**DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS**
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 349 – 355

# On linearly $pF$-planar mappings

Raad J. Al Lamy, Josef Mikeš, Marie Škodová

Abstract. In this paper we study linearly $pF$-planar mappings of $n$-dimensional spaces $A_n$ with a torsion-free affine connection and with an affinor structure $F$, for which $F^{p+1} = 0$. These mappings are certain generalizations of geodesic, holomorphically projective and $F$-planar mappings.

We investigate here also general aspects of linearly $pF$-planar mappings.

## 1. Introduction

This paper is concerned with certain generalizations of geodesic, quasigeodesic, holomorphically projective and $F$-planar mappings [**1, 4, 5, 6, 7, 8, 9, 10**].

We consider an $n$-dimensional space $A_n$ with a torsion-free affine connection $\nabla$ and an affinor structure $F$, i.e. a tensor field of type $\binom{1}{1}$.

Unless otherwise specified, all spaces, connections and mappings under consideration are differentiable of the sufficiently high class. The dimension $n$ of the spaces being considered is higher that three, as a rule. This fact is not specially stipulated. All spaces are assumed to be connected.

Definition 1. (Raad J. Al Lamy, I.N. Kurbatova [**2, 3**]) A curve $\ell$, which is given by the equations

$$\ell = \ell(t), \qquad \lambda(t) \stackrel{\text{def}}{=} d\ell(t)/dt \ (\neq 0), \quad t \in I \tag{1}$$

where $t$ is a parameter, is called *$pF$-planar* if its tangent vector $\lambda(t_0)$, for any initial value $t_0$ of the parameter $t$, remains, under the parallel translation along the curve $\ell$, in the distribution generated by the vector functions $\lambda$, $F\lambda$, $F^2\lambda$, $\dots$ , $F^p\lambda$ along $\ell$.

In particular, if $p = 0$, we obtain the definition of a geodesic parametrized by an arbitrary parameter, and for $p = 1$ we obtain the definition of an $F$-planar curve, [**1, 4, 5, 7, 8, 9, 10**].

In accordance with this definition, $\ell$ is $pF$-planar if and only if the following condition holds:

$$\nabla_{\lambda(t)} \lambda(t) = \sum_{\sigma=0}^{p} \overset{\sigma}{\varrho}(t)\, F^\sigma \lambda(t), \tag{2}$$

where $\overset{0}{\varrho}, \ \dots \ , \overset{p}{\varrho}$ are some functions of the parameter $t$.

2000 *Mathematics Subject Classification.* 53B05, 53C15.
*Key words and phrases.* linearly $pF$-planar mapping, space with affine connection.
This paper is in final form and no version of it will be published elswhere.

Note that the class of $pF$-planar curves of the space $A_n$ is very broad. Obviously through any point in each direction there passes a set of $pF$-planar curves depending on $p$ arbitrary continuous functions $\overset{1}{\varrho}, \ \dots \ ,\overset{p}{\varrho}$. The class of $pF$-planar curves contains: geodesic lines, $F$-planar curves, quasigeodesic curves, analytic planar curves (see [**1, 4, 5, 6, 7, 8, 9, 10**]).

In the following we are going to consider a special class of $pF$-planar curves which are defined as follows.

Definition 2. A $pF$-planar curve is called *linearly pF-planar* if the following condition holds for its tangent vector $\lambda(t)$:

$$\nabla_{\lambda(t)}\,\lambda(t) = \varrho(t)\lambda(t) + 2\sum_{\sigma=1}^{p} \overset{\sigma}{\varphi}(\lambda(t))\,F^{\sigma}\lambda(t), \tag{3}$$

where $\overset{\sigma}{\varphi}$, $\sigma = 1, \ \dots \ , p$, are some linear differential forms defined on $A_n$, and $\varrho(t)$ is a function of the parameter $t$.

If the linear differential forms $\overset{\sigma}{\varphi}$ are fixed in advance and the affinor structure $F$ is also fixed, then there is (up to a parametrization) a unique linearly $pF$-planar curve emanating from a point of $A_n$ in a given direction.

