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Covariant Hamiltonians for diffeomorphism-invariant Lagrangians on linear frame bundles

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Abstract. Let γ be an Ehresmann connection on the linear frame bundle of a manifold N. The notion of torsion associated to γ is defined. The covariant Hamiltonian Λ^γ associated with a Lagrangian density on $J^1(FN)$ is constructed and it is shown to be diffeomorphism-invariant for every diffeomorphism-invariant Lagrangian density Λ if and only if the torsion associted with γ is constant. The case of a linear connection is also studied.

1. Introduction and preliminaries

Let $p\colon M \to N$ be a fibred manifold over an orientable connected n-manifold N, oriented by a volume form $\mathbf{v}$, with $\dim M = m+n$. We confine ourselves to consider only coordinate systems (x^i) on N adapted to $\mathbf{v}$, that is

$$\mathbf{v} = dx^1 \wedge \cdots \wedge dx^n.$$

We also set

$$\mathbf{v}_i = dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Let γ be an Ehresmann (or non-linear) connection on p, i.e., γ is a differential 1-form on M with values in the vertical bundle $V(p)$, such that $\gamma(X) = X$, $\forall X \in V(p)$.

Given a Lagrangian density Λ on J^1M, we can write $\Lambda = L\mathbf{v}$, $L \in C^\infty(J^1M)$ being the Lagrangian function associated to Λ. The covariant Hamiltonian Λ^γ associated to Λ with respect to γ (e.g. see [**5**]) is the Lagrangian density defined by

$$\Lambda^\gamma = \big((p^{10})^*\gamma - \vartheta\big) \wedge \omega_\Lambda - \Lambda, \tag{1}$$

where,

(1) $p^{10}\colon J^1M \to M$ is the natural projection: $p^{10}(j^1_x s) = s(x)$,
(2) ϑ is the structure form on J^1M; that is,

$$\vartheta = \big(dy^\alpha - y^\alpha_i dx^i\big) \otimes \frac{\partial}{\partial y^\alpha} \tag{2}$$

on a fibred coordinate system (x^i, y^α), $1 \leqslant i \leqslant n$, $1 \leqslant \alpha \leqslant m$, for the submersion p, and

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(3) ω_Λ is the Legendre form attached to the Lagrangian density Λ, with local expression

$$\omega_\Lambda = (-1)^{i-1}\frac{\partial L}{\partial y_i^\alpha}\mathbf{v}_i \otimes dy^\alpha. \tag{3}$$

Given a subgroup $\mathcal{G} \subseteq \operatorname{Aut}(p)$ of the group of all automorphisms of the fibred manifold $p\colon M \to N$, a Lagrangian density Λ is said to be $\mathcal{G}$-invariant if

$$\left(\Phi^{(1)}\right)^* \Lambda = \Lambda, \quad \forall \Phi \in \mathcal{G}, \tag{4}$$

where $\Phi^{(1)}\colon J^1M \to J^1M$ denotes the 1-jet prolongation of Φ.

When a group $\mathcal{G}$ of transformations of M is given, a natural question arises: Is it possible to select a class of connections γ such that Λ^γ is $\mathcal{G}$-invariant for every $\mathcal{G}$-invariant Λ?

For example, the solution to this question for the gauge group of a principal bundle is obtained in [**1**].

The goal of this paper is to provide a complete answer to the question above in the case of the bundle of linear frames, i.e., when $p\colon M \to N$ is $\pi\colon FN \to N$ and $\mathcal{G} = \operatorname{Diff} N$.

