manifold (M, g) is pseudosymmetric [**7**], if and only if $R\cdot R = L_R Q(g,R)$ holds on M where L_R is a function. (The present notation and similar notations in the following lines are explained in more details in the next section). Because the property to be semisymmetric is written, in this notation, as $R\cdot R = 0$, every semisymmetric manifold is pseudosymmetric. The converse statement is not true ([**7**]).

Let M^n and $N^{n+d}(c)$ be an n-dimensional Riemmannian manifold and an (n+d)-dimensional space form, respectively. An isometric immersions $f : M^n \longrightarrow N^{n+d}(c)$ is called semi-parallel (see [**5**] and [**16**]) or semisymmetric (see [**13**] and [**14**]) if the condition $\bar{R}\cdot h = 0$ holds, where h is the second fundamental form of M, where $\bar{R}$

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denotes the curvature tensor belonging to the connection $\overline{\overline{\nabla}}$. The classification of semi-parallel hypersurfaces in $N^{n+1}(c)$ was given by Deprez for $c = 0$ in [**6**] and by Dillen for $c \neq 0$ in [**10**], (see also [**19**]). Intrinsically, every semi-parallel submanifold is a semisymmetric Riemannian manifold. Geometrically, semi-parallel submanifols are characterized as the second order envelopes of the parallel ones (see [**15**]). For the corresponding theorem some generalizations have been given in [**17**], [**18**].

In [**8**], the authors obtained some results in hypersurfaces in 4-dimensional space form $N^4(c)$ satisfying $\bar{R} \cdot h = LQ(g, h)$. The hypersurfaces satisfying this condition are called pseudo-parallel (see [**1**] and [**2**]) or extended semi-parallel (see [**22**]). In [**2**] it has been shown that a pseudo-parallel hypersurface of a space form is either quasi-umbilical or a cyclic of Dupin. All Cartan hypersurfaces in $S^4(c)$ are the examples of pseudosymmetic hypersurfaces which are not pseudo-parallel.

In [**1**], Asperti et. all considered the isometric immersions $f : M^n \longrightarrow N^{n+d}(c)$ satisfying the pseudo-parallelity condition $\bar{R} \cdot h = L_h Q(g, h)$. They have shown that if f is pseudo-parallel with $H(p) = 0$ and $L_h(p) \geqslant 0$ then the point p is a geodesic point of M, i.e. the second fundamental form vanishes identically, where H is the mean curvature vector of M.

In [**20**] we considered the Ricci generalized pseudo-parallel hypersurfaces, i.e. the hypersurfaces satisfying the condition $\bar{R} \cdot h = L_h Q(S, h)$.

In the present study we generalized our results for the case when M^n is a submanifold in the space form $N^{n+d}(c)$. We prove that if the isometric immersion $f : M^n \longrightarrow N^{n+d}(c)$ is Ricci pseudo-parallel with $H(p) = 0$ and $L(p)\tau(p) - nc \geqslant 0$, then the point p is a geodesic point, i.e. the second fundamental form vanishes at p.

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2. Basic Concepts

Let (M, g), $n \geqslant 3$ be a connected semi-Riemannian manifold of class C^∞. We denote by ∇, R, S and κ the Levi-Civita connection, Riemann curvature tensor, the Ricci tensor and scalar curvature of (M, g) respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \chi(M)$, $\chi(M)$ being Lie algebra of vector fields on M. We next define endomorphisms $\mathcal{R}(X, Y)$ and $X \wedge_A Y$ of $\chi(M)$ by

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \tag{2}$$

respectively, where $X, Y, Z \in \chi(M)$ and A is a symmetric $(0, 2)$-tensor. The Riemannian curvature tensor R and the $(0, 4)$-tensor G of (M, g) are defined by

$$R(X_i, X_j, X_k, X_l) = g(\mathcal{R}(X_i, X_j)X_k, X_l), \tag{3}$$

$$G(X_i, X_j, X_k, X_l) = g((X_i \wedge_g X_j)X_k, X_l) \tag{4}$$

respectively. Now, for a $(0, k)$-tensor field T, $k \geqslant 1$ and a $(0, 2)$ -tensor field A on (M, g) we define the tensors $R \cdot T$ and $Q(A, T)$ by

