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Variational sequences and Lepage forms

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Abstract. The notion of a Lepage form, introduced by Krupka, is generalized to differential forms of arbitrary degree in higher order mechanics. It is shown that the Lepage forms can be used to find a natural representation of classes of forms in a variational sequence as globally well-defined differential forms. The resulting expressions for classes agree with Krbek and Musilová. The construction of a Lepage form is based on the interior Euler-Lagrange operator introduced by Anderson within the variational bicomplex theory on infinite jet manifolds. We give a finite order definition of this operator, and prove its invariance. The structure of Lepage equivalents of the Euler-Lagrange forms is discussed in detail.

1. Introduction

The aim of this paper is to complete some recent results, generalizing the concept of a Lepage form to forms of *arbitrary* degree in *higher order fibered mechanics* [**13**], i.e., on fibered manifolds over a one dimensional basis.

The Lepage forms, as well as Lepage equivalents of lagrangians for multidimensional variational problems on fibered manifolds, were introduced by Krupka [**6**], [**7**]. These forms allow us to describe the geometric structure of the first variation formula, and to introduce the global concepts such as the Euler-Lagrange form and the Noether's current. The Lepage forms unify several interesting objects in the calculus of variations, among them the Poincaré-Cartan form, the Caratheodory form, and the so called fundamental form, and give us a generalization of these forms to higher order variational theory. They also represent a tool for generalizations of the Hamilton theory to higher order problems in fibered manifolds. In the context of the variational sequence theory, the exterior derivative of a Lepage form immediately gives the corresponding class, which turns up to be the Euler-Lagrange form.

Our main motivation for the study of possible generalizations is to achieve an analogous property for forms of degree higher than 1. In particular, we show that the exterior derivative of a Lepage 2-form will give us, as its class, the well known (invariant) Helmholtz form (see also Krupka [**5**] and Šeděnková [**13**]).

Recently, Krupková presented an extension of the notion of a Lepage form to 2-forms (in mechanics) and to $(n+1)$-forms (in field theory), and applied these forms to the inverse problem of the calculus of variations in higher order mechanics, and to the order reducibility problem. She defined a Lepage form as the closed counterpart of the Euler-Lagrange form (see [**10**], [**11**]).

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Our definition agrees with the structure of variational sequences (Krbek and Musilová [**4**], Krupka [**8**], [**9**], Vitolo [**16**]), but is more general. We use a (finite order) modification of the interior Euler operator presented by Anderson for the variational bicomplex theory [**1**] (for this operator, see also Bauderon [**2**], Dedecker and Tulczyjew [**3**], Kuperschmidt [**12**]). We give a new simple proof of basic properties of this operator for finite order jet spaces.

2. Preliminaries

Let us introduce our basic notation (see e.g. [**5**], [**13**]). In this paper, Y is a fibered manifold over a one-dimensional manifold X, of dimension $m+1$, and π is the projection of Y. We denote by J^rY the *r-jet prolongation* of Y, and by $\pi^{r,s} : J^rY \to J^sY$, $\pi^r : J^rY \to X$ the *canonical jet projections*. $J^r\gamma$ is the *r-jet prolongation* of a section γ of Y. Any fibered chart (V,ψ), $\psi=(t,q^\sigma)$, on Y, where $1 \leqslant \sigma \leqslant m$, induces the *associated charts* on X and on J^rY, (U,φ), $\varphi=(t)$, and (V^r,ψ^r), $\psi^r=(t,q^\sigma,q^\sigma_1,q^\sigma_2,\cdots,q^\sigma_r)$.

A differential k-form ρ on J^rY is called *contact*, if it vanishes along the r-jet prolongation $J^r\gamma$ of every section γ. Recall that for a fibered chart (V,ψ), $\psi=(t,q^\sigma)$, we have the *contact basis* of one-forms $(dt,\omega^\sigma,\omega^\sigma_1,\omega^\sigma_2,\dots,\omega^\sigma_r,\ dq^\sigma_{r+1})$ on $V^{r+1} = (\pi^{r+1,0})^{-1}V$, defined by the forms

$$\omega^\sigma_j = dq^\sigma_j - q^\sigma_{j+1}dt, \qquad 0 \leqslant j \leqslant r. \tag{1}$$

A differential form which contains exactly k exterior factors of the form (1) is called *k-contact*. Every form ρ, on J^rY can be uniquely decomposed, after the canonical lifting to $J^{r+1}Y$, as the sum of the *k-contact components* $p_k\rho$ (see e.g. [**8**]).