In the papers by Raad J. Al Lamy and I.N. Kurbatova [**2, 3**] a definition of *pF-planar mappings* between (pseudo-) Riemannian spaces $V_n$ and $\bar{V}_n$ was mentioned. The authors have found the fundamental equations of these mappings. Unfortunately, an error was found in their proof. Therefore in the second part of our paper we relate the term *pF-planar mappings* to the *linearly pF-planar mappings*, for which fundamental equations are identical with the equations in the papers mentioned above.

## 2. Linearly $pF$-planar mappings between two spaces with affine connection

We suppose two spaces $A_n$ and $\bar{A}_n$ with torsion-free affine connections $\nabla$ and $\bar{\nabla}$, respectivelly. An affinor structure $F$ and linear forms $\overset{1}{\varphi}, \ \dots \ ,\overset{p}{\varphi}$ are defined on $\bar{A}_n$.

Definition 3. A diffeomorphism $f\colon A_n \to \bar{A}_n$ between two manifolds with affine connection is called *linearly pF-planar* if any geodesic in $A_n$ is mapped onto a linear $pF$-planar curve in $\bar{A}_n$.

**Important convention.** Due to the diffeomorphism $f$ we always suppose that $\nabla$, $\bar{\nabla}$, $F$ and linear forms $\overset{1}{\varphi}, \ \dots \ ,\overset{p}{\varphi}$ are defined on $A_n$. Moreover, we always identify a given curve $\ell\colon I \to A_n$ and its tangent vector function $\lambda(t)$ with their images $\bar{\ell} = f \circ \ell$ and $\bar{\ell} = f_*(\lambda(t))$ in $\bar{A}_n$.

In the sequel we shall need the following lemma:

Lemma 1. *Let $V$ be a vector space, $Q\colon V \times V \to V$ be a symmetric bilinear mapping and $B\colon V \to V$ a linear mapping for which* $\operatorname{rank} B > 1$.

*If, for each vector $\lambda \in V\backslash\operatorname{Ker}(B)$*

$$Q(\lambda, \lambda) = \varrho^*(\lambda)\,B(\lambda) \tag{4}$$

*holds, where $\varrho^*(\lambda)$ is a function on $V\backslash\operatorname{Ker}(B)$, then there is a linear form $\varrho$ such that the condition*

$$Q(X, Y) = \varrho(X)B(Y) + \varrho(Y)B(X) \tag{5}$$

*holds for any* $X, Y \in V$*, and* $\varrho^*(\lambda) = 2\varrho(\lambda)$ *for each* $\lambda \in V\setminus\mathrm{Ker}(B)$.

**Proof.** The formula (4) has the following coordinate expression

$$Q^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = 2\varrho^*(\lambda)B^h_\alpha\lambda^\alpha, \tag{6}$$

where $\lambda^i, B^h_i, Q^h_{ij}$ are the components of $\lambda, B, Q$.

We multiply (6) with $B^i_\alpha\lambda^\alpha$ and then we make the alternation through indices $h$ and $i$. After the adaption we obtain

$$(Q^h_{\alpha\beta}B^i_\gamma - Q^i_{\alpha\beta}B^h_\gamma)\lambda^\alpha\lambda^\beta\lambda^\gamma = 0. \tag{7}$$

The term in brackets does not depend on $\lambda$ and (7) holds for any vector $\lambda \in V\setminus\mathrm{Ker}(B)$, therefore

$$Q^h_{(\alpha\beta}B^i_{\gamma)} - Q^i_{(\alpha\beta}B^h_{\gamma)} = 0 \tag{8}$$

holds, where the round brackets denote symmetrization of indices.

Because $B^h_i \neq 0$, it means, that there exist vectors $\varepsilon^i$ and $\nu_h$ such that $B^h_i\varepsilon^i\nu_h = 1$.

At the beginning we contract the formula (8) with $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma\nu_i$. We make sure of

$$Q^h_{\alpha\beta}\varepsilon^\alpha\varepsilon^\beta = a\,B^h,$$

where $a = Q^i_{\alpha\beta}\nu_i\varepsilon^\alpha\varepsilon^\beta$ and $B^h = B^h_\alpha\varepsilon^\alpha$.