2. Covariant Hamiltonians on FN

Each coordinate system (x^i) on an open domain $U \subseteq N$ induces a coordinate system (x^i, x^i_j) on $\pi^{-1}(U)$ by setting

$$u = \left(\left(\frac{\partial}{\partial x^1}\right)_x, \dots, \left(\frac{\partial}{\partial x^n}\right)_x\right) \cdot \begin{pmatrix} x^1_1(u) & \dots & x^1_n(u) \\ \vdots & \ddots & \vdots \\ x^n_1(u) & \dots & x^n_n(u) \end{pmatrix}, \quad x = \pi(u),$$

and a coordinate system $(x^i, x^i_j, x^i_{j,k})$, on J^1U, defined by

$$x^i_{j,k}(j^1_x s) = \frac{\partial(x^i_j \circ s)}{\partial x_k}(x).$$

Let γ be an Ehresmann connection on FN. On an induced coordinate system (x^i, x^i_j) for π, its local expression is

$$\gamma = \left(dx^i_j + \gamma^i_{jk} dx^k\right) \otimes \frac{\partial}{\partial x^i_j}, \quad \gamma^i_{jk} \in C^\infty(FN). \tag{5}$$

The Poincaré-Cartan form attached to Λ ([**2**, Definition 2.4], [**10**]) is defined to be the ordinary n-form on $J^1(FN)$ given by

$$\Theta_\Lambda = \vartheta \wedge \omega_\Lambda + \Lambda, \tag{6}$$

ϑ being the structure form on $J^1(FN)$, whose local expression, according to the general formula (2) above, is

$$\vartheta = \left(dx^i_j - x^i_{j,k} dx^k\right) \otimes \frac{\partial}{\partial x^i_j}, \tag{7}$$

and ω_Λ the Legendre form attached to Λ, whose local expression, according to the general formula (3) above, is

$$\omega_\Lambda = (-1)^{k-1}\frac{\partial L}{\partial x^i_{j,k}}\mathbf{v}_k \otimes dx^i_j. \tag{8}$$

Once an Ehresmann connection γ has been chosen, according to the general formula (1), the covariant Hamiltonian Λ^γ associated to a Lagrangian density Λ with respect to γ is given by the formula

$$\Lambda^\gamma = \left((\pi^{10})^*\gamma - \vartheta\right) \wedge \omega_\Lambda - \Lambda, \tag{9}$$

where $\pi^{10}\colon J^1(FN) \to N$ is the natural projection.

3. Diffeomorphism-invariant covariant Hamiltonians

A Lagrangian density Λ on $J^1(FN)$ is said to be $\mathrm{Diff}N$-invariant—or even, invariant under diffeomorphisms—if

$$(\tilde{\varphi}^{(1)})^*\Lambda = \Lambda, \quad \forall\varphi \in \mathrm{Diff}N, \tag{10}$$

where $\tilde{\varphi}$ is the natural lift of φ to FN ([**3**, VI, §§1,2]) and $\tilde{\varphi}^{(1)}$ is the 1-jet prolongation of $\tilde{\varphi}$.

If $\Phi_t\colon FN \to FN$ is the flow of a π-projectable vector field $X \in \mathfrak{X}(FN)$, then

$$\Phi_t^{(1)}\colon J^1(FN) \to J^1(FN)$$

is the flow of a vector field on $J^1(FN)$, denoted by $X^{(1)}$ and called the infinitesimal contact transformation associated to X. The map $X \mapsto X^{(1)}$ is an injection of Lie algebras. If

$$X = u^i\frac{\partial}{\partial x^i} + v^i_j\frac{\partial}{\partial x^i_j}, \quad u^i \in C^\infty(N),\ v^i_j \in C^\infty(FN),$$

is the local expression of X, then (e.g., see [**4, 6, 8, 9**]) we have

$$X^{(1)} = u^i\frac{\partial}{\partial x^i} + v^i_j\frac{\partial}{\partial x^i_j} + v^i_{jk}\frac{\partial}{\partial x^i_{j,k}}, \tag{11}$$

where

$$v^i_{jk} = \frac{\partial v^i_j}{\partial x^k} + x^r_{s,k}\frac{\partial v^i_j}{\partial x^r_s} - x^i_{j,h}\frac{\partial u^h}{\partial x^k}. \tag{12}$$