A vector field Ξ on Y is called *π-vertical* if $T\pi \cdot \Xi = 0$. In a fibered chart,

$$\Xi = \Xi^\sigma \frac{\partial}{\partial q^\sigma}. \tag{2}$$

The *r-jet prolongation* of Ξ is a vector field on J^rY given by

$$J^r\Xi = \sum_{i=0}^{r} \Xi^\sigma_i \frac{\partial}{\partial q^\sigma_i}, \qquad \Xi^\sigma_i = \frac{d\,\Xi^\sigma_{i-1}}{dt}. \tag{3}$$

Let Ω^r_k be the direct image of the sheaf of smooth k-forms over J^rY by the jet projection $\pi^{r,0}$. Denote

$$\Omega^r_{0,c} = \{0\}, \quad \Omega^r_{k,c} = \ker p_{k-1}, \quad \Theta^r_k = \Omega^r_{k,c} + d\Omega^r_{k-1,c}, \tag{4}$$

where $k \geqslant 1$, and $d\Omega^r_{k-1,c}$ is the image sheaf of $\Omega^r_{k-1,c}$ by d. Then for every open set $W \subset Y$, Ω^r_kW is the Abelian group of k-forms, and $\Omega^r_{k,c}W$ is the Abelian group of k-contact k-forms on $W^r = (\pi^{r,0})^{-1}(W)$. $d\Omega^r_{k-1,c}W$ is the Abelian group of forms which can locally be expressed as the differential of $(k-1)$-contact $(k-1)$-forms on W^r, and Θ^r_kW is a subgroup of Ω^r_kW. We get an exact sequence of soft sheaves

$$0 \to \Theta^r_1 \to \Theta^r_2 \to \Theta^r_3 \to \cdots \to \Theta^r_M \to 0, \tag{5}$$

where the morphisms are the exterior derivative, and $M = mr+1$. This sequence is a subsequence of the De Rham sequence

$$\begin{aligned} 0 \to \mathbb{R} \to \Omega^r_0 \to \Omega^r_1 \to \Omega^r_2 \to \cdots \to \Omega^r_M \\ \to \Omega^r_{M+1} \to \cdots \to \Omega^r_{N-1} \to \Omega^r_N \to 0, \end{aligned} \tag{6}$$

where $N = \dim J^rY = 1 + m(r+1)$. The quotient sequence

$$\begin{aligned} 0 \to \mathbb{R} \to \Omega_0^r \to \Omega_1^r/\Theta_1^r \to \Omega_2^r/\Theta_2^r \to \cdots \to \Omega_M^r/\Theta_M^r \\ \to \Omega_{M+1}^r \to \cdots \to \Omega_{N-1}^r \to \Omega_N^r \to 0 \end{aligned} \tag{7}$$

is also exact. (7) is the *r-th order variational sequence*. The class of a differential form $\rho \in \Omega_k^r$ is denoted by $[\rho]$. The quotient mapping $E_k : \Omega_k^r/\Theta_k^r \to \Omega_{k+1}^r/\Theta_{k+1}^r$ is defined by

$$E_k([\rho]) = [d\rho]. \tag{8}$$

E_1 is the well-known *Euler-Lagrange mapping*, and E_2 is the *Helmholtz-Sonin mapping.*

A *lagrangian* of order r for Y is a π^r-horizontal 1-form λ on J^rY. In fibered coordinates,

$$\lambda = \mathcal{L}dt, \tag{9}$$

where $\mathcal{L}$ is the *Lagrange function.* A form $\rho \in \Omega_1^r Y$ is a *Lepage* 1*-form* if $p_1 d\rho$ is a $\pi^{r+1,0}$-horizontal. A Lepage form ρ is a *Lepage equivalent* of λ, if $h\rho = \lambda$.

It is known that in higher order mechanics, Lepage equivalents are uniquely determined by lagrangians (Krupka [**6**]). We denote by θ_λ the Lepage equivalent of λ. For $r = 1$, θ_λ is the well-known *Cartan form*, if $r > 1$, we get the *generalized Cartan form.* Computing $E_\lambda = p_1 d\theta_\lambda$, we obtain the *Euler-Lagrange form*

$$E_\lambda = E_\sigma(\mathcal{L})\omega^\sigma \wedge dt, \tag{10}$$

where the components

$$E_\sigma(\mathcal{L}) = \sum_{l=0}^{r} (-1)^l \frac{d^l}{dt^l} \frac{\partial \mathcal{L}}{\partial q_l^\sigma} \tag{11}$$

are the *Euler-Lagrange expressions.*

3. The Interior Euler-Lagrange operator

In this section, we give a new coordinate-free definition of the interior Euler-Lagrange operator. First we need the following lemma.

LEMMA 1. *For every* 1*-contact* 2*-form* ρ *on* J^rY *there exist a unique* 1*-contact,* ω^σ*-generated* 2*-form* ρ_0 *on* $J^{2r+1}Y$ *and a unique contact* 1*-form* ρ' *on* $J^{2r}Y$ *such that*

$$\left(\pi^{2r+1,r}\right)^* \rho = \rho_0 + p_1 d\rho'. \tag{12}$$

PROOF. To prove existence, write in a fibered chart

$$\left(\pi^{2r+1,r}\right)^* \rho = \sum_{i=0}^{r} P_\sigma^i \omega_i^\sigma \wedge dt. \tag{13}$$

By a direct computation,

$$p_1 d(Q_\sigma^i \omega_{i-1}^\sigma) = -Q_\sigma^i \omega_i^\sigma \wedge dt - \frac{dQ_\sigma^i}{dt} \omega_{i-1}^\sigma \wedge dt \tag{14}$$

for any Q^i_σ. Applying this formula to (13), we can eliminate all terms with $i \geqslant 1$. We obtain

$$\left(\pi^{2r+1,r}\right)^* \rho = \sum_{i=0}^{r}(-1)^i \frac{d^i P^i_\sigma}{dt^i}\omega^\sigma \wedge dt + p_1 d\left(\sum_{i=1}^{r}\sum_{j=1}^{i}(-1)^j \frac{d^{j-1}P^i_\sigma}{dt^{j-1}}\omega^\sigma_{i-j}\right). \tag{15}$$

Setting

$$\rho_0 = \sum_{i=0}^{r}(-1)^i \frac{d^i P^i_\sigma}{dt^i}\omega^\sigma \wedge dt, \qquad \rho' = \sum_{i=1}^{r}\sum_{j=1}^{i}(-1)^j \frac{d^{j-1}P^i_\sigma}{dt^{j-1}}\omega^\sigma_{i-j}, \tag{16}$$

we obtain (12).