Now we contract (8) with $\varepsilon^\beta\varepsilon^\gamma\nu_i$ and once more make sure of

$$Q^h_{\alpha\beta}\varepsilon^\beta = \frac{1}{2}a\,B^h_\alpha + B^h a_\alpha,$$

where $a_\alpha = Q^i_{\alpha\beta}\nu_i\varepsilon^\beta - \frac{1}{2}aB^i_\alpha\nu_i$.

Finally we contract (8) with $\varepsilon^\gamma\nu_i$ and obtain

$$Q^h_{\alpha\beta} = \varrho_\alpha B^h_\beta + \varrho_\beta B^h_\alpha + B^h a_{\alpha\beta}, \tag{9}$$

where $a_{\alpha\beta} = Q^i_{\alpha\beta}\nu_i - a_\alpha B^i_\beta\nu_i - a_\beta B^i_\alpha\nu_i$ and $\varrho_\alpha = Q^i_{\alpha\beta}\varepsilon^\beta\nu_i - \frac{1}{2}aB^i_\alpha\nu_i$. Here $\varrho_\alpha$ are components of a 1-form $\varrho$ defined on $V\setminus\mathrm{Ker}(B)$ and hence on $V$.

Now we substitute (9) to (8) and obtain

$$B^h a_{(\alpha\beta}B^i_{\gamma)} - B^i a_{(\alpha\beta}B^h_{\gamma)} = 0.$$

It is easy to see from the previous formula that $a_{\alpha\beta} \neq 0$ implies $\operatorname{rank} B \leqslant 1$, a contradiction.

Then $a_{\alpha\beta} = 0$ and the formula (9) has the expression $Q^h_{ij} = \varrho_i B^h_j + \varrho_j B^h_i$ which is the coordinate expression of (5). From (4) we obtain, that $\varrho^*(\lambda) = 2\varrho(\lambda)$ for each $\lambda \in V\setminus\mathrm{Ker}(B)$.

The following holds.

Theorem 1. *A linearly pF-planar mapping $f$ from $A_n$ onto $\bar{A}_n$ is characterized by the condition*

$$P(X,Y) = \sum_{\sigma=0}^{p}(\overset{\sigma}{\varphi}(X)F^\sigma Y + \overset{\sigma}{\varphi}(Y)F^\sigma X), \tag{10}$$

*for any vector fields* $X, Y$*, where* $P \overset{\mathrm{def}}{=} \bar{\nabla} - \nabla$ *is the deformation tensor field of* $f$ *(of type* $\binom{1}{2}$*), and* $\overset{0}{\varphi}, \overset{1}{\varphi}, \dots, \overset{p}{\varphi}$ *are some linear differential forms on* $A_n$.

Let us recall that on each tangent space $T_xA_n$, $P(X,Y)$ is a symmetric bilinear mapping $T_xA_n \times T_xA_n \to T_xA_n$.

**Proof.** Fix a point $x \in A_n$, and a tangent vector $\lambda$ at the point $x$. Choose the unique geodesic $\ell$ (with the canonical parametrization) for which $x$ is the initial point and $\lambda$ the initial tangent vector, i.e. $\lambda = \lambda(0)$. Then we get $\nabla_{\lambda(t)}\lambda(t) = 0$ for all $t$. Consider a linearly $pF$-planar mapping $f\colon A_n \to \bar{A}_n$. The image $\bar{\ell}$ of $\ell$ with respect to $f$ can be naturally identified with $\ell$ and it is characterized by the condition

$$\bar{\nabla}_{\lambda(t)}\,\lambda(t) = \bar{\varrho}(t)\lambda(t) + 2\sum_{\sigma=1}^{p} \overset{\sigma}{\varphi}(\lambda(t))\, F^{\sigma}\lambda(t). \tag{11}$$

Here $\bar{\varrho}$ is a function of $t$, and its initial value at $t = 0$ is *uniquely determined* by the prescribed tangent vector $\lambda$. Now, *at a given point* $x \in A_n$ we have

$$P(\lambda,\lambda) - 2\sum_{\sigma=1}^{p} \overset{\sigma}{\varphi}(\lambda)F^{\sigma}\lambda = \bar{\varrho}(0)\,\lambda. \tag{12}$$

The coefficient of $\lambda$ on the right-hand side is a constant uniquely determined by $\lambda$. Hence, in the tangent space $T_xA_n$, we can write the formula (12) in the form

$$Q(\lambda,\lambda) = \theta(\lambda)\,\lambda,$$

where $Q(\lambda,\lambda)$ is the left-hand side of (12) and $\theta(\lambda)$ is a function of $\lambda$ on $T_xA_n$. From Lemma 1, taking for $B$ the identity operator Id, the formula (10) follows.