$$\begin{aligned} (R \cdot T)(X_1, ..., X_k; X, Y) &= -T(\mathcal{R}(X, Y)X_1, X_2, ..., X_k) \\ &\quad - ... - T(X_1, ..., X_{k-1}, \mathcal{R}(X, Y)X_k), \end{aligned} \tag{5}$$

$$Q(A,T)(X_1,...,X_k;X,Y) = -T((X\wedge_A Y)X_1,X_2,...,X_k) - ... - T(X_1,...,X_{k-1},(X\wedge_A Y)X_k), \tag{6}$$

respectively. Putting into the above formulas $T=R$, or $T=S$, respectively, and $A=g$, or $A=S$, respectively, we obtain the tensors $R\cdot R$, $R\cdot S$, $Q(g,h)$, $Q(g,R)$, $Q(S,h)$ etc.

Let $f: M^n \longrightarrow N^{n+d}(c)$ be an isometric immersion of an n-dimensional Riemannian manifold M into $(n+d)$-dimensional space form $N^{n+d}(c)$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita Connections of M and $N^{n+d}(c)$ respectively, and by $N(M)$ its normal bundle. Then for vector fields X,Y which are tangent to M, the second fundamental form h is given by the formula $h(X,Y)=\widetilde{\nabla}_X Y-\nabla_X Y$. Furthermore, for $\xi\in N(M)$, $A_\xi: TM \longrightarrow TM$ will denote the Weingarten operator in the ξ direction, $A_\xi X=\nabla^\perp_X\xi-\widetilde{\nabla}_X\xi$, where $\nabla^\perp$ denotes the normal connection of M. The second fundamental form h and A_ξ are related by $\widetilde{g}(h(X,Y),\xi)=g(A_\xi X,Y)$, where g is the induced metric of $\widetilde{g}$ for any vector fields X,Y tangent to M. The mean curvature vector H of M is defined to be

$$H=\frac{1}{n}Tr(h).$$

The covariant derivative $\overline{\nabla}h$ of h is defined by

$$(\overline{\nabla}_X h)(Y,Z)=\nabla^\perp_X(h(Y,Z))-h(\nabla_X Y,Z)-h(Y,\nabla_X Z), \tag{7}$$

where, $\overline{\nabla}h$ is a normal bundle valued tensor of type $(0,3)$ and is called the third fundamental form of M. The equation of Codazzi implies that $\overline{\nabla}h$ is symmetric hence

$$(\overline{\nabla}_X h)(Y,Z)=(\overline{\nabla}_Y h)(X,Z)=(\overline{\nabla}_Z h)(X,Y). \tag{8}$$

Here, $\overline{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M. If $\overline{\nabla}h=0$, then f is called parallel, [**11**].

The basic equations of Gauss and Ricci are;

$$R(X,Y,Z,W)=cg(X\wedge Y(Z),W)+g(h(X,Y),h(W,Z))-g(h(X,Z),h(Y,W)) \tag{9}$$

$$\widetilde{g}(R^\perp(X,Y)\xi,\eta)=g([A_\xi,A_\eta]X,Y);\xi,\eta\in N(M). \tag{10}$$

Here $R^\perp$ is the curvature operator of the normal connection defined by

$$R^\perp(X,Y)Z=\nabla^\perp_X\nabla^\perp_Y Z-\nabla^\perp_Y\nabla^\perp_X Z-\nabla^\perp_{[X,Y]}Z.$$

An isometric immersion f (or the submanifold M) is said to have *flat normal connection* (or trivial normal connection) if $R^\perp=0$. If M has flat normal connection then shortly we call it to be normally flat. The relation (10) shows that the triviality of the normal connection of M into space form $\mathbb{N}^{n+d}(c)$ (and more generally, for submanifolds in a locally conformally flat space) is equivalent to the fact that all second fundamental tensors mutually commute, or that all second fundamental tensors are mutually diagonalizable, [**4**].