To prove uniqueness, suppose that

$$\left(\pi^{2r+1,r}\right)^* \rho = \rho_0 + p_1 d\rho' = \eta_0 + p_1 d\eta', \tag{17}$$

where

$$\begin{aligned} \rho_0 &= A_\sigma \omega^\sigma \wedge dt, \qquad \rho' = \sum_{i=0}^{2r-1} A^i_\sigma \omega^\sigma_i, \\ \eta_0 &= B_\sigma \omega^\sigma \wedge dt, \qquad \eta' = \sum_{i=0}^{2r-1} B^i_\sigma \omega^\sigma_i. \end{aligned} \tag{18}$$

Then from (17)

$$(A_\sigma - B_\sigma)\omega^\sigma \wedge dt + p_1 d\left(\sum_{i=0}^{2r-1}(A^i_\sigma - B^i_\sigma)\omega^\sigma_i\right) = 0. \tag{19}$$

From this equation it follows that $A^i_\sigma - B^i_\sigma = 0$ and $A_\sigma - B_\sigma = 0$, proving that $\rho' = \eta'$ and $\rho_0 = \eta_0$. □

Now we are in a position to introduce two operators I and J, acting on contact forms, in three steps:

(a) If ρ is a 1-contact 2-form on J^rY, we set

$$I\rho = \rho_0, \qquad J\rho = \rho'. \tag{20}$$

(b) Let ρ be a 2-contact 3-form on J^rY, let Ξ_1, Ξ_2 be π-vertical vector fields. We define a 2-contact 3-form $I\rho$ on $J^{2r+1}Y$ by

$$i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I\rho = \frac{1}{2}\left(i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_1}\rho) - i_{J^{2r+1}\Xi_1} I(i_{J^r\Xi_2}\rho)\right), \tag{21}$$

and a 2-contact 2-form $J\rho$ on $J^{2r}Y$ by

$$i_{J^{2r}\Xi_2} i_{J^{2r}\Xi_1} J\rho = -\frac{1}{2}\left(i_{J^{2r}\Xi_2} J(i_{J^r\Xi_1}\rho) - i_{J^{2r}\Xi_1} J(i_{J^r\Xi_2}\rho)\right). \tag{22}$$

(c) Let ρ be a k-contact $(k+1)$-form on J^rY, let $\Xi_1, \Xi_2, \dots, \Xi_k$ be π-vertical vector fields. We define a k-contact $(k+1)$-form $I\rho$ on $J^{2r+1}Y$ by

$$\begin{aligned}
&i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I\rho \\
&= \frac{1}{k}\Big(i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_1}\rho) \\
&- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_1} I(i_{J^r\Xi_2}\rho) \\
&- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_{k-1}} \dots i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_3}\rho) \\
&- \dots \\
&- i_{J^{2r+1}\Xi_k} i_{J^{2r+1}\Xi_1} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_{k-1}}\rho) \\
&- i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_{k-1}} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_k}\rho)\Big).
\end{aligned} \tag{23}$$

Analogously, we define a k-contact k-form $J\rho$ on $J^{2r}Y$ by

$$\begin{aligned}
&i_{J^{2r}\Xi_k} \dots i_{J^{2r}\Xi_2} i_{J^{2r}\Xi_1} J\rho \\
&= -\frac{1}{k}\Big(i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}} \dots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_2} J(i_{J^r\Xi_1}\rho) \\
&- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}} \dots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_1} J(i_{J^r\Xi_2}\rho) \\
&- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_{k-1}} \dots i_{J^{2r}\Xi_1} i_{J^{2r}\Xi_2} J(i_{J^r\Xi_3}\rho) \\
&- \dots \\
&- i_{J^{2r}\Xi_k} i_{J^{2r}\Xi_1} \dots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_2} J(i_{J^r\Xi_{k-1}}\rho) \\
&- i_{J^{2r}\Xi_1} i_{J^{2r}\Xi_{k-1}} \dots i_{J^{2r}\Xi_3} i_{J^{2r}\Xi_2} J(i_{J^r\Xi_k}\rho)\Big).
\end{aligned} \tag{24}$$

Note that in (23) and (24), we apply *alternations* to all vector arguments.

