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Minimal metrics on nilmanifolds

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Abstract. A left invariant metric on a nilpotent Lie group is called *minimal*, if it minimizes the norm of the Ricci tensor among all left invariant metrics with the same scalar curvature. Such metrics are unique up to isometry and scaling and the groups admitting a minimal metric are precisely the nilradicals of (standard) Einstein solvmanifolds. If N is endowed with an invariant symplectic, complex or hypercomplex structure, then minimal compatible metrics are also unique up to isometry and scaling. The aim of this paper is to give more evidence of the existence of minimal metrics, by presenting several explicit examples. This also provides many continuous families of symplectic, complex and hypercomplex nilpotent Lie groups. A list of all known examples of Einstein solvmanifolds is also given.

1. Introduction

A nilpotent Lie group N can never admit an Einstein left invariant metric, unless it is abelian. A way of getting as close as possible to this would be by defining a left invariant metric $\langle\cdot,\cdot\rangle$ on N to be *minimal* if

$$||\operatorname{ric}_{\langle\cdot,\cdot\rangle}|| = \min\left\{||\operatorname{ric}_{\langle\cdot,\cdot\rangle'}|| : \operatorname{sc}(\langle\cdot,\cdot\rangle') = \operatorname{sc}(\langle\cdot,\cdot\rangle)\right\},$$

where $\langle\cdot,\cdot\rangle'$ runs over all left invariant metrics on N and $\operatorname{ric}_{\langle\cdot,\cdot\rangle}$, $\operatorname{sc}(\langle\cdot,\cdot\rangle)$ denote the Ricci tensor and the scalar curvature , respectively. Indeed,

$$||\operatorname{ric}_{\langle\cdot,\cdot\rangle} - (1/n)\operatorname{sc}(\langle\cdot,\cdot\rangle)\langle\cdot,\cdot\rangle||^2 = ||\operatorname{ric}_{\langle\cdot,\cdot\rangle}||^2 - \operatorname{sc}(\langle\cdot,\cdot\rangle)^2/n.$$

A left invariant metric is always identified with the corresponding inner product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{n}$ of N. The following conditions on $\langle\cdot,\cdot\rangle$ are equivalent to minimality and show that such metrics are special from many other points of view:

(i) $\langle\cdot,\cdot\rangle$ is a Ricci soliton metric: the solution $\langle\cdot,\cdot\rangle_t$ with initial point $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle$ to the normalized Ricci flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\cdot,\cdot\rangle_t = -2\operatorname{ric}_{\langle\cdot,\cdot\rangle_t} - 2||\operatorname{ric}_{\langle\cdot,\cdot\rangle_t}||^2\langle\cdot,\cdot\rangle_t,$$

(under which $\operatorname{sc}(\langle\cdot,\cdot\rangle_t)$ is constant in time) remains isometric to $\langle\cdot,\cdot\rangle$, that is, $\langle\cdot,\cdot\rangle_t = \varphi_t^*\langle\cdot,\cdot\rangle$ for some one parameter group of diffeomorphisms $\{\varphi_t\}$ of N (see [**L3**]).

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(ii) $(N, \langle\cdot,\cdot\rangle)$ admits a standard metric solvable extension $(S, \langle\cdot,\cdot\rangle')$ which is Einstein: the Lie algebra $\mathfrak{s}$ of S is given by the orthogonal decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a}$ is abelian, $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $\langle\cdot,\cdot\rangle'|\mathfrak{n} \times \mathfrak{n} = \langle\cdot,\cdot\rangle$ (see Section 6).

(iii) $\langle\cdot,\cdot\rangle$ is a quasi-Einstein metric:

$$\mathrm{ric}_{\langle\cdot,\cdot\rangle} = c\langle\cdot,\cdot\rangle + L_X\langle\cdot,\cdot\rangle$$

for some C^∞ vector field X in N and $c \in \mathbb{R}$, where $L_X\langle\cdot,\cdot\rangle$ denotes the usual Lie derivative (see [**L3**]).

(iv) $\mathrm{Ric}_{\langle\cdot,\cdot\rangle} = cI + D$ for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{n})$, where $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}$ is the Ricci operator and $\mathrm{Der}(\mathfrak{n})$ is the space of all derivations of $\mathfrak{n}$.

The uniqueness up to isometry and scaling of minimal metrics on a given nilpotent Lie group was proved in [**L3**]. The existence question is still nebulous; the only known obstruction until now is that $\mathfrak{n}$ has to admit an $\mathbb{N}$-gradation, that is, a direct sum decomposition $\mathfrak{n} = \mathfrak{n}_{k_1} \oplus ... \oplus \mathfrak{n}_{k_r}$, $k_i \in \mathbb{N}$, such that $[\mathfrak{n}_{k_i}, \mathfrak{n}_{k_j}] \subset \mathfrak{n}_{k_i+k_j}$. Such gradation is defined by the symmetric derivation D from condition (iv) above, which was proved to have natural eigenvalues (up to scaling) by J. Heber [**Hb**] in the context of Einstein solvmanifolds.

Problem. Does every $\mathbb{N}$-graded nilpotent Lie algebra admit a minimal metric?

In view of equivalence (ii) above, what we are wondering is if any $\mathbb{N}$-graded nilpotent Lie group can be the nilradical of a standard Einstein solvmanifold. It is perhaps too optimistic to expect an affirmative answer to this question, even in the two-step nilpotent case. The interplay with Einstein solvmanifolds provides a lot of examples of nilpotent Lie groups admitting minimal metrics. The rather long list of known examples given in Section 6 shows that although the answer to the above question might be no, there is a great deal of nilpotent Lie groups admitting a minimal metric and also, most of those which are distinguished in some way do so.

The search for a canonical metric also makes sense when there is a given invariant geometric structure on N, as for example a symplectic, complex or hypercomplex structure. With these structures in mind, we can define an invariant geometric structure as a tensor on N defined by left translation of a tensor γ on $\mathfrak{n}$ (or a set of tensors), usually non-degenerate in some way, which satisfies a suitable integrability condition

$$\mathrm{IC}(\gamma, \mu) = 0, \tag{1}$$

involving only γ and the Lie bracket μ of $\mathfrak{n}$. This will allow us to study these three classes of structures and maybe some other ones of similar characteristics at the same time.

The pair (N, γ) will be called a *class-γ nilpotent Lie group*, and N will be assumed to be simply connected for simplicity. A left invariant Riemannian metric on N is said to be *compatible* with (N, γ) if the corresponding inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ satisfies an orthogonality condition

$$\mathrm{OC}(\gamma, \langle\cdot,\cdot\rangle) = 0, \tag{2}$$

in which only $\langle\cdot,\cdot\rangle$ and γ are involved. We denote by $\mathcal{C} = \mathcal{C}(N, \gamma)$ the set of all left invariant metrics on N which are compatible with (N, γ). The pair $(\gamma, \langle\cdot,\cdot\rangle)$ with $\langle\cdot,\cdot\rangle \in \mathcal{C}$ will often be referred to as a *class-γ metric structure*.

The Ricci tensor has always been a very useful tool to deal with the existence of distinguished metrics, and since the geometric structure under consideration should be involved in the definition of such a metric, we consider the *invariant Ricci operator* $\mathrm{Ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle}$ (and the *invariant Ricci tensor* $\mathrm{ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle} = \langle \mathrm{Ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle}\cdot,\cdot\rangle$), that is, the orthogonal projection of the Ricci operator $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}$ onto the subspace of those symmetric maps of $\mathfrak{n}$ leaving γ invariant. D. Blair, S. Ianus and A. Ledger [**BI, BL, B**] have proved in the compact case that metrics satisfying

$$\mathrm{ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle} = 0 \tag{3}$$

are very special in symplectic (so called metrics with hermitian Ricci tensor) and contact geometry, as they are precisely the critical points of two very natural curvature functionals on $\mathcal{C}$: the total scalar curvature functional S and a functional K measuring how far are the metrics of being Kähler or Sasakian, respectively (see Section 3 for further information).

Unfortunately, for a non-abelian nilpotent Lie group, condition (3) is too strong for the classes of structures we have in mind, and hence it is natural to try to get as close as possible to this unattainable goal. In this light, a metric $\langle\cdot,\cdot\rangle \in \mathcal{C}(N,\gamma)$ is called *minimal* if it minimizes the functional $||\,\mathrm{ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle}\,||^2 = \mathrm{tr}(\mathrm{Ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle})^2$ on the set of all compatible metrics with the same scalar curvature. Recall that for $\gamma = 0$ (i.e. when we are not considering any structure) the Ricci invariant tensor coincides with the usual Ricci tensor and so we get precisely minimal metrics as defined at the beginning of this section.