A Lagrangian density Λ on $J^1(FN)$ is said to be $\mathfrak{X}(N)$-invariant if

$$L_{\tilde{X}^{(1)}}\Lambda = 0, \quad \forall X \in \mathfrak{X}(N), \tag{13}$$

where $\tilde{X}^{(1)}$ denotes the 1-jet prolongation of $\tilde{X}$. If the local expression of X is

$$X = u^i\frac{\partial}{\partial x^i}, \quad u^i \in C^\infty(N),$$

then we have

$$\tilde{X} = u^i\frac{\partial}{\partial x^i} + x^h_j\frac{\partial u^i}{\partial x^h}\frac{\partial}{\partial x^i_j}. \tag{14}$$

If $\theta = (\theta^1, \dots, \theta^n)$ is the canonical 1-form on FN ([**3**, III, §2, p. 118]), then we can write

$$\Lambda = \mathcal{L}\theta^1 \wedge \cdots \wedge \theta^n, \tag{15}$$

where $\mathcal{L}$ is a smooth function on $J^1(FN)$, called the canonical Lagrangian associated to Λ.

A Lagrangian density Λ on $J^1(FN)$ is $\mathrm{Diff}N$-invariant (resp. $\mathfrak{X}(N)$-invariant) if and only if $\mathcal{L} \circ J^1(\tilde{\varphi}) = \mathcal{L}$, $\forall \varphi \in \mathrm{Diff}N$ (resp. $\tilde{X}^{(1)}\mathcal{L} = 0$, $\forall X \in \mathfrak{X}(N)$). These results are proved in [**7**, §2.1]. Basically, they follow from the fact that θ is both $\mathrm{Diff}N$-invariant and $\mathfrak{X}(N)$-invariant. Hence the problem of determining invariant Lagrangian densities is reduced to that of determining invariant Lagrangian functions. Moreover, $\mathrm{Diff}N$-invariance implies $\mathfrak{X}(N)$-invariance and both notions are equivalent except when N is orientable and admits an orientation-reversing diffeomorphism (also see [**7**, §2.1]). Because of this, we consider only $\mathfrak{X}(N)$-invariance throughout this paper.

Let Λ^γ be the covariant Hamiltonian associated to a Lagrangian density Λ on $J^1(FN)$ with respect to an Ehresmann connection γ. As the covariant Hamiltonian Λ^γ is a Lagrangian density, it also admits a canonical Lagrangian $\mathcal{L}^\gamma \in C^\infty(J^1(FN))$ defined, according to the formula (15), as follows:

$$\Lambda^\gamma = \mathcal{L}^\gamma \theta^1 \wedge \cdots \wedge \theta^n, \tag{16}$$

where

$$\mathcal{L}^\gamma = \left(\gamma^i_{jk} + x^i_{j,k}\right) \frac{\partial \mathcal{L}}{\partial x^i_{j,k}} - \mathcal{L}, \quad \gamma^i_{jk} \in C^\infty(FN). \tag{17}$$

3.1. Torsion of a first-order Ehresmann connection on FN. We define the notion of torsion associated with an Ehresmann conection γ on FN, by imitating that of a linear connection; e.g., see [**3**, III, §2].

Let $B(v)$ be the standard horizontal vector field associated to $v \in \mathbb{R}^n$ (see [**3**, III, §2]) with respect to γ; precisely, $B(v)$ is the only γ-horizontal vector field on FN such that $\pi_*(B(v)_u) = u(v)$ for every $u \in F_xN$, where, here, the linear frame u is considered to be a linear isomorphism $u\colon \mathbb{R}^n \to T_xN$. Then, we define the torsion of γ as being the map

$$T^\gamma\colon FN \to \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

given by

$$T^\gamma(u)(v,w) = d\theta(B(v), B(w))(u), \quad u \in FN,\ v, w \in \mathbb{R}^n.$$

Let $T^\gamma(u)^i$ be the components of $T^\gamma(u)$ with respect to the standard basis (v_i) of $\mathbb{R}^n$, i.e.,

$$T^\gamma(u)(v,w) = T^\gamma(u)^i(v,w)v_i.$$

Taking the local expressions of the standard horizontal vector field associated to v_a and that of the canonical form on FN, into account,

$$B(v_a) = x^k_a\left(\frac{\partial}{\partial x^k} - \gamma^i_{jk}\frac{\partial}{\partial x^i_j}\right),$$
$$\theta^i = y^i_j dx^j, \quad (y^i_j) = (x^i_j)^{-1},$$

we obtain

$$T^\gamma(u)^c(v_a, v_b) = \left(x^k_a \gamma^i_{bk} - x^k_b \gamma^i_{ak}\right) y^c_i. \tag{18}$$