For example, if $k = 2$ and ρ is a 2-contact 3-form on $J^{r+1}Y$,

$$\rho = \sum_{i,j=0}^{r} P^{ij}_{\sigma\nu} \omega^\sigma_i \wedge \omega^\nu_j \wedge dt, \tag{25}$$

we have

$$I\rho = \frac{1}{2} \sum_{i,j=0}^{r} \sum_{l=0}^{i} (-1)^i \binom{i}{l} \frac{d^{i-l}}{dt^{i-l}} (P^{ji}_{\nu\sigma} - P^{ij}_{\sigma\nu})\, \omega^\nu_{j+l} \wedge \omega^\sigma \wedge dt, \tag{26}$$

and

$$J\rho = -\frac{1}{2} \sum_{j=0}^{r} \sum_{i=1}^{r} \sum_{l=1}^{i} \sum_{k=0}^{l-1} (-1)^l \binom{l-1}{k} \frac{d^{l-1-k}}{dt^{l-1-k}} (P^{ji}_{\nu\sigma} - P^{ij}_{\sigma\nu})\, \omega^\nu_{j+k} \wedge \omega^\sigma_{i-l}. \tag{27}$$

THEOREM 1. *Let $k \geqslant 1$, let ρ be a k-contact $(k+1)$-form on J^rY. Then*

$$(\pi^{2r+1,r})^*\rho = I\rho + p_k dJ\rho. \tag{28}$$

PROOF. For $k = 1$, Theorem 1 reduces to the Lemma 1. Supposing that the theorem is true for some k, we want to show that then (28) also holds for $(k+1)$-forms.

Let ρ be a $(k+1)$-contact $(k+2)$-form on J^rY. Let $\Xi_1, \Xi_2, \dots, \Xi_k, \Xi_{k+1}$, be π-vertical vector fields on Y. To compute the contractions in (c) we use the identities

$$i_{J^{2r+1}\Xi}(\pi^{2r+1,r})^*\rho = I(i_{J^r\Xi}\rho) + p_k dJ(i_{J^r\Xi}\rho), \tag{29}$$

$$i_{J^{2r+1}\Xi} p_k dJ\rho = p_{k-1} d i_{J^{2r}\Xi} J\rho = p_{k-1} i_{J^{2r}\Xi} dJ\rho. \tag{30}$$

Then

$$\begin{aligned}
& i_{J^{2r+1}\Xi_{k+1}} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} (\pi^{2r+1,r})^* \rho \\
& = \frac{1}{k+1} (i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_2} (I(i_{J^r\Xi_1}\rho) + p_k dJ(i_{J^r\Xi_1}\rho)) \\
& - i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_3} i_{J^{2r+1}\Xi_1} (I(i_{J^r\Xi_2}\rho) + p_k dJ(i_{J^r\Xi_2}\rho)) \\
& - \dots \\
& - i_{J^{2r+1}\Xi_{k+1}} i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_{k-1}} \dots i_{J^{2r+1}\Xi_2} (I(i_{J^r\Xi_k}\rho) + p_k dJ(i_{J^r\Xi_k}\rho))) \\
& - i_{J^{2r+1}\Xi_1} i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_2} (I(i_{J^r\Xi_{k+1}}\rho) + p_k dJ(i_{J^r\Xi_{k+1}}\rho))) \\
& = i_{J^{2r+1}\Xi_{k+1}} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I\rho \\
& + h d i_{J^{2r+1}\Xi_{k+1}} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} J\rho \\
& = i_{J^{2r+1}\Xi_{k+1}} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} (I\rho + p_{k+1} dJ\rho).
\end{aligned} \tag{31}$$

Omitting contractions, we get (28). This completes the proof. □

Note that an equivalent definition of the operator I can be given by the following formula

$$\begin{aligned}
& i_{J^{2r+1}\Xi_k} \dots i_{J^{2r+1}\Xi_2} i_{J^{2r+1}\Xi_1} I\rho = \frac{1}{k} \big(i_{J^{2r+1}\Xi_k} I(i_{J^r\Xi_{k-1}} \dots i_{J^r\Xi_2} i_{J^r\Xi_1}\rho) \\
& \quad - i_{J^{2r+1}\Xi_{k-1}} I(i_{J^r\Xi_k} i_{J^r\Xi_{k-2}} \dots i_{J^r\Xi_2} i_{J^r\Xi_1}\rho) \\
& \quad - \dots - i_{J^{2r+1}\Xi_2} I(i_{J^r\Xi_{k-1}} \dots i_{J^r\Xi_3} i_{J^r\Xi_k} i_{J^r\Xi_1}\rho) \\
& \quad - i_{J^{2r+1}\Xi_1} I(i_{J^r\Xi_{k-1}} \dots i_{J^r\Xi_2} i_{J^r\Xi_k}\rho) \big) .
\end{aligned} \tag{32}$$

From k-contact $(k+1)$-forms, we extend the operator I to *all* $(k+1)$-forms ρ on J^rY. We set

$$\mathcal{I}\rho = I p_k \rho. \tag{33}$$

We get an $\mathbb{R}$-linear mapping $\mathcal{I} : \Omega^r_k Y \to \Omega^{2r+1}_k Y$, which we call the *interior Euler-Lagrange operator*. Main properties of $\mathcal{I}$ can be summarized as follows.