We may also try to improve the metric via the evolution flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\cdot,\cdot\rangle_t = \pm\,\mathrm{ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle_t}, \tag{4}$$

whose fixed points are precisely metrics satisfying (3) (the choice of the right sign depend on the class of structure). In the symplectic case, this flow is called the anticomplexified Ricci flow and has been recently studied by H-V Le and G. Wang [**LW**]. Of particular significance are then those metrics for which the solution to the normalized flow (under which the scalar curvature is constant in time) remains isometric in time to the initial metric. Such special metrics will be called *invariant Ricci solitons*. The following theorem was obtained by using deep results from geometric invariant theory concerning the moment map for a linear action of a reductive Lie group (see Section 2).

THEOREM 1. [**L8**] *Let (N,γ) be a nilpotent Lie group endowed with an invariant geometric structure γ (non-necessarily integrable). Then the following conditions on a left invariant Riemannian metric $\langle\cdot,\cdot\rangle$ which is compatible with (N,γ) are equivalent:*

(i) *$\langle\cdot,\cdot\rangle$ is minimal.*
(ii) *$\langle\cdot,\cdot\rangle$ is an invariant Ricci soliton.*
(iii) *$\mathrm{Ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle} = cI + D$ for some $c \in \mathbb{R}$, $D \in \mathrm{Der}(\mathfrak{n})$.*

Moreover, there is at most one compatible left invariant metric on (N,γ) up to isometry (and scaling) satisfying any of the above conditions.

COROLLARY 1. [**L8**] *Let γ, γ' be two geometric structures on a nilpotent Lie group N, and assume that they admit minimal compatible metrics $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$, respectively. Then γ is isomorphic to γ' if and only if there exists $\varphi \in \mathrm{Aut}(\mathfrak{n})$ and*

$c>0$ such that $\gamma'=\varphi.\gamma$ and

$$\langle\varphi X,\varphi Y\rangle' = c\langle X,Y\rangle \qquad \forall X,Y\in\mathfrak{n}.$$

In particular, if γ and γ' are isomorphic then their respective minimal compatible metrics are necessarily isometric up to scaling (recall that $c=1$ when $\mathrm{sc}(\langle\cdot,\cdot\rangle)=\mathrm{sc}(\langle\cdot,\cdot\rangle')$).

A major obstacle to classify geometric structures is the lack of invariants. The uniqueness result in the above theorem and its corollary gives rise to a useful tool to distinguish two geometric structures; indeed, if they are isomorphic then their respective minimal compatible metrics (if any) have to be isometric. One therefore can eventually distinguish geometric structures with Riemannian data, which suddenly provides us with a great deal of invariants. This will be used in this paper to give explicit continuous families of pairwise non-isomorphic geometric structures in low dimensions, mainly by using only one Riemannian invariant: the eigenvalues of the Ricci operator. To actually find the candidates for such families we apply a variational method which is explained in Section 2.

Existence of minimal compatible metrics is proved for all 4-dimensional symplectic structures and a curve in dimension 6, two curves of abelian complex structures on the Iwasawa manifold and several continuous families depending on various parameters of abelian and non-abelian hypercomplex structures in dimension 8. It is finally showed in Section 6 that if one considers no structure (i.e. $\gamma=0$), then the 'moment map' approach proposed in [**L8**] can be also applied to the study of Einstein solvmanifolds, obtaining many of the uniqueness and structure results proved by J. Heber in [**Hb**].

The existence problem is also far to be solved in this case; the theorem does not even suggest when such a distinguished metric does exist. How special are the symplectic or complex structures admitting a minimal metric?. So far, we know how to deal with this 'existence question' only by giving several explicit examples, which is the aim of this paper. The neat 'algebraic' characterization (iii) in Theorem 1 will be very useful. It turns out that in low dimensions the structures in general tend to admit a minimal compatible metric. At the moment, the only counterexamples we have to the existence question are the characteristically nilpotent Lie algebras (i.e. $\mathrm{Der}(\mathfrak{n})$ is nilpotent) admitting a symplectic structure recently found by D. Burde in [**Bu**].

Remark 1. In [**L9**], by taking advantage of the interplay with invariant theory, we describe the moduli space of all isomorphism classes of geometric structures on nilpotent Lie groups of a given class and dimension admitting a minimal compatible metric, as the disjoint union of semi-algebraic varieties which are homeomorphic to categorical quotients of suitable linear actions of reductive Lie groups. Such special geometric structures can therefore be distinguished by using invariant polynomials.

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2. Variety of compatible metrics

Let us consider as a parameter space for the set of all real nilpotent Lie algebras of a given dimension n, the set $\mathcal{N}$ of all nilpotent Lie brackets on a fixed n-dimensional real vector space $\mathfrak{n}$. If

$$V = \Lambda^2\mathfrak{n}^* \otimes \mathfrak{n} = \{\mu : \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{n} : \mu \text{ skew-symmetric bilinear map}\},$$

then

$$\mathcal{N} = \{\mu \in V : \mu \text{ satisfies Jacobi and is nilpotent}\}$$

is an algebraic subset of V. Indeed, the Jacobi identity and the nilpotency condition are both determined by zeroes of polynomials.

We fix a tensor γ on $\mathfrak{n}$ (or a set of tensors), and let G_γ denote the subgroup of $\Lambda(n)$ preserving γ. These groups act naturally on V by

$$g.\mu(X,Y) = g\mu(g^{-1}X, g^{-1}Y), \qquad X,Y \in \mathfrak{n}, \quad g \in \Lambda(n), \quad \mu \in V, \tag{5}$$

and leave $\mathcal{N}$ invariant. Consider the subset $\mathcal{N}_\gamma \subset \mathcal{N}$ given by

$$\mathcal{N}_\gamma = \{\mu \in \mathcal{N} : \mathrm{IC}(\gamma,\mu) = 0\},$$

that is, those nilpotent Lie brackets for which γ is integrable (see (1)). $\mathcal{N}_\gamma$ is also an algebraic variety since $\mathrm{IC}(\gamma,\mu)$ is always linear on μ (at least in the cases we have in mind: symplectic, complex and hypercomplex). Recall that

$$W_\gamma = \{\mu \in V : \mathrm{IC}(\gamma,\mu) = 0\}$$

is a G_γ-invariant linear subspace of V, and $\mathcal{N}_\gamma = \mathcal{N} \cap W_\gamma$.

For each $\mu \in \mathcal{N}$, let N_μ denote the simply connected nilpotent Lie group with Lie algebra $(\mathfrak{n},\mu)$. Fix an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ compatible with γ, that is, such that (2) holds. We identify each $\mu \in \mathcal{N}_\gamma$ with a class-γ metric structure

$$\mu \longleftrightarrow (N_\mu, \gamma, \langle\cdot,\cdot\rangle), \tag{6}$$

where all the structures are defined by left invariant translation. Therefore, each $\mu \in \mathcal{N}_\gamma$ can be viewed in this way as a metric compatible with the class-γ nilpotent Lie group (N_μ, γ), and two metrics μ, λ are compatible with the same geometric structure if and only if they live in the same G_γ-orbit. Indeed, the action of G_γ on $\mathcal{N}_\gamma$ has the following interpretation: each $\varphi \in G_\gamma$ determines a Riemannian isometry preserving the geometric structure

$$(N_{\varphi.\mu}, \gamma, \langle\cdot,\cdot\rangle) \mapsto (N_\mu, \gamma, \langle\varphi\cdot,\varphi\cdot\rangle)$$

by exponentiating the Lie algebra isomorphism $\varphi^{-1} : (\mathfrak{n}, \varphi.\mu) \mapsto (\mathfrak{n}, \mu)$. We then have the identification $G_\gamma.\mu = \mathcal{C}(N_\mu, \gamma)$, and more in general the following

Proposition 1. [**L8**] *Every class-γ metric structure $(N', \gamma', \langle\cdot,\cdot\rangle')$ on a nilpotent Lie group N' of dimension n is isometric-isomorphic to a $\mu \in \mathcal{N}_\gamma$.*

According to the above proposition and identification (6), the orbit $G_\gamma.\mu$ parameterizes all the left invariant metrics which are compatible with (N_μ, γ) and hence we may view $\mathcal{N}_\gamma$ as the space of all class-γ metric structures on nilpotent Lie groups of dimension n. Since two metrics $\mu, \lambda \in \mathcal{N}_\gamma$ are isometric if and only if they live in the same K_γ-orbit, where $K_\gamma = G_\gamma \cap \mathrm{O}(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ (see [**L8**, Appendix]), we have that $\mathcal{N}_\gamma/K_\gamma$ parameterizes class-γ metric nilpotent Lie groups of dimension n up to isometry and $G_\gamma.\mu/K_\gamma$ do the same for all the compatible metrics on (N_μ, γ).

In the search for the best compatible metric, it is natural to consider the functional $F : \mathcal{N}_\gamma \mapsto \mathbb{R}$ given by $F(\mu) = \mathrm{tr}(\mathrm{Ric}^\gamma{}_\mu)^2$, which in some sense measures how far the metric μ is from satisfying (3). The critical points of $F/||\mu||^4$ on the projective algebraic variety $\mathbb{P}\mathcal{N}_\gamma \subset \mathbb{P}V$ (which is equivalent to normalize by the scalar curvature since $\mathrm{sc}(\mu) = -\frac{1}{4}||\mu||^2$), may therefore be considered compatible metrics of particular significance.