4. The main results

THEOREM 1. *Let γ be an Ehresmann connection on $\pi\colon J^1(FN) \to N$. The covariant Hamiltonian Λ^γ associated with every $\mathfrak{X}(N)$-invariant Lagrangian density Λ on $J^1(FN)$ is $\mathfrak{X}(N)$-invariant, if and only if the torsion associated with γ is constant.*

PROOF. Let $\mathcal{L}^\gamma \in C^\infty(J^1(FN))$ be the canonical Lagrangian associated to the covariant Hamiltonian Λ^γ. We need to prove that $\tilde{X}^{(1)}(\mathcal{L}^\gamma) = 0$, for every $X \in \mathfrak{X}(N)$ and every invariant Lagrangian $\mathcal{L}$, if and only if the torsion of γ is constant. We first recall that every $\mathfrak{X}(N)$-invariant Lagrangian can be written as a differentiable function of the following $\frac{1}{2}n^2(n-1)$ Lagrangians:

$$\mathcal{L}^c_{ab} = (x^d_a x^e_{b,d} - x^d_b x^e_{a,d})y^c_e, \quad a < b. \tag{19}$$

See [**7**, 16, section 2.2] for the details of this fact and for the geometric meaning of such Lagrangians. Hence, we only need to impose

$$\tilde{X}^{(1)}((\mathcal{L}^c_{ab})^\gamma) = 0, \quad \forall X \in \mathfrak{X}(N),$$

and every system of indices $a < b$, c. From the local expression of $\mathcal{L}^c_{ab}$ in the formula (19), we have

$$\frac{\partial \mathcal{L}^c_{ab}}{\partial x^i_{j,k}} = (x^k_a \delta^j_b - x^k_b \delta^j_a)y^c_i. \tag{20}$$

Hence

$$x^i_{j,k} \frac{\partial \mathcal{L}^c_{ab}}{\partial x^i_{j,k}} = \mathcal{L}^c_{ab}.$$

From (20) and (18), we obtain

$$\begin{aligned}(\mathcal{L}^c_{ab})^\gamma &= \left(\gamma^i_{jk} + x^i_{j,k}\right) \frac{\partial \mathcal{L}^c_{ab}}{\partial x^i_{j,k}} - \mathcal{L}^c_{ab} \\ &= \gamma^i_{jk} \frac{\partial \mathcal{L}^c_{ab}}{\partial x^i_{j,k}} \\ &= (T^\gamma)^c_{ab},\end{aligned}$$

and, therefore $\tilde{X}^{(1)}(\mathcal{L}^c_{ab})^\gamma = \tilde{X}((T^\gamma)^c_{ab})$, which allows us to conclude the proof. □

COROLLARY 1. *Let γ be a linear connection on $\pi\colon J^1(FN) \to N$. The covariant Hamiltonian Λ^γ associated with every $\mathfrak{X}(N)$-invariant Lagrangian density Λ on $J^1(FN)$ is $\mathfrak{X}(N)$-invariant, if and only if γ is torsion free.*

PROOF. The torsion of a linear connection is equivariant with respect to the natural actions of $GL(n;\mathbb{R})$ on FN and on $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$; that is,

$$T^\gamma(u \cdot A) = A^{-1} \cdot T^\gamma(u), \qquad \forall A \in GL(n;\mathbb{R}).$$

From the theorem above, we know that the statement holds if and only if T^γ is constant; hence,

$$\begin{aligned}T^\gamma(u) &= T^\gamma(u \cdot A) \\ &= A^{-1} \cdot T^\gamma(u),\end{aligned}$$

for every $A \in GL(n;\mathbb{R})$. Accordingly, $T^\gamma(u)$ is $GL(n;\mathbb{R})$-invariant; hence $T^\gamma(u)$ must vanish. □

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