Conversely, let the formula (10) hold for a diffeomorphism $f\colon A_n \to \bar{A}_n$. Then for every tangent vector $\lambda$ we get $Q(\lambda,\lambda) = 2\,\overset{0}{\varphi}(\lambda)$ and for each geodesic $\ell = \ell(t)$ (parametrized in arbitrary way) and its image $\bar{\ell} = f \circ \ell$ we obtain

$$\bar{\nabla}_{\lambda(t)}\lambda(t) - \nabla_{\lambda(t)}\lambda(t) = 2\,\overset{0}{\varphi}(\lambda(t))\lambda(t) + 2\sum_{\sigma=1}^{n} \overset{\sigma}{\varphi}(\lambda(t))\, F^{\sigma}\lambda(t).$$

Hence, if $\nabla_{\lambda(t)}\lambda(t) = \varrho(t)\lambda(t)$, then $\bar{\nabla}_{\lambda(t)}\lambda(t)$ is of the form (11), where $\bar{\varrho}(t) = 2\,\overset{0}{\varphi}(\lambda(t)) + \varrho(t)$. Thus $\bar{\ell}$ is linearly $pF$-planar, which had to be proved.

**Remarks.** From the formula (10) characterizing a linearly $pF$-planar mapping we obtain that a linear $pF$-planar curve in $A_n$ is mapped onto a linear $pF$-planar curve in $\bar{A}_n$; and using also formula (2), we also see that a $pF$-planar curve in $A_n$ is mapped onto a $pF$-planar curve in $\bar{A}_n$.

In general, from the condition that each $pF$-planar curve in $A_n$ is mapped on a $pF$-planar curve in $\bar{A}_n$ under $f$, the equation (10) can not be deduced. Thus the mapping is not linearly $pF$-planar in our sense.

## 3. Linearly $pF$-planar mappings satisfying $F^{p+1} = 0$

We have the following

THEOREM 2. *Suppose $f\colon A_n \to \bar{A}_n$ is a diffeomorphism and $F$ is an affinor structure on $A_n$ which satisfies*

$$F^{p+1} = 0, \qquad \text{rank } F^p > 1. \tag{13}$$

*If any pF-planar curve in $A_n$ is mapped on a pF-planar curve in $\bar{A}_n$, then $f$ is a linearly pF-planar mapping.*

**Proof.** Fix a point $x \in A_n$, and a tangent vector $\lambda$ at the point $x$. Choose the unique geodesic $\ell$ (with the canonical parametrization) for which $x$ is the initial point and $\lambda$ the initial tangent vector, i.e. $\lambda = \lambda(0)$. Then we get $\nabla_{\lambda(t)}\lambda(t) = 0$ for all $t$.

Consider a mapping $f\colon A_n \to \bar{A}_n$. It is obvious, that geodesics $\ell$ are the special case of $pF$-planar curves. The image $\bar{\ell}$ of $\ell$ with respect to $f$ can be naturally identified with $\ell$ and it is characterized by the condition (analogical (2))

$$\bar{\nabla}_{\lambda(t)}\,\lambda(t) = \sum_{\sigma=0}^{p} \overset{\sigma}{\bar{\varrho}}(t)\, F^{\sigma}\lambda(t).$$

Here $\overset{\sigma}{\bar{\varrho}}$ $(\sigma = 0, \ \ldots \ , p)$ are functions of $t$, and their initial value at $t = 0$ are *uniquely determined* by the prescribed tangent vector $\lambda$. Now, *at a given point* $x \in A_n$ we have

$$P(\lambda,\lambda) = \sum_{\sigma=0}^{p} \overset{\sigma}{\bar{\varrho}}(0) F^{\sigma}\lambda \tag{14}$$

where $P = \bar{\nabla} - \nabla$.