For an orthonormal frame field $\{e_1,e_2,...,e_n\}$ of M, the Ricci tensor S is defined by

$$S(X,Y)=\sum_{k=1}^n g(\mathcal{R}(e_k,X)Y,e_k). \tag{11}$$

Consequently, by the use of (9) the equation (11) turns into

$$S(X,Y) = (n-1)cg(X,Y) + \sum_{a=1}^{n} TrA_a g(A_a X, Y) - \sum_{a=1}^{n} g(A_a X, A_a Y). \tag{12}$$

The second covariant derivative $\overline{\nabla}^2 h$ of h is defined by

$$\begin{aligned} (\overline{\nabla}^2 h)(Z,W,X,Y) &= (\overline{\nabla}_X \overline{\nabla}_Y h)(Z,W) \\ &= \nabla_X^{\perp}((\overline{\nabla}_Y h)(Z,W)) - (\overline{\nabla}_Y h)(\nabla_X Z, W) \\ &\quad -(\overline{\nabla}_X h)(Z, \nabla_Y W) - (\overline{\nabla}_{\nabla_X Y} h)(Z,W). \end{aligned} \tag{13}$$

Then we have

$$\begin{aligned} (\overline{\nabla}_X \overline{\nabla}_Y h)(Z,W) - (\overline{\nabla}_Y \overline{\nabla}_X h)(Z,W) &= (\bar{R}(X,Y) \cdot h)(Z,W) \\ &= \mathcal{R}^{\perp}(X,Y)h(Z,W) - h(\mathcal{R}(X,Y)Z,W) \\ &\quad -h(Z, \mathcal{R}(X,Y)W). \end{aligned} \tag{14}$$

where $\bar{R}$ is the curvature tensor belonging to the connection $\overline{\nabla}$.

3. Ricci Generalized Pseudo-parallel Immersions

In the present section we consider Ricci generalized pseudo-parallel submanifolds in real space forms.

Definition 1. *An isometric immersion $f : M^n \longrightarrow N^{n+d}(c)$ is Ricci pseudo-parallel if there exists a real valued function L_h on M such that*

$$\bar{R} \cdot h = L_h Q(S,h) \tag{15}$$

holds for all $p \in M$ and $X, Y \in TM$, where $\bar{R}$ denotes the curvature tensor belonging to the connection $\overline{\nabla}$ and $Q(S,h)$ is the tensor field defined by

$$\begin{aligned} Q(S,h)(X,Y,Z,W) &= [(X \wedge_S Y)h](Z,W) \\ &= -h((X \wedge_S Y)Z, W) - h(Z, (X \wedge_S Y)W). \end{aligned} \tag{16}$$

We prove the following result;

Theorem 1. *Let $f : M^n \longrightarrow N^{n+d}(c)$ is a Ricci pseudo-parallel immersion, then*

$$R \cdot R = L_h[Q(S,R) - cQ(S,G)].$$

where the (0,4)- tensor G is defined by

$$G(X,Y,Z,W) = g((X \wedge Y)Z, W).$$

Proof : First, we observe that by (5) the tensor $R(U,V) \cdot R$ can be written as

$$\begin{aligned} (R(U,V) \cdot R)(X,Y,Z,W) &= -R(\mathcal{R}(U,V)X,Z,W) - R(X, \mathcal{R}(U,V)Y, Z, W) \\ &\quad -R(X,Y,\mathcal{R}(U,V)Z,W) - R(X,Y,Z,\mathcal{R}(U,V)W) \end{aligned}$$

So, by the use of (9), we get

$$
\begin{aligned}
(R(U,V)\cdot R)(X,Y,Z,W) = &-c\{g((X\wedge Y)Z,\mathcal{R}(U,V)W)+g((X\wedge Y)\mathcal{R}(U,V)Z,W)\\
&+g(([\mathcal{R}(U,V)X]\wedge Y)Z,W)+g((X\wedge[\mathcal{R}(U,V)Y])Z,W)\}\\
&+g(h(X,Z),h(\mathcal{R}(U,V)W,Y)+h(W,\mathcal{R}(U,V)Y))\\
&-g(h(Y,Z),h(\mathcal{R}(U,V)X,W)+h(X,\mathcal{R}(U,V)W))\\
&+g(h(W,Y),h(\mathcal{R}(U,V)X,Z)+h(X,\mathcal{R}(U,V)Z))\\
&-g(h(X,W),h(\mathcal{R}(U,V)Y,Z)+h(Y,\mathcal{R}(U,V)Z)).
\end{aligned}\tag{17}
$$

Furthermore, making use of (9), (10) and (14), the Ricci pseudo-parallelity condition (15) turns into

$$
\begin{aligned}
\mathcal{R}^{\perp}(U,V)h(X,Y) &= h(\mathcal{R}(U,V)X,Y)+h(X,\mathcal{R}(U,V)Y)\\
&\quad -L_hS(V,X)h(U,Y)+L_hS(U,X)h(V,Y)\\
&\quad -L_hS(V,Y)h(X,U)+L_hS(U,Y)h(X,V).
\end{aligned}\tag{18}
$$