A crucial fact of this approach is that the moment map $m_\gamma : V \mapsto \mathfrak{p}_\gamma$ for the action of G_γ on V is proved to be

$$m_\gamma(\mu) = 8\,\mathrm{Ric}^\gamma{}_\mu, \qquad \forall\, \mu \in \mathcal{N}_\gamma,$$

where $\mathfrak{p}_\gamma$ is the space of symmetric maps of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ leaving γ invariant (i.e. $\mathfrak{g}_\gamma = \mathfrak{k}_\gamma \oplus \mathfrak{p}_\gamma$ is a Cartan decomposition). This allows us to use strong and well-known results on the moment map due to F. Kirwan [**K**] and L. Ness [**N**], and proved by A. Marian [**M**] in the real case (we also refer to [**L8**, Section 3] and [**L7**] for further information). Indeed, since F becomes a scalar multiple of the square norm of the moment map, we obtain the following

THEOREM 2. [**M**] *Let $F : \mathbb{P}\mathcal{N}_\gamma \mapsto \mathbb{R}$ be defined by $F([\mu]) = \mathrm{tr}(\mathrm{Ric}^\gamma{}_\mu)^2/||\mu||^4$. Then for $\mu \in \mathcal{N}_\gamma$ the following conditions are equivalent:*

(i) *$[\mu]$ is a critical point of F.*
(ii) *$F|_{G_\gamma.[\mu]}$ attains its minimum value at $[\mu]$.*
(iii) *$\mathrm{Ric}^\gamma{}_\mu = cI + D$ for some $c \in \mathbb{R}$, $D \in \mathrm{Der}(\mu)$.*

Moreover, all the other critical points of F in the orbit $G_\gamma.[\mu]$ lie in $K_\gamma.[\mu]$.

The equivalence between (i) and (iii) in Theorem 1, as well as the uniqueness result, follow then almost directly from the above theorem. We note that Theorem 2 also gives a variational method to find minimal compatible metrics, by characterizing them as the critical points of a natural curvature functional (see [**L8**, Example 5.2] for an explicit application).

Most of the results obtained in this paper are still valid for general Lie groups, although some considerations have to be carefully taken into account (see [**L8**, Remark 4.6]).

3. Symplectic structures

Let (M, ω) be a symplectic manifold, that is, a differentiable manifold M endowed with a global 2-form ω which is closed ($\mathrm{d}\,\omega = 0$) and non-degenerate ($\omega^n \neq 0$). A Riemannian metric g on M is said to be *compatible* with ω if there exists an almost-complex structure J_g (i.e. a $(1,1)$-tensor field with $J_g^2 = -I$) such that

$$\omega = g(\cdot, J_g\cdot).$$

In that case J_g is uniquely determined by g, and one may also define that an almost-complex structure J is *compatible* with ω if

$$g_J = \omega(\cdot, J\cdot)$$

determines a Riemannian metric, which is again uniquely determined by J. In such a way we are really talking about compatible pairs $(g, J) = (g, J_g) = (g_J, J)$, and the triple (ω, g, J_g) is called an *almost-Kähler* structure on M.

It is well known that for any symplectic manifold there always exist a compatible metric. Moreover, the space $\mathcal{C} = \mathcal{C}(M, \omega)$ of all compatible metrics is usually huge; recall for instance that the group of all symplectomorphisms (i.e. diffeomorphisms φ of M such that $\varphi^*\omega = \omega$) acts on $\mathcal{C}$.

We fix from now on a symplectic manifold (M, ω). Let Ric_g and ∇_g denote the Ricci operator and the Levi-Civita conexion of a compatible metric g, respectively.

The most famous conditions to ask g to satisfy are Einstein (i.e. $\mathrm{Ric}_g = cI$) and Kähler (i.e. $\nabla_g J_g = 0$), which are both very strong and share the following property.

DEFINITION 1. We say that g has *hermitian* (or *J-invariant*) *Ricci tensor* or that J_g is *harmonic*, if $\mathrm{Ric}_g J_g = J_g \mathrm{Ric}_g$. Examples of compatible metrics with hermitian Ricci tensor which are neither Einstein nor Kähler are known in any dimension $2n \geqslant 6$ (see [**AG, DM, LW**]). It is proved in [**L8**] that a symplectic nilpotent Lie group can never admit a compatible metric with hermitian Ricci tensor unless it is abelian.

A classical approach to searching for distinguished metrics is the variational one, that is, to consider critical points of natural functionals of the curvature on the space of all metrics of a given class. For instance, if M is compact, D. Hilbert [**Hl**] proved that Einstein metrics on M are precisely the critical points of the total scalar curvature functional

$$S : \mathcal{M}_1 \mapsto \mathbb{R}, \qquad S(g) = \int_M \mathrm{sc}(g)\, \mathrm{d}\,\nu_g,$$

where $\mathcal{M}_1$ is the space of all Riemannian metrics on M with volume equal to 1. Since the set of compatible metrics $\mathcal{C}$ is smaller, one should expect a weaker critical point condition for $S : \mathcal{C} \mapsto \mathbb{R}$. Another natural functional in our setup would be

$$K : \mathcal{C} \mapsto \mathbb{R}, \qquad K(g) = \int_M ||\nabla_g J_g||^2\, \mathrm{d}\,\nu_g,$$

for which Kähler metrics are precisely the global minima. D. Blair and S. Ianus proved that, curiously enough, both functionals S and K have the same critical points on $\mathcal{C}$.

THEOREM 3. [**BI**] *Let (M, ω) be a compact symplectic manifold and $\mathcal{C}$ the set of all compatible metrics. Then $g \in \mathcal{C}$ is a critical point of $S : \mathcal{C} \mapsto \mathbb{R}$ or $K : \mathcal{C} \mapsto \mathbb{R}$ if and only if g has hermitian Ricci tensor.*

This result and the above considerations do suggest that the compatible metrics with hermitian Ricci tensor (if any) are really 'good friends' of the symplectic structure.

In [**LW**], H-V Le and G. Wang approach the problem of the existence of such metrics by considering an evolution flow inspired in the Ricci flow introduced by R. Hamilton [**H**]. If ric_g is the Ricci tensor of a compatible metric g, then consider the orthogonal decomposition

$$\mathrm{ric}_g = \mathrm{ric^{ac}}_g + \mathrm{ric^{c}}_g, \tag{7}$$

where $\mathrm{ric^{ac}}_g = \frac{1}{2}(\mathrm{ric}_g - \mathrm{ric}_g(J_g\cdot, J_g\cdot))$ and $\mathrm{ric^{c}}_g = \frac{1}{2}(\mathrm{ric}_g + \mathrm{ric}_g(J_g\cdot, J_g\cdot))$ are the *anti-complexified* and *complexified* parts of ric_g, respectively. In this way, g has hermitian Ricci tensor if and only if $\mathrm{ric^{ac}}_g = 0$, and since the gradient of the functional K equals $-\mathrm{ric^{ac}}_g$ it is natural to consider the negative gradient flow equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \mathrm{ric^{ac}}_{g(t)}, \tag{8}$$

for a curve $g(t)$ of metrics, which is called in [**LW**] the *anti-complexified Ricci flow*. Recall that the fixed points of (8) are precisely the metrics with hermitian Ricci tensor. The main result in [**LW**] is the short time existence and uniqueness of the solution to (8) when M is compact.

Let N be a real $2n$-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n}$, whose Lie bracket is denoted by $\mu : \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{n}$. An invariant *symplectic* structure on N is defined by a 2-form ω on $\mathfrak{n}$ satisfying

$$\omega(X, \cdot) \equiv 0 \quad \text{if and only if} \quad X = 0 \quad \text{(non-degenerate)},$$

and for all $X, Y, Z \in \mathfrak{n}$

$$\omega(\mu(X,Y),Z) + \omega(\mu(Y,Z),X) + \omega(\mu(Z,X),Y) = 0 \quad (\text{closed}, \mathrm{d}\,\omega = 0). \tag{9}$$

Fix a symplectic nilpotent Lie group (N, ω). A left invariant Riemannian metric which is compatible with (N, ω) is determined by an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ such that if

$$\omega(X, Y) = \langle X, J_{\langle \cdot, \cdot \rangle} Y \rangle \quad \forall\, X, Y \in \mathfrak{n} \quad \text{then} \quad J_{\langle \cdot, \cdot \rangle}^2 = -I.$$

For the geometric structure $\gamma = \omega$ we have that

$$G_\gamma = \mathrm{S}p(n, \mathbb{R}) = \{g \in \Lambda(2n) : g^t J g = J\}, \qquad K_\gamma = \mathrm{U}(n),$$

and the Cartan decomposition of $\mathfrak{g}_\gamma = \mathfrak{sp}(n, \mathbb{R}) = \{A \in \mathfrak{gl}(2n) : A^t J + JA = 0\}$ is given by

$$\mathfrak{sp}(n, \mathbb{R}) = \mathfrak{u}(n) \oplus \mathfrak{p}_\gamma, \qquad \mathfrak{p}_\gamma = \{A \in \mathfrak{p} : AJ = -JA\}.$$

Thus the invariant Ricci tensor ric^γ coincides with the anti-complexified Ricci tensor (see [**LW**]) and for any $\langle \cdot, \cdot \rangle \in \mathcal{C}$,

$$\mathrm{Ric}^\gamma{}_{\langle \cdot, \cdot \rangle} = \mathrm{Ric}^{\mathrm{ac}}{}_{\langle \cdot, \cdot \rangle} = \frac{1}{2} \left(\mathrm{Ric}_{\langle \cdot, \cdot \rangle} + J_{\langle \cdot, \cdot \rangle} \mathrm{Ric}_{\langle \cdot, \cdot \rangle} J_{\langle \cdot, \cdot \rangle} \right). \tag{10}$$

This implies that our 'goal' condition $\mathrm{Ric}^\gamma{}_{\langle \cdot, \cdot \rangle} = 0$ (see (3)) is equivalent to have hermitian Ricci tensor. Also, the evolution flow (4) is not other than the anti-complexified Ricci flow.