The coefficients of $F^{\sigma}\lambda$ on the right-hand side are constants determined by $\lambda$. Hence, in the tangent space $T_x A_n$, we can write the formula (14) in the form

$$P(\lambda,\lambda) = \sum_{\sigma=0}^{p} \overset{\sigma}{\varrho}(\lambda) F^{\sigma}\lambda \tag{15}$$

where $\overset{\sigma}{\varrho}(\lambda)$ $(\sigma = 0, \ \ldots \ , p)$ are function of $\lambda$ on $T_x A_n$.

In each tangent space $T_x$ we put $W_x = \operatorname{Ker}((F_x)^p)$.

We multiply (15) by $F^p$:

$$F^p P(\lambda,\lambda) = \overset{0}{\varrho}(\lambda) F^p\lambda.$$

Based on Lemma 1 it follows, that there exists a linear form $\overset{0}{\varphi}$, for which

$$\overset{0}{\varrho}(\lambda) = 2\,\overset{0}{\varphi}(\lambda) \quad \text{for each } \lambda \in T_x \backslash W_x. \tag{16}$$

After that we multiply the term (15) by $F^{p-1}$ and use (16). Thus for each $\lambda \in T_x \backslash W_x$:

$$F^{p-1}P(\lambda,\lambda) - 2\,\overset{0}{\varphi}(\lambda)\lambda = \overset{1}{\varrho}(\lambda) F^p\lambda.$$

Based on Lemma 1 we obtain by analogy, that there exists a linear form $\overset{1}{\varphi}$, for which $\overset{1}{\varrho}(\lambda) = 2\,\overset{1}{\varphi}(\lambda)$ for each $\lambda \in T_x \backslash W_x$.

Then we multiply the formula (15) step-by-step by $F^{p-2}, F^{p-3}, \ \ldots \ , F$, and we check easily by induction that

$$\overset{2}{\varrho}(\lambda) = 2\,\overset{2}{\varphi}(\lambda), \quad \ldots \quad , \overset{p-1}{\varrho}(\lambda) = 2\,\overset{p-1}{\varphi}(\lambda)$$

for each $\lambda \in T_x \backslash W_x$, where $\overset{\sigma}{\varphi}$ $(\sigma = 2, \ \ldots \ , p-1)$ are linear forms.

The formula (15) takes on the following form for each $\lambda \in T_x \backslash W_x$:

$$P(\lambda,\lambda) - 2\sum_{\sigma=0}^{p-1} \overset{\sigma}{\varphi}(\lambda) F^{\sigma}(\lambda) = \overset{p}{\varrho}(\lambda) F^p(\lambda).$$

Hence from Lemma 1 it follows, that there exists a linear form $\overset{p}{\varphi}$ for which $\overset{p}{\varrho}(\lambda) = 2\overset{p}{\varphi}(\lambda)$ for each $\lambda \in T_x \backslash W_x$, and formula (10) holds. Clearly the mapping $f$: $A_n \to \bar{A}_n$ is linearly $pF$-planar. The Theorem 2 was proved completely.

## 4. Linearly $pF$-planar mappings satisfying $F^{p+1} = 0$ and preserving the covariant derivative $\nabla F$

Finally we will ponder over the problem of linearly $pF$-planar mappings in which the conditions (13) hold and

$$\bar{\nabla} F = \nabla F, \tag{17}$$

i.e. the covariant derivative of the structure $F$ is preserved.

These conditions hold, for example, if the structure is covariantly constant both in $A_n$ and $\bar{A}_n$.

With respect to the definition of the covariant derivative and based on formulas (10) we have

$$\bar{\nabla}_X F(Y) = \nabla_X F(Y) + \sum_{\sigma=0}^{p} (\overset{\sigma}{\varphi}(FX) F^{\sigma} Y - \overset{\sigma}{\varphi}(X) F^{\sigma+1} Y) \tag{18}$$

for each $X, Y$.