Substituting (17) into (18) and using again equation (9), we obtain

$$
\begin{aligned}
(R(U,V)\cdot R)(X,Y,Z,W) &= L_h\{-g(\mathcal{R}(X,Y)Z,(U\wedge_S V)W)\\
&\quad -g(\mathcal{R}((U\wedge_S V)X,Y)Z,W)\\
&\quad -g(\mathcal{R}(X,(U\wedge_S V)Y)Z,W)\\
&\quad -g(\mathcal{R}(X,Y)(U\wedge_S V)Z,W)\\
&\quad +cg((X\wedge Y)Z,(U\wedge_S V)W)\\
&\quad +cg([(U\wedge_S V)X]\wedge Y)Z,W)\\
&\quad +cg((X\wedge[(U\wedge_S V)Y])Z,W)\\
&\quad +cg((X\wedge Y)(U\wedge_S V)Z,W)\}.
\end{aligned}\tag{19}
$$

Consequently, by the use of (6) after some computations we get

$$(R(U,V)\cdot R)(X,Y,Z,W)=L_h[Q(S,R)-cQ(S,G)](X,Y,Z;V,W).$$

Our theorem is thus proved.

THEOREM 2. *If the isometric immersion $f:M^n\longrightarrow N^{n+d}(c)$ is Ricci pseudo-parallel with $H(p)=0$ and $L(p)\tau(p)-nc\geqslant 0$, then the point p is a geodesic point, i.e. the second fundamental form vanishes at p.*

Proof : Let $f:M^n\longrightarrow N^{n+d}(c)$ be an isometric immersion. We choose a local orthonormal frame field $\{e_1,e_2,...,e_n,e_{n+1},...,e_{n+d}\}$ adapted to f. Then for $1\leqslant i,j\leqslant n$ the components of the second fundamental form h are given by

$$h^{\alpha}_{ij}=g(h(e_i,e_j),e_\alpha).\tag{20}$$

Similarly, the components of the first and the second covariant derivative of h are given by

$$h^{\alpha}_{ijk}=g((\overline{\nabla}_{e_k}h)(e_i,e_j),e_\alpha)=\overline{\nabla}_{e_k}h^{\alpha}_{ij},\tag{21}$$

and

$$
\begin{aligned}
h^{\alpha}_{ijkl} &= g((\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}h)(e_i,e_j),e_\alpha)\\
&= \overline{\nabla}_{e_l}h^{\alpha}_{ijk}\\
&= \overline{\nabla}_{e_l}\overline{\nabla}_{e_k}h^{\alpha}_{ij}
\end{aligned}\tag{22}
$$

respectively. If f is Ricci pseudo-parallel, then by definition, the condition

$$\bar{R}(e_l, e_k) \cdot h = L_h[(e_l \wedge_S e_k)]h \tag{23}$$

is fulfilled where

$$[(e_l \wedge_S e_k)h]\,(e_i, e_j) = -h((e_l \wedge_S e_k)e_i, e_j) - h(e_i, (e_l \wedge_S e_k)e_j) \tag{24}$$

for $1 \leqslant i, j, k, l \leqslant n$.

Substituting (2) into (24), we get

$$\begin{aligned}[(e_l \wedge_S e_k)h](e_i, e_j) &= -S(e_k, e_i)h(e_l, e_i) + S(e_l, e_i)h(e_k, e_i) \\ &\quad -S(e_k, e_j)h(e_l, e_i) - S(e_l, e_j)h(e_k, e_i).\end{aligned} \tag{25}$$

By (14) we have

$$(\bar{R}(e_l, e_k) \cdot h)(e_i, e_j) = (\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}h)(e_i, e_j) - (\overline{\nabla}_{e_k}\overline{\nabla}_{e_l}h)(e_i, e_j). \tag{26}$$

Making use of (20), (22), (25) and (26), the Ricci generalized pseudo-parallelity condition (23) turns into

$$h^\alpha_{ijkl} = h^\alpha_{ijlk} - L_h\{S_{ki}h^\alpha_{lj} + S_{li}h^\alpha_{hj} - S_{kj}h^\alpha_{kj} - S_{lj}h^\alpha_{ki}\} \tag{27}$$

where $S(e_i, e_j) = S_{ij}$ and $1 \leqslant i, j, k, l \leqslant n,\ n+1 \leqslant \alpha \leqslant n+d$.