THEOREM 2. *Let $\pi : Y \to X$ be a fibered manifold over one-dimensional base X. Let $k \geqslant 0$.*

(a) *For every open set $V \subset Y$ and every $\rho \in \Omega^r_{k+1} V$, $\mathcal{I}\rho$ lies in the same class as $(\pi^{2r+1,r})^*\rho$.*

(b) *$\mathcal{I}^2 = \mathcal{I}$ (up to the canonical jet projection).*

(c) *For every open set $V \subset Y$, the kernel of the mapping $\mathcal{I} : \Omega^r_{k+1} V \to \Omega^{2r+1}_{k+1} V$ coincides with $\Theta^r_{k+1} V$.*

From (c) it follows that the image of $\Omega^r_{k+1} V$ by $\mathcal{I}$, $\mathcal{I}(\Omega^r_{k+1} V) \subset \Omega^{2r+1}_{k+1} Y$, is canonically isomorphic with the quotient group $\Omega^r_{k+1} V / \Theta^r_{k+1} V$. If $\rho \in \Omega^r_{k+1} V$, we call the form $\mathcal{I}\rho$ the *canonical representative* of ρ.

Anderson [**1**] defines the interior Euler operator by means of the theory of differential operators on the *infinite* jet prolongation of a fibered manifold. Similar approach is used by Krbek and Musilová [**4**] for finite prolongations. On the contrary, in our definition only the concept of the Lie derivative on finite order prolongations is needed, and the standard rules for computation with Lie derivatives are applied. Formally, both definitions give the same result.

4. Lepage forms

In this section, we introduce Lepage forms, and Lepage equivalents of the canonical representatives of differential forms.

Let $k \geqslant 0$. A form $\rho \in \Omega^r_{k+1}Y$ is called a *Lepage form*, if

$$p_{k+1} d\rho = \mathcal{I} d\rho. \tag{34}$$

Let $\beta \in \Omega^s_{k+1}/\Theta^s_{k+1}$ be a class, i.e., let $\beta = \mathcal{I}\eta$ for some $\eta \in \Omega^s_{k+1}$. A form $\rho \in \Omega^r_{k+1}$ is said to be a *Lepage equivalent* of β, if ρ is a Lepage form, and

$$p_k \rho = \beta. \tag{35}$$

The meaning of a Lepage form consists in a generalization of formulas (10), (11), Section 2. For $k = 0$, this definition reduces to the original one (Krupka [**6**], [**7**]); in this case, $p_1 d\rho$ is the Euler-Lagrange form. For $k = 1$, $p_2 d\rho$ is the *Helmholtz form* (for more details we refer to Šeděnková [**15**]).

If ρ is a Lepage form, then $d\rho$, and for any η, the form $\rho + d\eta$ is also a Lepage form.

We now analyze the structure of Lepage 2-forms in higher order mechanics. Let (V, ψ), $\psi = (t, q^\sigma)$, be a fibered chart on Y. To simplify notation, we define the *cut total derivative* of a function $f : V^r \to \mathbb{R}$ by

$$\frac{d' f}{dt} = \frac{\partial f}{\partial t} + \sum_{i=0}^{r-1} \frac{\partial f}{\partial q^\sigma_i} q^\sigma_{i+1} = \frac{df}{dt} - \frac{\partial f}{\partial q^\sigma_r} q^\sigma_{r+1}. \tag{36}$$

We also use the notation (V^3, ψ^3), $\psi^3 = (t, q^\sigma, \dot q^\sigma, \ddot q^\sigma, \dddot q^\sigma)$, for the associated charts on J^3Y (and analogously on J^1Y and J^2Y), and write

$$\omega^\sigma = dq^\sigma - q^\sigma dt, \qquad \dot\omega^\sigma = d\dot q^\sigma - \ddot q^\sigma dt, \qquad \ddot\omega^\sigma = d\ddot q^\sigma - \dddot q^\sigma dt. \tag{37}$$

In the following three theorems, we characterize separately the Lepage 2-forms on J^1Y, J^2Y, and, in general, on J^rY. The proofs are based on direct computations.

Let $\rho \in \Omega^1_2 Y$, and let

$$\begin{aligned} \rho = {} & a_\sigma \omega^\sigma \wedge dt + b_\sigma d\dot q^\sigma \wedge dt + c_{\sigma\nu} \omega^\sigma \wedge \omega^\nu \\ & + d_{\sigma\nu} d\dot q^\sigma \wedge \omega^\nu + e_{\sigma\nu} d\dot q^\sigma \wedge d\dot q^\nu \end{aligned} \tag{38}$$

be the chart expressions of ρ. The coefficients $c_{\sigma\nu}$, $e_{\sigma\nu}$ in this expression are supposed to be antisymmetric.