We now review the variational approach developed in Section 2. Fix a non-degenerate 2-form ω on $\mathfrak{n}$, and let $\mathrm{S}p(n, \mathbb{R})$ denote the subgroup of $\Lambda(2n)$ preserving ω, that is,

$$\mathrm{S}p(n, \mathbb{R}) = \{\varphi \in \Lambda(2n) : \omega(\varphi X, \varphi Y) = \omega(X, Y) \quad \forall\, X, Y \in \mathfrak{n}\}.$$

Consider the algebraic subvariety $\mathcal{N}_s := \mathcal{N}_\gamma \subset \mathcal{N}$ given by

$$\mathcal{N}_s = \{\mu \in \mathcal{N} : \mathrm{d}_\mu \omega = 0\},$$

that is, those nilpotent Lie brackets for which ω is closed (see (9)). By fixing an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ satisfying that

$$\omega = \langle \cdot, J \cdot \rangle \qquad \text{with} \qquad J^2 = -I,$$

(6) identifies each $\mu \in \mathcal{N}_s$ with the almost-Kähler manifold $(N_\mu, \omega, \langle \cdot, \cdot \rangle, J)$. The action of $\mathrm{S}p(n, \mathbb{R})$ on $\mathcal{N}_s$ has the following interpretation: each $\varphi \in \mathrm{S}p(n, \mathbb{R})$ determines a Riemannian isometry which is also a symplectomorphism

$$(N_{\varphi.\mu}, \omega, \langle \cdot, \cdot \rangle, J) \mapsto (N_\mu, \omega, \langle \varphi \cdot, \varphi \cdot \rangle, \varphi^{-1} J \varphi)$$

by exponentiating the Lie algebra isomorphism $\varphi^{-1} : (\mathfrak{n}, \varphi.\mu) \mapsto (\mathfrak{n}, \mu)$.

Let $\mathfrak{n}$ be a $2n$-dimensional vector space with basis $\{X_1, ..., X_{2n}\}$ over $\mathbb{R}$, and consider the non-degenerate 2-form

$$\omega = \alpha_1 \wedge \alpha_{2n} + ... + \alpha_n \wedge \alpha_{n+1},$$

where $\{\alpha_1, ..., \alpha_{2n}\}$ is the dual basis of $\{X_i\}$. For the compatible inner product $\langle X_i, X_j\rangle = \delta_{ij}$ we have that $\omega = \langle \cdot, J\cdot\rangle$ for

$$J = \begin{bmatrix} & 0 & & & \cdot^{\cdot^{\cdot}} & -1 \\ & & 1 & -1 & & \\ 1 & \cdot^{\cdot^{\cdot}} & & & 0 & \end{bmatrix}.$$

In all the following examples the symplectic structure will be ω, the almost-complex structure J and the compatible metric $\langle \cdot, \cdot\rangle$. We will vary Lie brackets and use constantly identification (6).

Example. Let μ_n the $2n$-dimensional Lie algebra whose only non-zero bracket is

$$\mu_n(X_1, X_2) = X_3,$$

that is, μ_n is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^{2n-3}$, where $\mathfrak{h}_3$ is the 3-dimensional Heisenberg Lie algebra. Recall that (N_{μ_2}, ω) is precisely the simply connected cover of the famous Kodaira-Thurston manifold. It is easy to prove that $\overline{\mathrm{S}p(n, \mathbb{R}).\mu_n} = \mathrm{S}p(n, \mathbb{R}).\mu_n \cup \{0\}$, and so the orbit $\mathrm{S}p(n, \mathbb{R}).[\mu_n]$ is closed in $\mathbb{P}V$. This implies that the functional F from Theorem 2 must attain its minimum value on $\mathrm{S}p(n, \mathbb{R}).[\mu_n]$ and hence there exists a metric compatible with (N_{μ_n}, ω) which is minimal. In fact, the inner product $\langle X_i, X_j\rangle = \delta_{ij}$ satisfies

$$\mathrm{Ric^{ac}}_{\langle\cdot,\cdot\rangle} = -\tfrac{1}{4}\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & -1 & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & -1 & \\ & & & & & & & & -1 \end{bmatrix} = -\tfrac{3}{4}I + \tfrac{1}{4}\begin{bmatrix} 2 & & & & & & & & \\ & 2 & & & & & & & \\ & & 4 & & & & & & \\ & & & 3 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 3 & & & \\ & & & & & & 2 & & \\ & & & & & & & 4 & \\ & & & & & & & & 4 \end{bmatrix}, \qquad 2n \geqslant 8,$$

$$\mathrm{Ric^{ac}}_{\langle\cdot,\cdot\rangle} = -\tfrac{1}{4}\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{bmatrix} = -\tfrac{3}{4}I + \tfrac{1}{4}\begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 4 & & & \\ & & & 2 & & \\ & & & & 4 & \\ & & & & & 4 \end{bmatrix}, \qquad 2n = 6,$$

$$\mathrm{Ric^{ac}}_{\langle\cdot,\cdot\rangle} = -\tfrac{1}{4}\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & -2 & \\ & & & -1 \end{bmatrix} = -\tfrac{5}{4}I + \tfrac{1}{4}\begin{bmatrix} 4 & & & \\ & 3 & & \\ & & 7 & \\ & & & 6 \end{bmatrix}, \qquad 2n = 4,$$

and hence $\mathrm{Ric^{ac}}_{\langle\cdot,\cdot\rangle} \in \mathbb{R}I + \mathrm{Der}(\mu_n)$ in all the cases. Moreover, it follows from the closeness of $\mathrm{S}p(n, \mathbb{R}).[\mu_n]$ that F must also attain its maximum value, and therefore $\mathrm{S}p(n, \mathbb{R}).[\mu_n] = \mathrm{U}(n).[\mu_n]$ by uniqueness in Theorem 2. This implies that there is only one left invariant metric compatible with (N_{μ_n}, ω) up to isometry, often called the Abbena metric in the case $n = 2$.

Example. Consider the 4-dimensional Lie algebra given by

$$\lambda(X_1, X_2) = X_3, \qquad \lambda(X_1, X_3) = X_4.$$

The compatible metric $\langle X_i, X_j\rangle = \delta_{ij}$ is minimal for (N_λ, ω) since

$$\mathrm{Ric^{ac}}_{\langle\cdot,\cdot\rangle} = -\frac{1}{4}\begin{bmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{bmatrix} = -\frac{5}{4}I + \frac{1}{2}\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{bmatrix} \in \mathbb{R}I + \mathrm{Der}(\lambda).$$

It is well-known that (N_{μ_2}, ω) and (N_λ, ω) are the only symplectic nilpotent Lie groups in dimension 4, and then the existence of minimal compatible metrics in the case $2n = 4$ follows.