Recall that the Laplacian Δh^α_{ij} of h^α_{ij} is defined by

$$\Delta h^\alpha_{ij} = \sum_{i,j,k=1}^{n} h^\alpha_{ijkk}. \tag{28}$$

Then we obtain

$$\frac{1}{2}\Delta(\|h\|^2) = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} h^\alpha_{ij}h^\alpha_{ijkk} + \left\|\overline{\nabla} h\right\|^2, \tag{29}$$

where

$$\|h\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} (h^\alpha_{ij})^2, \tag{30}$$

and

$$\left\|\overline{\nabla} h\right\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} (h^\alpha_{ijkk})^2, \tag{31}$$

are the square of the length of second and the third fundamental forms of M, respectively. In addition, making use of (20) and (22), we obtain

$$\begin{aligned} h^\alpha_{ij}h^\alpha_{ijkk} &= g(\alpha(e_i, e_j), e_\alpha)g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), e_\alpha) \\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j)g(h(e_i, e_j), e_\alpha), e_\alpha) \\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), h(e_i, e_j)). \end{aligned} \tag{32}$$

Therefore, due to (32), the equation (29) becomes

$$\frac{1}{2}\Delta(\|h\|^2) = \sum_{i,j,k=1}^{n} g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), h(e_i, e_j)) + \left\|\overline{\nabla} h\right\|^2. \tag{33}$$

Further, by the use of (23), (25) and (26), we get

$$
\begin{aligned}
g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i,e_j),h(e_i,e_j) &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_i}h)(e_k,e_j),h(e_i,e_j)) \\
&= g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_k}h)(e_j,e_k),h(e_i,e_j)) \\
&\quad -L_h\{S(e_i,e_j)g(h(e_k,e_k)h(e_i,e_j)) \\
&\quad -S(e_k,e_j)g(h(e_k,e_i),h(e_i,e_j)) \\
&\quad +S(e_k,e_i)g(h(e_j,e_k),h(e_i,e_j)) \\
&\quad -S(e_k,e_k)g(h(e_k,e_j),h(e_i,e_j))\}.
\end{aligned} \tag{34}
$$

Substituting (34) into (33), we have

$$
\begin{aligned}
\frac{1}{2}\Delta(\|h\|^2) &= \sum_{i,j,k=1}^{n} g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}h)(e_k,e_k),h(e_i,e_j)) \\
&\quad -L_h\{S(e_j,e_i)g(h(e_k,e_k),h(e_i,e_j)) \\
&\quad -S(e_k,e_j)g(h(e_i,e_k),h(e_i,e_j)) \\
&\quad +S(e_k,e_i)g(h(e_j,e_k),h(e_i,e_j)) \\
&\quad -S(e_k,e_k)g(h(e_j,e_k),h(e_i,e_j))\} + \left\|\overline{\nabla}h\right\|^2.
\end{aligned} \tag{35}
$$

Thus, making use of (12), we obtain

$$
\begin{aligned}
\sum_{i,j,k=1}^{n} S(e_j,e_i)g(h(e_k,e_k),h(e_i,e_j)) &= \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} S(e_j,e_i)g(A_\alpha e_k,e_k)g(A_\alpha e_i,e_j) \\
&= \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} S(e_j,e_i)Tr(A_\alpha)g(A_\alpha e_i,e_j),
\end{aligned} \tag{36}
$$

and

$$
\begin{aligned}
\sum_{i,j,k=1}^{n} S(e_k,e_j)g(h(e_i,e_k),h(e_i,e_j)) &= \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} S(e_k,e_j)g(A_\alpha e_i,e_k)g(A_\alpha e_i,e_j) \\
&= \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} S(e_k,e_j)g(A_\alpha e_k,e_i)g(A_\alpha e_j,e_i) \\
&= \sum_{j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} S(e_k,e_j)g(A_\alpha e_k,A_\alpha e_j) \\
&= \sum_{j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} [(n-1)cg(e_k,e_j)g(A_\alpha e_k,A_\alpha e_j)] \\
&\quad + \sum_{\alpha=n+1}^{n+d} TrA_\alpha g(A_\alpha e_k,e_j)g(A_\alpha e_k,A_\alpha e_j) \\
&\quad - \sum_{\alpha=n+1}^{n+d} g(A_\alpha e_k,A_\alpha e_j)g(A_\alpha e_k,A_\alpha e_j).
\end{aligned} \tag{37}
$$