THEOREM 3. *Let $\rho \in \Omega^1_2 Y$ be expressed by* (38). *The following three conditions are equivalent:*

(a) *ρ is a Lepage form.*

(b) *The components of ρ satisfy*

$$\begin{aligned} & \frac{\partial e_{\sigma\nu}}{\partial \dot q^\lambda} + \frac{\partial e_{\nu\lambda}}{\partial \dot q^\sigma} + \frac{\partial e_{\lambda\sigma}}{\partial \dot q^\nu} = 0, \\ & d_{\sigma\nu} - d_{\nu\sigma} - \frac{\partial b_\sigma}{\partial \dot q^\nu} + \frac{\partial b_\nu}{\partial \dot q^\sigma} + 2\frac{d' e_{\sigma\nu}}{dt} = 0, \\ & \frac{\partial d_{\sigma\nu}}{\partial \dot q^\lambda} - \frac{\partial d_{\nu\sigma}}{\partial \dot q^\lambda} + \frac{\partial d_{\lambda\sigma}}{\partial \dot q^\nu} - \frac{\partial d_{\lambda\nu}}{\partial \dot q^\sigma} + 2\left(\frac{\partial e_{\lambda\sigma}}{\partial q^\nu} - \frac{\partial e_{\lambda\nu}}{\partial q^\sigma}\right) = 0, \\ & \frac{\partial a_\nu}{\partial \dot q^\sigma} - \frac{\partial a_\sigma}{\partial \dot q^\nu} + \frac{\partial b_\nu}{\partial q^\sigma} - \frac{\partial b_\sigma}{\partial q^\nu} + \frac{d' d_{\sigma\nu}}{dt} - \frac{d' d_{\nu\sigma}}{dt} + 4c_{\sigma\nu} = 0. \end{aligned} \tag{39}$$

(c) *There exist functions A_σ, and a 1-form η such that*

$$\rho = A_\sigma \omega^\sigma \wedge dt + \frac{1}{4}\left(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma}\right)\omega^\sigma \wedge \omega^\nu + d\eta. \tag{40}$$

Let now ρ be a 2-form on J^2Y,

$$\begin{aligned}\rho = {} & a_\sigma \omega^\sigma \wedge dt + b_\sigma \dot{\omega}^\sigma \wedge dt + c_\sigma d\ddot{q}^\sigma \wedge dt + d_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + e_{\sigma\nu}\dot{\omega}^\sigma \wedge \omega^\nu \\ & + f_{\sigma\nu}\dot{\omega}^\sigma \wedge \dot{\omega}^\nu + g_{\sigma\nu} d\ddot{q}^\sigma \wedge \omega^\nu + h_{\sigma\nu} d\ddot{q}^\sigma \wedge \dot{\omega}^\nu + k_{\sigma\nu} d\ddot{q}^\sigma \wedge d\ddot{q}^\nu,\end{aligned} \tag{41}$$

where the coefficients $d_{\sigma\nu}$, $f_{\sigma\nu}$, $k_{\sigma\nu}$ are antisymmetric. One can easily see that

$$\begin{aligned}p_2 d\rho = {} & (P_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + Q_{\sigma\nu}\dot{\omega}^\sigma \wedge \omega^\nu + R_{\sigma\nu}\ddot{\omega}^\sigma \wedge \omega^\nu \\ & + S_{\sigma\nu}\dot{\omega}^\sigma \wedge \dot{\omega}^\nu + T_{\sigma\nu}\ddot{\omega}^\sigma \wedge \dot{\omega}^\nu + U_{\sigma\nu}\ddot{\omega}^\sigma \wedge \ddot{\omega}^\nu) \wedge dt,\end{aligned} \tag{42}$$

where

$$\begin{aligned}
P_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial a_\nu}{\partial q^\sigma} - \frac{\partial a_\sigma}{\partial q^\nu} + 2\frac{dd_{\sigma\nu}}{dt} + \left(\frac{\partial g_{\lambda\sigma}}{\partial q^\nu} - \frac{\partial g_{\lambda\nu}}{\partial q^\sigma}\right)\dddot{q}^\lambda\right),\\
Q_{\sigma\nu} &= \frac{\partial a_\nu}{\partial \dot{q}^\sigma} - \frac{\partial b_\sigma}{\partial q^\nu} + 2d_{\sigma\nu} + \frac{de_{\sigma\nu}}{dt} - \left(\frac{\partial g_{\lambda\nu}}{\partial \dot{q}^\sigma} - \frac{\partial h_{\lambda\sigma}}{\partial q^\nu}\right)\dddot{q}^\lambda,\\
R_{\sigma\nu} &= \frac{\partial a_\nu}{\partial \ddot{q}^\sigma} - \frac{\partial c_\sigma}{\partial q^\nu} + e_{\sigma\nu} + \frac{dg_{\sigma\nu}}{dt} - \left(\frac{\partial g_{\lambda\nu}}{\partial \ddot{q}^\sigma} - 2\frac{\partial k_{\lambda\sigma}}{\partial q^\nu}\right)\dddot{q}^\lambda,\\
S_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial b_\nu}{\partial \dot{q}^\sigma} - \frac{\partial b_\sigma}{\partial \dot{q}^\nu} + e_{\sigma\nu} - e_{\nu\sigma} + 2\frac{df_{\sigma\nu}}{dt} + \left(\frac{\partial h_{\lambda\sigma}}{\partial \dot{q}^\nu} - \frac{\partial h_{\lambda\nu}}{\partial \dot{q}^\sigma}\right)\dddot{q}^\lambda\right),\\
T_{\sigma\nu} &= \frac{\partial b_\nu}{\partial \ddot{q}^\sigma} - \frac{\partial c_\sigma}{\partial \dot{q}^\nu} + 2f_{\sigma\nu} + g_{\sigma\nu} + \frac{dh_{\sigma\nu}}{dt} - \left(\frac{\partial h_{\lambda\nu}}{\partial \ddot{q}^\sigma} - 2\frac{\partial k_{\lambda\sigma}}{\partial \dot{q}^\nu}\right)\dddot{q}^\lambda,\\
U_{\sigma\nu} &= \frac{1}{2}\left(\frac{\partial c_\nu}{\partial \ddot{q}^\sigma} - \frac{\partial c_\sigma}{\partial \ddot{q}^\nu} + h_{\sigma\nu} - h_{\nu\sigma} + 2\frac{dk_{\sigma\nu}}{dt} + 2\left(\frac{\partial k_{\lambda\sigma}}{\partial \ddot{q}^\nu} - \frac{\partial k_{\lambda\nu}}{\partial \ddot{q}^\sigma}\right)\dddot{q}^\lambda\right).
\end{aligned} \tag{43}$$