Example. Let $\mu = \mu(a,b,c)$ be the 6-dimensional 2-step nilpotent Lie algebra defined by

$$\mu(X_1,X_2) = aX_4, \quad \mu(X_1,X_3) = bX_5, \quad \mu(X_2,X_3) = cX_6.$$

It is easy to check that $\mu \in \mathcal{N}_s$ if and only if $a-b+c=0$. We can also get from a simple calculation that

$$\begin{aligned} \mathrm{Ric^{ac}}_\mu &= -\tfrac{1}{4}(a^2+b^2+c^2)\begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& -1 && \\ &&&& -1 & \\ &&&&& -1 \end{bmatrix} \\ &= -\tfrac{3}{4}(a^2+b^2+c^2)I + \tfrac{1}{2}(a^2+b^2+c^2)\begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 2 && \\ &&&& 2 & \\ &&&&& 2 \end{bmatrix} \in \mathbb{R}I + \mathrm{Der}(\mu), \end{aligned}$$

and so the whole family $\{[\mu(a,b,c)] : a-b+c=0\} \subset \mathbb{P}\mathcal{N}_s$ consists of critical points of F. We assume that $a^2+b^2+c^2=2$ in order to avoid homothetical changes, which is equivalent to $\mathrm{sc}(\mu) = -1$. The Ricci operator on the center $\mathfrak{z} = \langle X_4, X_5, X_6\rangle_{\mathbb{R}}$ is given by

$$\mathrm{Ric}_\mu = \frac{1}{2}\begin{bmatrix} a^2 && \\ & b^2 & \\ && c^2 \end{bmatrix},$$

and thus the curve $\{\mu_{st} = \mu(s,s+t,t) : s^2+st+t^2 = 1,\ 0 \leqslant t \leqslant \frac{1}{\sqrt{3}}\}$ is pairwise non-isometric. It then follows from Corollary 1 that $(N_{\mu_{st}}, \omega)$ is a curve of pairwise non-isomorphic symplectic nilpotent Lie groups. In terms of the notation in [**S**, Table A.1], we have that $\mu_{01} \simeq (0,0,0,0,12,13)$ and $\mu_{st} \simeq (0,0,0,12,13,23)$ for any $0 < t \leqslant \frac{1}{\sqrt{3}}$. We note that this curve coincides with the curve of pairwise non-isomorphic symplectic structures denoted by $\omega_1(t)$ in [**KGM**, Theorem 3.1, 18], and then this example shows that any symplectic structure in such a curve admits a compatible metric which is minimal

4. Complex structures

Let N be a real $2n$-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n}$, whose Lie bracket is denoted by $\mu : \mathfrak{n}\times\mathfrak{n} \mapsto \mathfrak{n}$. An invariant *almost-complex* structure on N is defined by a map $J : \mathfrak{n} \mapsto \mathfrak{n}$ satisfying $J^2 = -I$. If in addition J satisfies the integrability condition

$$\mu(JX,JY) = \mu(X,Y) + J\mu(JX,Y) + J\mu(X,JY), \qquad \forall X,Y \in \mathfrak{n}, \tag{11}$$

then J is said to be a *complex* structure.

Fix an almost-complex nilpotent Lie group (N,J). A left invariant Riemannian metric which is *compatible* with (N,J), also called an *almost-hermitian metric*, is given by an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ such that

$$\langle JX, JY\rangle = \langle X,Y\rangle \qquad \forall X,Y \in \mathfrak{n}.$$

We have for this particular geometric structure $\gamma = J$ that

$$G_\gamma = \Lambda(n,\mathbb{C}) = \{\varphi \in \Lambda(2n) : \varphi J = J\varphi\}, \qquad K_\gamma = \mathrm{U}(n),$$

and the Cartan decomposition of $\mathfrak{g}_\gamma = \mathfrak{gl}(n,\mathbb{C}) = \{A \in \mathfrak{gl}(2n) : AJ = JA\}$ is given by

$$\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p}_\gamma, \qquad \mathfrak{p}_\gamma = \{A \in \mathfrak{p} : AJ = JA\}.$$

The invariant Ricci operator is then given by the complexified Ricci operator

$$\mathrm{Ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle} = \mathrm{Ric}^{\mathrm{c}}{}_{\langle\cdot,\cdot\rangle} = \frac{1}{2}\left(\mathrm{Ric}_{\langle\cdot,\cdot\rangle} - J\,\mathrm{Ric}_{\langle\cdot,\cdot\rangle} J\right)$$

(see (7)). In this way, condition $\mathrm{Ric}^{\gamma}{}_{\langle\cdot,\cdot\rangle} = 0$ is equivalent to the Ricci operator anti-commute with J. We do not know if this property has any special significance in complex geometry, but for instance it holds for a Kähler metric if and only if the metric is Ricci flat. Anyway, as in the symplectic case, the condition $\mathrm{Ric}^{\mathrm{c}}{}_{\langle\cdot,\cdot\rangle} = 0$ is also forbidden for non-abelian N since $\mathrm{tr}\,\mathrm{Ric}^{\mathrm{c}}{}_{\langle\cdot,\cdot\rangle} = \mathrm{sc}(\langle\cdot,\cdot\rangle) < 0$.

We now fix a map $J : \mathfrak{n} \mapsto \mathfrak{n}$ satisfying $J^2 = -I$ and consider the algebraic subvariety $\mathcal{N}_c := \mathcal{N}_\gamma \subset \mathcal{N}$ given by

$$\mathcal{N}_c = \{\mu \in \mathcal{N} : (11) \text{ holds}\},$$

that is, those nilpotent Lie brackets for which J is integrable and so define a complex structure on N_μ, the simply connected nilpotent Lie group with Lie algebra $(\mathfrak{n}, \mu)$.

Fix also an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ compatible with J, then (6) identifies each $\mu \in \mathcal{N}_c$ (or $\mathcal{N}$) with the hermitian (or almost-hermitian) manifold $(N_\mu, J, \langle\cdot,\cdot\rangle)$. If we use the same triple $(\omega, J, \langle\cdot,\cdot\rangle)$ to define and identify $\mathcal{N}_s$ (see Section 3) and $\mathcal{N}_c$, then the intersection of these varieties is $\mathcal{N}_s \cap \mathcal{N}_c = \{0\}$ since no non-abelian nilpotent Lie group can admit a left invariant Kähler metric.

We now give some examples.

Example. Let μ_n be the $2n$-dimensional Lie algebra considered in Example 3. It is easy to check that $\langle\cdot,\cdot\rangle$ is also minimal as a compatible metric for the almost-complex nilpotent Lie group (N_{μ_n}, J). For the 4-dimensional Lie algebra in Example 3, we have that $\mathrm{Ric}^{\mathrm{c}}{}_{\langle\cdot,\cdot\rangle} = -\frac{1}{4}I$ and hence this metric is minimal for the almost-complex nilpotent Lie group (N_λ, J) as well.

For $\mathfrak{n}_1 = \mathbb{R}^4$ and $\mathfrak{n}_2 = \mathbb{R}^2$, consider the vector space $W = \Lambda^2\mathfrak{n}_1^* \otimes \mathfrak{n}_2$ of all skew-symmetric bilinear maps $\mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \mapsto \mathfrak{n}_2$. Any 6-dimensional 2-step nilpotent Lie algebra with $\dim \mu(\mathfrak{n}, \mathfrak{n}) \leqslant 2$ can be modelled in this way. Fix basis $\{X_1, ..., X_4\}$ and $\{Z_1, Z_2\}$ of $\mathfrak{n}_1$ and $\mathfrak{n}_2$, respectively. Each element in W will be described as $\mu = \mu(a_1, a_2, ..., f_1, f_2)$, where

$$\begin{array}{lll} \mu(X_1, X_2) = a_1Z_1 + a_2Z_2, & \mu(X_1, X_4) = c_1Z_1 + c_2Z_2, & \mu(X_2, X_4) = e_1Z_1 + e_2Z_2, \\ \mu(X_1, X_3) = b_1Z_1 + b_2Z_2, & \mu(X_2, X_3) = d_1Z_1 + d_2Z_2, & \mu(X_3, X_4) = f_1Z_1 + f_2Z_2. \end{array}$$

The complex structure and the compatible metric will be always defined by

$$J = \begin{bmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{bmatrix}, \qquad \langle X_i, X_j\rangle = \langle Z_i, Z_j\rangle = \delta_{ij}.$$

If $A = (a_1, a_2)$, ... , $F = (f_1, f_2)$ and $JA = (-a_2, a_1)$, ... , $JF = (-f_2, f_1)$, then it is easy to check that J is integrable on N_μ (or (N_μ, J) is a complex nilpotent Lie group) if and only if

$$E = B + JD + JC, \tag{12}$$

J is *bi-invariant* (i.e. $\mu(JX, Y) = J\mu(X, Y)$) if and only if

$$A = F = 0, \qquad C = D = JB, \qquad E = -B, \tag{13}$$

and J is *abelian* (i.e. $\mu(JX, JY) = \mu(X, Y)$) if and only if

$$E = B, \qquad D = -C. \tag{14}$$

We note that the above conditions determine $\Lambda(2,\mathbb{C})\times\Lambda(1,\mathbb{C})$-invariant linear subspaces of W of dimensions 10, 2 and 8, respectively. For each $\mu\in W$, it is easy to see that the Ricci operator of the almost-hermitian manifold $(N_\mu, J, \langle\cdot,\cdot\rangle)$ (see identification (6)) restricted to $\mathfrak{n}_1$, $\mathrm{Ric}_\mu|_{\mathfrak{n}_1}$, is given by

$$-\frac{1}{2}\begin{bmatrix} \|A\|^2+\|B\|^2+\|C\|^2 & \langle B,D\rangle+\langle C,E\rangle & -\langle A,D\rangle+\langle C,F\rangle & -\langle A,E\rangle-\langle B,F\rangle \\ \langle B,D\rangle+\langle C,E\rangle & \|A\|^2+\|D\|^2+\|E\|^2 & \langle A,B\rangle+\langle E,F\rangle & \langle A,C\rangle-\langle D,F\rangle \\ -\langle A,D\rangle+\langle C,F\rangle & \langle A,B\rangle+\langle E,F\rangle & \|B\|^2+\|D\|^2+\|F\|^2 & \langle B,C\rangle+\langle D,E\rangle \\ -\langle A,E\rangle-\langle B,F\rangle & \langle A,C\rangle-\langle D,F\rangle & \langle B,C\rangle+\langle D,E\rangle & \|C\|^2+\|E\|^2+\|F\|^2 \end{bmatrix} \tag{15}$$

and

$$\mathrm{Ric}_\mu|_{\mathfrak{n}_2}=\frac{1}{2}\begin{bmatrix} \|v_1\|^2 & \langle v_1,v_2\rangle \\ \langle v_1,v_2\rangle & \|v_2\|^2\end{bmatrix},\quad v_i=(a_i,b_i,c_i,d_i,e_i,f_i),\quad i=1,2.$$