Furthermore, by definition

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)),$$

and

$$\tau = \sum_{k=1}^{n} S(e_k, e_k),$$

where τ is the scalar curvature of M. Hence, we can write

$$\sum_{i,j,k=1}^{n} S(e_k, e_k) g(h(e_i, e_j), h(e_i, e_j)) = \tau \|h\|^2 . \tag{38}$$

Moreover, substituting (36)-(38) into (35), we get

$$\begin{aligned} \frac{1}{2}\Delta(\|h\|^2) = &\sum_{i,j,k=1}^{n} g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} h)(e_k, e_k), h(e_i, e_j)) \\ &-L_h\{\sum_{j,k=1}^{n}\sum_{\alpha=n+1}^{n+d} S(e_i, e_j) Tr(A_\alpha) g(A_\alpha e_i, e_j) - \tau \|h\|^2\} + \left\|\overline{\nabla} h\right\|^2 . \end{aligned} \tag{39}$$

If we take $H^\alpha = \sum_{k=1}^{n} h^\alpha_{kk}$ the equation (39) reduces to

$$\begin{aligned} \frac{1}{2}\Delta(\|h\|^2) &= \sum_{i,j=1}^{n}\sum_{\alpha=n+1}^{n+d} h^\alpha_{ij}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} H^\alpha) \\ &\quad -L_h\{\sum_{i,j=1}^{n}\sum_{\alpha=n+1}^{n+d} S(e_i, e_j) Tr(A_\alpha) g(A_\alpha e_i, e_j) - \tau \|h\|^2\} + \left\|\overline{\nabla} h\right\|^2 . \end{aligned}$$

B. Y. Chen has shown in, [**4**], that

$$\begin{aligned} \frac{1}{2}\Delta(\|h\|^2) &= \sum_{i,j=1}^{n}\sum_{\alpha=n+1}^{n+d} h^\alpha_{ij}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} H^\alpha) + nc(\|h\|^2 - n\|H\|^2) \\ &\quad - \sum_{\alpha,\beta=n+1}^{n+d} \{[Tr(A_\alpha \circ A_\beta)]^2 + \|[A_\alpha, A_\beta]\|^2 \\ &\quad + H^\beta Tr(A_\alpha \circ A_\beta \circ A_\alpha)\} + \left\|\overline{\nabla} h\right\|^2 . \end{aligned}$$

where

$$\|H\|^2 = \frac{1}{n^2}\sum_{\alpha=n+1}^{n+d} (H^\alpha)^2. \tag{40}$$

Hence comparing the last two equations, one can get

$$\begin{aligned} 0 = \ &nc\left(\|h\|^2 - n\|H\|^2\right) + L_h\left\{\sum_{i,j=1}^{n}\sum_{\alpha=n+1}^{n+d} S(e_i, e_j) Tr(A_\alpha) g(A_\alpha e_i, e_j) - \tau \|h\|^2\right\} \\ &- \sum_{\alpha,\beta=n+1}^{n+d}\left\{[Tr(A_\alpha \circ A_\beta)]^2 + \|A_\alpha \circ A_\beta\|^2 + H^\beta Tr(A_\alpha \circ A_\beta \circ A_\alpha)\right\}. \end{aligned}$$

Now, if $H(p) = 0$ then by (40), we have $H^\alpha = 0$ at every point p of M, which easily implies $Tr(A_\alpha) = 0$. So

$$\|h(p)\|^2 \{L_h(p)\tau(p) - nc\} + \sum_{\alpha,\beta=n+1}^{n+d} \{[Tr(A_\alpha \circ A_\beta)]^2 + \|[A_\alpha \circ A_\beta]\|^2\} = 0.$$

If $L_h(p)\tau(p) - nc \geqslant 0$ then $Tr(A_\alpha \circ A_\beta) = 0$. In particular, $\|A_\alpha\|^2 = Tr(A_\alpha \circ A_\alpha) = 0$, hence $h(p) = 0$. This completes the proof of our theorem.

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