We have the following result.

THEOREM 4. *Let $\rho \in \Omega_2^2 Y$ be a form, expressed by* (41). *The following three conditions are equivalent:*

(a) *ρ is a Lepage form.*

(b) *The components of $p_2 d\rho$ satisfy*

$$\begin{aligned}&U_{\sigma\nu} - U_{\nu\sigma} = 0, \qquad T_{\sigma\nu} = 0, \qquad S_{\sigma\nu} - S_{\nu\sigma} = 0,\\ &R_{\sigma\nu} + R_{\nu\sigma} = 0, \qquad Q_{\sigma\nu} - Q_{\nu\sigma} - 2\frac{dR_{\sigma\nu}}{dt} = 0.\end{aligned} \tag{44}$$

(c) *There exist functions A_σ and a 1-form η such that*

$$\frac{\partial}{\partial \ddot{q}^\tau}\left(\frac{\partial A_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial A_\nu}{\partial \ddot{q}^\sigma}\right) = 0, \tag{45}$$

and

$$\begin{aligned}\rho = {} & A_\sigma \omega^\sigma \wedge dt + \frac{1}{4}\left(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma} - \frac{d}{dt}\left(\frac{\partial A_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial A_\nu}{\partial \ddot{q}^\sigma}\right)\right)\omega^\sigma \wedge \omega^\nu \\ & - \frac{1}{2}\left(\frac{\partial A_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial A_\nu}{\partial \ddot{q}^\sigma}\right)\dot{\omega}^\sigma \wedge \omega^\nu + d\eta.\end{aligned} \tag{46}$$

Note that conditions (44) can be expressed in terms of the coefficients of the form (41) by means of (43).

To characterize the Lepage 2-forms on J^rY, consider a form $\rho \in \Omega_2^r Y$. Analogous results can also be given for r-th order 2-forms

$$\begin{aligned}\rho = {} & \sum_{i=0}^{r-1} a_\sigma^i \omega_i^\sigma \wedge dt + b_\sigma^r dq_r^\sigma \wedge dt + \sum_{i,j=0}^{r-1} c_{\sigma\nu}^{ij} \omega_i^\sigma \wedge \omega_j^\nu \\ & + \sum_{j=0}^{r-1} d_{\sigma\nu}^{rj} dq_r^\sigma \wedge \omega_j^\nu + e_{\sigma\nu}^{rr} dq_r^\sigma \wedge dq_r^\nu,\end{aligned} \tag{47}$$

where the coefficients $c_{\sigma\nu}^{ij}$ are antisymmetric in the pairs $\binom{i}{\sigma}, \binom{j}{\nu}$, and the coefficients $e_{\sigma\nu}^{rr}$ are antisymmetric in σ, ν. Then we have

$$p_2 d\rho = \left(\sum_{j=0}^{r} H_{\sigma\nu}^{0j} \omega^\sigma \wedge \omega_j^\nu + \sum_{i,j=1}^{r} H_{\sigma\nu}^{ij} \omega_i^\sigma \wedge \omega_j^\nu \right) \wedge dt, \tag{48}$$