It means that under conditions (17) and (18) it follows:

$$\sum_{\sigma=0}^{p} (\overset{\sigma}{\varphi}(FX) F^{\sigma} Y - \overset{\sigma}{\varphi}(X) F^{\sigma+1} Y) = 0 \tag{19}$$

for each $X, Y$.

Applying, step-by-step, the operators $F^p, F^{p-1}, \ \dots\ , F$ to (19), we show from the conditions $F^{p+1} = 0$ and $F^p \neq 0$ that (19) is equivalent to the following series of conditions:

$$\overset{0}{\varphi}(FX) = 0 \quad \text{and} \quad \overset{\sigma}{\varphi}(X) = \overset{\sigma+1}{\varphi}(FX), \quad (\sigma = 0, \ \dots\ , p-1) \quad \text{for each } X.$$

If we denote $\varphi \overset{\text{def}}{=} \overset{p}{\varphi}$, then clearly

$$\overset{\sigma}{\varphi}(X) = \varphi(F^{p-\sigma} X), \quad \text{for each } X, \quad \sigma = 0, \ \dots\ , p. \tag{20}$$

On the other hand from (20) follows (19). We can formulate

THEOREM 3. *Suppose that $f$: $A_n \to \bar{A}_n$ is a linearly $pF$-planar mapping and $F$ is an affinor structure on $A_n$ for which (10) is satisfied. The covariant derivative of $F$ is preserved (i.e. (17) holds), if and only if, for the linear forms $\overset{\sigma}{\varphi}$ defined by (10) the equalities (20) hold.*

**Acknowledgements.** The authors wish to thank prof. O. Kowalski for his valuable comments. Supported by grant No. 201/05/2707 of The Grant Agency of Czech Republic and by the Council of Czech Government MSM 6198959214.

## References


[1] L.P. Eisenhart. *Riemannian geometry.* Princeton Univ. Press, 1926.

[2] Raad J. Al Lamy. *About 2F-plane mappings of affine connection spaces.* Coll. on Diff. Geom., Eger (Hungary)1989, 20-25.

[3] Raad J. Al Lamy, I.N. Kurbatova. *Invariant geometric objects of 2F-planar mappings of affine connection spaces and Riemannian spaces with affine structure of III order.* Dep. of UkrNIINTI (Kiev), 1990, No. 1004Uk90, 51p.

[4] J. Mikeš. *Geodesic mappings of affine-connected and Riemannian spaces.* J. Math. Sci., New York, 1996, 78, 3, 311-333.

[5] J. Mikeš. *Holomorphically projective mappings and their generalizations.* J. Math. Sci., 1998, 89, 3, 1334-1353.

[6] J. Mikeš, O. Pokorná. *On holomorphically projective mappings onto almost Hermitian spaces.* Diff. Geom. and Its Appl. Proc. Conf., Opava, Aug. 27-31, 2001. Silezian Univ., Opava, 2001, 43-48.

[7] J. Mikeš, N.S. Sinyukov. *On quasiplanar mappings of spaces of affine connection.* (Russian) Izv. Vyssh. Uchebn. Zaved., Mat., 1983, 1(248), 55-61; (English) Sov. Math., 1983, 27, 1, 63-70.

[8] A.Z. Petrov. *Simulation of physical fields.* In: Gravitation and the Theory of Relativity, Vol. 4–5, Kazan' State Univ., Kazan, 1968, 7-21.

[9] N.S. Sinyukov. *Geodesic mappings of Riemannian spaces.* Nauka, Moscow, 1979.

[10] K. Yano. *Differential geometry on complex and almost complex spaces.* Oxford-London-New York-Paris-Frankfurt: Pergamon Press. XII, 1965, 323p.



Raad J. Al Lamy
Dept. of Basic Science, Fac. of Sci. & IT.,
Al Balqa' Applied Univ., Jordan,

e-mail: raad@bau.edu.jo

J. Mikeš & M. Škodová
Dept. of Algebra and Geometry,
Fac. of Sci., Palacky Univ., Tomkova 40,
779 00 Olomouc, Czech Rep.,

e-mail: josef.mikes@upol.cz; marie.skodova@upol.cz