Recall that if the complexified Ricci operator satisfied ${\mathrm{Ric}^{\mathrm{c}}}_\mu|_{\mathfrak{n}_1}=pI$, $p\in\mathbb{R}$, then μ is minimal. Indeed, since always ${\mathrm{Ric}^{\mathrm{c}}}_\mu|_{\mathfrak{n}_2}=qI$ for some $q\in\mathbb{R}$, we would have that

$${\mathrm{Ric}^{\mathrm{c}}}_\mu=\begin{bmatrix} pI & \\ & qI\end{bmatrix}=(2p-q)I+\begin{bmatrix} (q-p)I & \\ & 2(q-p)I\end{bmatrix}\in\mathbb{R}I+\mathrm{Der}(\mu). \tag{16}$$

In particular, any bi-invariant complex nilpotent Lie group (N_μ, J) (see (13)) admits a compatible metric which is minimal.

We will now focus on the abelian complex case (see (14)). It is not hard to see that these conditions imply that ${\mathrm{Ric}^{\mathrm{c}}}_\mu|_{\mathfrak{n}_1}=\mathrm{Ric}_\mu$, and so in this case, to get ${\mathrm{Ric}^{\mathrm{c}}}_\mu|_{\mathfrak{n}_1}\in\mathbb{R}I$ is necessary and sufficient that

$$\langle A+F,B\rangle=0,\qquad \langle A+F,C\rangle=0,\qquad \|A\|^2=\|F\|^2.$$

In order to avoid homothetical changes we will always ask for $\|v_1\|^2+\|v_2\|^2=2$, which is equivalent to $\mathrm{sc}(\mu)=-1$.

Example. If we put $A=(s,t)$, $F=(-s,t)$, $B=C=D=E=0$, $s^2+t^2=1$, then the corresponding curve μ_{st} of minimal compatible metrics satisfies

$$\mathrm{Ric}_{\mu_{st}}|_{\mathfrak{n}_2}=\begin{bmatrix} s^2 & 0\\ 0 & t^2\end{bmatrix},$$

proving that $\{\mu_{st}: s^2+t^2=1,\ 0\leqslant s\leqslant\frac{1}{\sqrt{2}}\}$ is a curve of pairwise non-isometric metrics. It then follows from Corollary 1 that $(N_{\mu_{st}}, J)$ is a curve of pairwise non-isomorphic abelian complex nilpotent Lie groups. Recall that $\mu_{st}\simeq\mathfrak{h}_3\oplus\mathfrak{h}_3$ for all $0<s$ and $\mu_{01}\simeq\mathfrak{h}_3\oplus\mathbb{R}^3$.

Example. For $A=(s,t)$, $F=(-s,t)$, $B=(\frac{1}{2},0)=C=-D=E$, $s^2+t^2=\frac{1}{2}$, the curve μ_{st} of minimal compatible metrics satisfies

$$\mathrm{Ric}_{\mu_{st}}|_{\mathfrak{n}_2}=\begin{bmatrix} s^2+\frac{1}{2} & 0\\ 0 & t^2\end{bmatrix},$$

which implies that the family $\{\mu_{st}: s^2+t^2=\frac{1}{2}\}$ is pairwise non-isometric. It is easy to see that for $t\neq 0$, μ_{st} is isomorphic to the complex Heisenberg Lie algebra, and hence $(N_{\mu_{st}}, J)$ defines a curve of pairwise non-isomorphic abelian complex structures on the Iwasawa manifold. Since $j_{\mu_{st}}(Z_2)^2\notin\mathbb{R}I$, we have that the hermitian manifolds $(N_{\mu_{st}}, J, \langle\cdot,\cdot\rangle)$ are not modified H-type (see [**L1**]).

Example. Consider the abelian complex structures defined by $A=-F$, $E=B$ and $D=-C$. In this case, the Hermitian manifolds $(N_\mu, J,\langle\cdot,\cdot\rangle)$ are modified H-type and μ is always isomorphic to the complex Heisenberg Lie algebra when $v_1, v_2\neq 0$. In fact, by assuming for simplicity that $\langle v_1,v_2\rangle=0$, then

$$j_\mu(Z)^2=-\frac{1}{2}(\langle Z,Z_1\rangle^2\|v_1\|^2+\langle Z,Z_2\rangle^2\|v_2\|^2)I,\qquad\forall Z\in\mathfrak{n}_2.$$

For $A = (s, 0) = -F$, $B = (0, t) = E$, $s^2 + t^2 = 1$, $D = C = 0$, the corresponding curve μ_{st} of minimal compatible metrics satisfies

$$\operatorname{Ric}_{\mu_{st}}|_{\mathfrak{n}_2} = \left[\begin{smallmatrix} s^2 & 0 \\ 0 & t^2 \end{smallmatrix}\right],$$

and so the family $\{\mu_{st} : s^2 + t^2 = 1,\ 0 \leqslant s \leqslant \frac{1}{\sqrt{2}}\}$ is pairwise non-isometric and the abelian complex structures $(N_{\mu_{st}}, J)$ are pairwise non-isomorphic. Each modified H-type metric is compatible with two spheres of abelian complex structures of this type which can be described by

$$\{\pm v_1 \times v_2 : v_i \in \mathbb{R}^3,\ ||v_1||^2 = 2s^2,\ ||v_2||^2 = 2t^2,\ \langle v_1, v_2\rangle = 0\},$$

where $v_1 \times v_2$ denotes the vectorial product, but one can see that these structures are all isomorphic to $(N_{\mu_{st}}, J)$ (compare with [**KS**]). We finally recall that $\mu_{01} \simeq \mathfrak{h}_5 \oplus \mathbb{R}$, and so $\langle\cdot,\cdot\rangle$ is a minimal compatible metric for the abelian complex nilpotent Lie group $(N_{\mu_{01}}, J)$.

Although it has not been mentioned, most of the curves given in this section have been obtained via the variational method provided by Theorem 2, by using an approach very similar to that in [**L8**, Example 5.2].

5. Hypercomplex structures

Let N be a real $4n$-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n}$, whose Lie bracket is denoted by $\mu : \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{n}$. An invariant *hypercomplex* structure on N is defined by a triple $\{J_1, J_2, J_3\}$ of complex structures on $\mathfrak{n}$ (see Section 4) satisfying the quaternion identities

$$J_i^2 = -I, \quad i = 1, 2, 3, \qquad J_1J_2 = J_3 = -J_2J_1. \tag{17}$$

An inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ is said to be *compatible* with $\{J_1, J_2, J_3\}$, also called an *hyper-hermitian metric*, if

$$\langle J_iX, J_iY\rangle = \langle X, Y\rangle \qquad \forall X, Y \in \mathfrak{n},\ i = 1, 2, 3. \tag{18}$$

Two hypercomplex nilpotent Lie groups $(N, \{J_1, J_2, J_3\})$ and $(N', \{J_1', J_2', J_3'\})$ are said to be *isomorphic* if there exists an automorphism $\varphi : \mathfrak{n}' \mapsto \mathfrak{n}$ such that

$$\varphi J_i' \varphi^{-1} = J_i, \quad i = 1, 2, 3.$$

For $\gamma = \{J_1, J_2, J_3\}$ we therefore have that

$$G_\gamma = \Lambda(n, \mathbb{H}) = \{\varphi \in \Lambda(4n) : \varphi J_i = J_i\varphi,\ i = 1, 2, 3\}, \qquad K_\gamma = \mathrm{S}p(n),$$

and the Cartan decomposition of

$$\mathfrak{g}_\gamma = \mathfrak{gl}(n, \mathbb{H}) = \{A \in \mathfrak{gl}(4n) : AJ_i = J_iA,\ i = 1, 2, 3\}$$

is given by

$$\mathfrak{gl}(n, \mathbb{H}) = \mathfrak{sp}(n) \oplus \mathfrak{p}_\gamma, \qquad \mathfrak{p}_\gamma = \{A \in \mathfrak{p} : AJ_i = J_iA,\ i = 1, 2, 3\}.$$

The invariant Ricci operator for a compatible metric $\langle\cdot,\cdot\rangle \in \mathcal{C}$ is then given by

$$\operatorname{Ric}^\gamma{}_{\langle\cdot,\cdot\rangle} = \frac{1}{4}(\operatorname{Ric}_{\langle\cdot,\cdot\rangle} - J_1 \operatorname{Ric}_{\langle\cdot,\cdot\rangle} J_1 - J_2 \operatorname{Ric}_{\langle\cdot,\cdot\rangle} J_2 - J_3 \operatorname{Ric}_{\langle\cdot,\cdot\rangle} J_3),$$

and hence condition $\operatorname{Ric}^\gamma{}_{\langle\cdot,\cdot\rangle} = 0$ can never holds since $\operatorname{tr} \operatorname{Ric}^\gamma{}_{\langle\cdot,\cdot\rangle} = \operatorname{sc}(\langle\cdot,\cdot\rangle) < 0$ for a non-abelian nilpotent Lie group.