where

$$\begin{aligned}
H_{\sigma\nu}^{00} &= \frac{1}{2}\left(\frac{\partial a_\nu^0}{\partial q_0^\sigma} - \frac{\partial a_\sigma^0}{\partial q_0^\nu} + 2\frac{dc_{\sigma\nu}^{00}}{dt} + \left(\frac{\partial d_{\lambda\sigma}^{r0}}{\partial q_0^\nu} - \frac{\partial d_{\lambda\nu}^{r0}}{\partial q_0^\sigma} \right) q_{r+1}^\lambda \right), \\
H_{\sigma\nu}^{0j} &= -\frac{\partial a_\sigma^0}{\partial q_j^\nu} + \frac{\partial a_\nu^j}{\partial q_0^\sigma} + 2\frac{dc_{\sigma\nu}^{0j}}{dt} + c_{\sigma\nu}^{0\,j-1} - c_{\nu\ \sigma}^{j-1\,0} + \left(\frac{\partial d_{\lambda\sigma}^{r0}}{\partial q_j^\lambda} - \frac{\partial d_{\lambda\nu}^{rj}}{\partial q_0^\sigma} \right) q_{r+1}^\lambda, \\
H_{\sigma\nu}^{0r} &= \frac{\partial a_\sigma^0}{\partial q_r^\nu} + \frac{\partial b_\nu^r}{\partial q_0^\sigma} + c_{\sigma\nu}^{0\,r-1} - c_{\nu\ \sigma}^{r-1\,0} - \frac{dd_{\nu\sigma}^{r0}}{dt} + \left(\frac{\partial d_{\lambda\sigma}^{r0}}{\partial q_r^\nu} + 2\frac{\partial e_{\nu\lambda}^{rr}}{\partial q_0^\sigma} \right) q_{r+1}^\lambda, \\
H_{\sigma\nu}^{ij} &= \frac{1}{2}\left(\frac{\partial a_\nu^j}{\partial q_i^\sigma} - \frac{\partial a_\sigma^i}{\partial q_j^\nu} + 2\frac{dc_{\sigma\nu}^{ij}}{dt} + 2(c_{\sigma\ \nu}^{i-1\,j} - c_{\nu\ \sigma}^{j-1\,i}) + \left(\frac{\partial d_{\lambda\sigma}^{ri}}{\partial q_j^\nu} - \frac{\partial d_{\lambda\nu}^{rj}}{\partial q_i^\sigma} \right) q_{r+1}^\lambda \right), \\
H_{\sigma\nu}^{ir} &= \frac{1}{2}\left(-\frac{\partial a_\sigma^i}{\partial q_r^\nu} + \frac{\partial b_\nu^r}{\partial q_i^\sigma} + 2c_{\sigma\nu}^{i\,r-1} - d_{\nu\sigma}^{r\,i-1} - \frac{dd_{\nu\sigma}^{ri}}{dt} + \left(\frac{\partial d_{\lambda\sigma}^{ri}}{\partial q_r^\nu} + 2\frac{\partial e_{\nu\lambda}^{rr}}{\partial q_i^\sigma} \right) q_{r+1}^\lambda \right) \\
&= -H_{\nu\sigma}^{ri}, \\
H_{\sigma\nu}^{rr} &= \frac{1}{2}\left(\frac{\partial b_\nu^r}{\partial q_r^\sigma} - \frac{\partial b_\sigma^r}{\partial q_r^\nu} + d_{\sigma\nu}^{r\,r-1} - d_{\nu\sigma}^{r\,r-1} + 2\frac{de_{\sigma\nu}^{rr}}{dt} + \left(\frac{\partial e_{\nu\lambda}^{rr}}{\partial q_r^\sigma} - \frac{\partial e_{\sigma\lambda}^{rr}}{\partial q_r^\nu} \right) q_{r+1}^\lambda \right).
\end{aligned} \tag{49}$$

THEOREM 5. *Let $\rho \in \Omega_2^r V$ be a form, expressed by (47). Then ρ is Lepage if and only if*

$$\begin{aligned}
& H_{\sigma\nu}^{ij} - H_{\nu\sigma}^{ji} = 0, \qquad 1 \leqslant i, j \leqslant r, \\
& H_{\sigma\nu}^{0j} + (-1)^j H_{\nu\sigma}^{0j} + \sum_{l=j+1}^{r} (-1)^l \binom{l}{j} \frac{d^{l-j}}{dt^{l-j}} H_{\nu\sigma}^{0l} = 0, \qquad 1 \leqslant j \leqslant r.
\end{aligned} \tag{50}$$

5. 2nd order dynamical forms

We know that the general definition of a Lepage form includes the Lepage equivalent of the canonical representatives of a 2-forms, that is of the *dynamical form*. Consider as an example a *second order* dynamical form

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge dt. \tag{51}$$

One can easily show that the *Lepage equivalent* ρ_ε of ε is of the form

$$\begin{aligned}\rho_\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge dt + \frac{1}{4}\left(\frac{\partial \varepsilon_\sigma}{\partial \dot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \dot{q}^\sigma} - \frac{d}{dt}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\right)\omega^\sigma \wedge \omega^\nu \\ - \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\dot{\omega}^\sigma \wedge \omega^\nu\end{aligned} \tag{52}$$

(compare with the second order Lepage form (46), Section 4).

The *Helmholtz form* of ε is the class $H_\varepsilon = [d\rho_\varepsilon] = p_2 d\rho_\varepsilon$. Explicitly,

$$\begin{aligned}H_\varepsilon = \frac{1}{2}\Bigg(\left(\frac{\partial \varepsilon_\sigma}{\partial q^\nu} - \frac{\partial \varepsilon_\nu}{\partial q^\sigma} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_\sigma}{\partial \dot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \dot{q}^\sigma}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\right)\omega^\nu \wedge \omega^\sigma \\ + \left(\frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial \dot{q}^\nu} + \frac{\partial \varepsilon_\nu}{\partial \dot{q}^\sigma}\right) - \frac{d}{dt}\frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\dot{\omega}^\nu \wedge \omega^\sigma + \frac{1}{2}\left(\frac{\partial \varepsilon_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial \varepsilon_\nu}{\partial \ddot{q}^\sigma}\right)\ddot{\omega}^\nu \wedge \omega^\sigma\Bigg) \wedge dt.\end{aligned} \tag{53}$$

It is known from the variational sequence theory that ε is locally variational if and only if $H_\varepsilon = 0$. Explicit expression for the Helmholtz form in field theory can be found in [**5**].

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