We now fix $\{J_1, J_2, J_3, \langle\cdot,\cdot\rangle\}$ satisfying (17) and (18) and consider the algebraic subvariety $\mathcal{N}_h := \mathcal{N}_\gamma \subset \mathcal{N}$ given by

$$\mathcal{N}_h = \{\mu \in \mathcal{N} : J_i \text{ is integrable for } i = 1, 2, 3\},$$

that is, those nilpotent Lie brackets for which $\{J_1, J_2, J_3\}$ is an hypercomplex structure on the corresponding nilpotent Lie group N_μ. Thus (6) identifies each $\mu \in \mathcal{N}_h$ with the hyper-hermitian manifold $(N_\mu, \{J_1, J_2, J_3\}, \langle\cdot,\cdot\rangle)$.

It is proved in [**L8**] that any hypercomplex 8-dimensional nilpotent Lie group admits a minimal compatible metric, and that the moduli space of all hypercomplex 8-dimensional nilpotent Lie groups up to isomorphism is 9-dimensional, and the moduli space of the abelian ones has dimension 5.

We will give now explicit continuous families of hypercomplex structures on some particular nilpotent Lie groups.

For $\mathfrak{n}_1 = \mathbb{R}^4$ and $\mathfrak{n}_2 = \mathbb{R}^4$ consider the vector space $W = \Lambda^2\mathfrak{n}_1^* \otimes \mathfrak{n}_2$ of all skew-symmetric bilinear maps $\mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \mapsto \mathfrak{n}_2$. Any 8-dimensional 2-step nilpotent Lie algebra with $\dim \mu(\mathfrak{n}, \mathfrak{n}) \leqslant 4$ can be modelled in this way. Fix basis $\{X_1, X_2, X_3, X_4\}$ and $\{Z_1, Z_2, Z_3, Z_4\}$ of $\mathfrak{n}_1$ and $\mathfrak{n}_2$, respectively. Each element in W will be denoted as $\mu = \mu(a_1, ..., a_4, ..., f_1, ..., f_4)$, where

$$\begin{array}{ll}
\mu(X_1, X_2) = a_1Z_1 + a_2Z_2 + a_3Z_3 + a_4Z_4, & \mu(X_2, X_3) = d_1Z_1 + d_2Z_2 + d_3Z_3 + d_4Z_4,\\
\mu(X_1, X_3) = b_1Z_1 + b_2Z_2 + b_3Z_3 + b_4Z_4, & \mu(X_2, X_4) = e_1Z_1 + e_2Z_2 + e_3Z_3 + e_4Z_4,\\
\mu(X_1, X_4) = c_1Z_1 + c_2Z_2 + c_3Z_3 + c_4Z_4, & \mu(X_3, X_4) = f_1Z_1 + f_2Z_2 + f_3Z_3 + f_4Z_4.
\end{array}$$

The compatible metric will be $\langle X_i, X_j\rangle = \langle Z_i, Z_j\rangle = \delta_{ij}$ and the hypercomplex structure will always act on $\mathfrak{n}_i$ by

$$J_1 = \begin{bmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} & & -1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & -1 & & \end{bmatrix}, \quad J_3 = \begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

If $A = (a_1, ..., a_4)$, ..., $F = (f_1, ..., f_4)$, then it is easy to prove that J_i is integrable for all $i = 1, 2, 3$ on N_μ (or $(N_\mu, \{J_1, J_2, J_3\})$ is a hypercomplex nilpotent Lie group) if and only if

$$E = B + J_1D + J_1C, \qquad D = -C - J_2A - J_2F, \qquad F = -A - J_3B + J_3E.$$

If we define $T := D + C$, then the above conditions are equivalent to

$$D = -C + T, \qquad E = B + J_1T, \qquad F = -A + J_2T. \tag{19}$$

In order to use a notation as similar as possible to [**DF1, DF2**], we should put $T = (t_3, t_2, -t_1, t_4)$. It is easy to check that $(N_\mu, \{J_1, J_2, J_3\})$ is *abelian* (i.e. $\mu(J_i\cdot, J_i\cdot) = \mu$, $i = 1, 2, 3$) if and only if $T = 0$. We note that $\dim W = 24$, and so condition (19) determine a $\Lambda(1, \mathbb{H}) \times \Lambda(1, \mathbb{H})$-invariant linear subspace W_h of W of dimension 16, and a 12-dimensional subspace W_{ah} if we ask in addition abelian.

For each $\mu \in W$, the Ricci operator of $(N_\mu, \{J_1, J_2, J_3\}, \langle\cdot,\cdot\rangle)$ (see identification (6)) restricted to $\mathfrak{n}_1$, $\mathrm{Ric}_\mu|_{\mathfrak{n}_1}$, is given by formula (15), and

$$\mathrm{Ric}_\mu|_{\mathfrak{n}_2} = \frac{1}{2}[\langle v_i, v_j\rangle], \quad 1 \leqslant i, j \leqslant 4, \quad v_i := (a_i, b_i, c_i, d_i, e_i, f_i).$$

Since the only symmetric transformations of $\mathfrak{n}_i = \mathbb{R}^4$ commuting with all the $J_i's$ are the multiplies of the identity, we obtain that the invariant Ricci operator satisfies $\mathrm{Ric}^\gamma{}_\mu|_{\mathfrak{n}_i} \in \mathbb{R}I$ for any $\mu \in W$. By arguing as in (16), one obtains that any $\mu \in W$ is minimal.

Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$ denote the 8-dimensional Lie algebras obtained as the direct sum of an abelian factor and the following H-type Lie algebras: the 5-dimensional

Heisenberg Lie algebra, the 6-dimensional complex Heisenberg Lie algebra and the 7-dimensional quaternionic Heisenberg Lie algebra. In order to avoid homothetical changes we will always ask for $||v_1||^2+...+||v_4||^2=2$, which is equivalent to $\mathrm{sc}(\mu)=-1$.

EXAMPLE. If we put $T=0$, $A=(0,r,0,0)$, $B=(0,0,s,0)$, $C=(0,0,0,t)$, we have for each μ_{rst} that

$$\mathrm{Ric}_{\mu_{rst}}|_{\mathfrak{n}_2}=\frac{1}{2}\begin{bmatrix}0&&&\\&r^2&&\\&&s^2&\\&&&t^2\end{bmatrix},$$

and thus the family

$$\{(N_{\mu_{rst}},\{J_1,J_2,J_3\},\langle\cdot,\cdot\rangle):0\leqslant r\leqslant s\leqslant t,\ r^2+s^2+t^2=2\}$$

of minimal compatible metrics is pairwise non-isometric. This gives rise then a 2-parameter family of pairwise non-isomorphic abelian hypercomplex nilpotent Lie groups (see Corollary 1). If $0<r$ then $\mu_{rst}\simeq\mathfrak{g}_3$, for $r=0<s$ we get a curve on $\mathfrak{g}_2$ and for $r=s=0$, $t=1$, a single structure on $\mathfrak{g}_1$.

EXAMPLE. We now set $T=0$ and choose A,B,C such that

$$v_1=(\frac{1}{\sqrt{2}},0,0,0,0,-\frac{1}{\sqrt{2}}),\ v_2=(0,\sqrt{\frac{3}{8}},0,0,\sqrt{\frac{3}{8}},0),\ ||v_3||^2+||v_4||^2=\frac{1}{4},\ ||v_3||^2>||v_4||^2.$$

Assume that two of such elements $\lambda=\mu(v_3,v_4)$ and $\lambda'=\mu(v_3',v_4')$ are in the same $\mathrm{S}p(1)\times\mathrm{S}p(1)$-orbit, say $\lambda'=\varphi.\lambda$ for some $\varphi=(\varphi_1,\varphi_2)\in\mathrm{S}p(1)\times\mathrm{S}p(1)$. Recall that

$$j_{\lambda'}(Z)=\varphi_1 j_\lambda(\varphi_2^{-1}Z)\varphi_1^{-1},\qquad\forall Z\in\mathfrak{n}_2,$$

(see [**L8**, Appendix]) and $\langle v_i,v_j\rangle=-\frac{1}{2}\operatorname{tr}j_\mu(Z_i)j_\mu(Z_j)$, $1\leqslant i,j\leqslant 4$. It then follows from

$$||v_1||>||v_2||>||v_3||>||v_4||,\qquad ||v_1'||>||v_2'||>||v_3'||>||v_4'||$$

that $j_{\varphi.\lambda}(Z_i)=\pm j_\lambda(Z_i)$ for all $i=1,...,4$, and hence $v_3'=\pm v_3$ and $v_4'=\pm v_4$. Thus we have a family of pairwise non-isomorphic abelian hypercomplex structures on $\mathfrak{g}_3$ depending on 5 parameters (see Corollary 1). Analogously, we get a 5-dimensional family on $\mathfrak{g}_2$ by putting $v_1=v_2=0$.

EXAMPLE. Let μ_t be the curve defined for $0\leqslant t\leqslant\frac{1}{\sqrt{3}}$ by

$$\begin{array}{ll}\mu_t(X_1,X_2)=\sqrt{1-3t^3}Z_1+tZ_2, & \mu_t(X_2,X_3)=tZ_4,\\ \mu_t(X_1,X_3)=tZ_3, & \mu_t(X_2,X_4)=-tZ_3,\\ \mu_t(X_1,X_4)=tZ_4, & \mu_t(X_3,X_4)=-\sqrt{1-3t^3}Z_1+tZ_2.\end{array}$$

It is easy to check that $(N_{\mu_t},\{J_1,J_2,J_3\})$ is always non-abelian hypercomplex (recall that $t_1=t_2=t_3=0$, $t_4=2t$) and the curve is pairwise non-isomorphic since it follows from

$$\mathrm{Ric}_{\mu_t}|_{\mathfrak{n}_2}=\begin{bmatrix}1-3t^2&&&\\&t^2&&\\&&t^2&\\&&&t^2\end{bmatrix}$$

that the curve $(N_{\mu_t},\langle\cdot,\cdot\rangle)$ is pairwise non-isometric (see Corollary 1). The starting and ending points are $\mu_0\simeq\mathfrak{g}_1$ and $\mu_{\frac{1}{\sqrt{3}}}\simeq\mathfrak{g}_3$, respectively, and $\mu_t\simeq\mathfrak{u}(2)\oplus\mathbb{C}^2$ for any $0<t<\frac{1}{\sqrt{3}}$ (see [**L2**] for further information on these 2-step nilpotent Lie algebras constructed via representations of compact Lie groups).

6. Einstein solvmanifolds

Our goal in this section is to show that the 'moment map' approach proposed in [**L8**] can also be applied to the study of Einstein solvmanifolds. After a brief overview of such spaces, we will follow the same path used to study compatible metrics in the previous sections, but by considering the Ricci operator $\operatorname{Ric}_{\langle\cdot,\cdot\rangle}$ itself. In other words, none geometric structure is considered (or $\gamma = 0$).

A *solvmanifold* is a solvable Lie group S endowed with a left invariant Riemannian metric, and S is called *standard* if $\mathfrak{a} := \mathfrak{n}^{\perp}$ is abelian, where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{s}$ is the Lie algebra of S. All known examples of non-compact homogeneous Einstein manifolds are isometric to standard Einstein solvmanifolds. These spaces have been extensively studied by J. Heber in [**Hb**], obtaining remarkable structure and uniqueness results.

Let N be a nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension n, whose Lie bracket is denoted by $\mu : \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{n}$. We have in this case $\gamma = 0$, thus any inner product is 'compatible', $G_\gamma = \Lambda(n)$, $K_\gamma = \mathrm{O}(n)$, $\mathfrak{p}_\gamma = \mathfrak{p}$, $\mathcal{N}_\gamma = \mathcal{N}$, $\operatorname{Ric}^\gamma = \operatorname{Ric}$ and then condition $\operatorname{Ric}^\gamma = 0$ is clearly forbidden for non-abelian N. Moreover, the evolution equation is precisely the Ricci flow and the corresponding invariant Ricci solitons coincide with minimal metrics.

Given a metric nilpotent Lie algebra $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$, a metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle\cdot,\cdot\rangle')$ is called a *metric solvable extension* of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ if the restrictions of the Lie bracket of $\mathfrak{s}$ and the inner product $\langle\cdot,\cdot\rangle'$ to $\mathfrak{n}$ coincide with the Lie bracket of $\mathfrak{n}$ and $\langle\cdot,\cdot\rangle$, respectively. It turns out that for each $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ there exists a unique rank-one (i.e. $\dim \mathfrak{a} = 1$) metric solvable extension of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ which stand a chance of being an Einstein space (see [**L4**]). This fact turns the study of rank-one Einstein solvmanifolds into a problem on nilpotent Lie algebras. More specifically, a nilpotent Lie algebra $\mathfrak{n}$ is the nilradical of a standard Einstein solvmanifold if and only if $\mathfrak{n}$ admits an inner product $\langle\cdot,\cdot\rangle$ such that $\operatorname{Ric}_{\langle\cdot,\cdot\rangle} = cI + D$ for some $c \in \mathbb{R}$ and $D \in \operatorname{Der}(\mathfrak{n})$, that is, $\langle\cdot,\cdot\rangle$ is minimal.

Let $\mathcal{N}$ be the variety of all nilpotent Lie algebras of dimension n. Fix an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$. Each $\mu \in \mathcal{N}$ is then identified via (6) with the Riemannian manifold $(N_\mu, \langle\cdot,\cdot\rangle)$, but we also have in this case another identification with a solvmanifold: for each $\mu \in \mathcal{N}$, there exists a unique rank-one metric solvable extension $S_\mu = (S_\mu, \langle\cdot,\cdot\rangle)$ of $(N_\mu, \langle\cdot,\cdot\rangle)$ standing a chance of being Einstein, and every $(n+1)$-dimensional rank-one Einstein solvmanifold can be modelled as S_μ for a suitable $\mu \in \mathcal{N}$. We recall that the study of standard solvmanifolds reduces essentially to the rank-one case (see [**Hb**, 4.18]).

The functional $F : \mathbb{P}\mathcal{N} \mapsto \mathbb{R}$ given by $F([\mu]) = \operatorname{tr} \operatorname{Ric}_\mu^2 / ||\mu||^4$ measures how far is the metric μ from being Einstein (see [**L3**]).

From Theorem 2 we then obtain the uniqueness up to isometry of Einstein metrics on standard solvable Lie groups proved in [**Hb**], as well as the variational result given in [**L4**] characterizing Einstein solvmanifolds as critical points of a natural curvature functional.

Theorem 1 gives the relationship between Ricci soliton metrics on nilpotent Lie groups and Einstein solvmanifolds proved in [**L3**]. Part (i) is a new characterization of these privileged metrics, claiming that they are precisely minimal metrics.

As far as we know, the following is a complete list of the known examples of nilpotent Lie groups admitting a minimal metric, or equivalently, of the nilradicals of standard Einstein solvmanifolds:

- Iwasawa N-groups: G/K irreducible symmetric space of noncompact type and $G = KAN$ the Iwasawa decomposition.
- [**PS**] Nilradicals of normal j-algebras (i.e. of noncompact homogeneous Kähler Einstein spaces).
- [**D**] Certain 2-step nilpotent Lie algebras for which there is a basis with very uniform properties (see also [**Wo**, 1.9]).
- [**Bo**] H-type Lie groups (see also [**La**]).
- [**C**] Nilradicals of homogeneous quaternionic Kähler spaces.
- [**EH, L2**] Naturally reductive nilpotent Lie groups.
- [**Hb**] Families of deformations of Iwasawa N-groups in the rank-one case.
- [**F1, F2**] Certain 2-step nilpotent Lie algebras constructed via Clifford modules.
- [**GK**] A 2-parameter family of 9-dimensional 2-step nilpotent Lie algebras (with 3-dimensional center) and certain modifications of Iwasawa N-groups (rank $\geqslant 2$).
- [**L6**] Nilpotent Lie algebras with a codimension one abelian ideal.
- [**L6**] A curve of 6-step nilpotent Lie algebras of dimension 7, which is the lowest possible dimension for a continuous family.
- [**Y, Mo**] Certain 2-step nilpotent Lie algebras defined from subsets of fundamental roots of complex simple Lie algebras.
- [**W, L6**] Nilpotent Lie algebras of dimension $\leqslant 6$.
- [**L5**] A curve of 10-dimensional 2-step nilpotent Lie algebras with 5-dimensional center.
- [**Ke**] A 2-parameter family of deformations of the 12-dimensional quaternionic hyperbolic space.
- [**T**] Nilradicals of parabolic subalgebras of semisimple Lie algebras which are 2-step or 3-step.

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