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# Contact symmetries and variational sequences

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Abstract. One of the results of the variational sequence theory, related to the inverse problem of the calculus of variations, states that a dynamical form $\varepsilon$, representing a system of partial differential equations, is locally variational if and only if the Helmholtz form $H(\varepsilon)$ vanishes. In this paper, a relationship between the Lie derivatives of $\varepsilon$ and $H(\varepsilon)$ is studied. It is shown that invariance of the Helmholtz form $H(\varepsilon)$ with respect to a vector field $Z$ preserving contact forms is equivalent with local variationality of the Lie derivative of $\varepsilon$ by $Z$.

## 1. Introduction

In this paper we study conditions which ensure that the Lie derivative $\partial_Z\varepsilon$ of a dynamical form $\varepsilon$ by a vector field $Z$, defined on the domain of $\varepsilon$, is a *variational* dynamical form. In particular, we find equations for the vector fields $Z$, transforming a non-variational dynamical form $\varepsilon$ to variational ones, $\partial_Z\varepsilon$. We call such vector fields *variational vector fields* for the dynamical form $\varepsilon$.

We use the theory of global higher order variational functionals in fibered spaces, our main sources are Goldschmidt and Sternberg [5], Krupka [6], [7], [8], and Trautman [18]. We also recall the variational sequence theory due to Krupka [9], [10]. For different aspects of the inverse variational problem and its connection to closed differential forms we refer to Crampin, Prince and Thompson [2], and Krupková [12], [13], [14].

Then we explain a new idea, namely that there should exist, in a certain sense, a close correspondence between the notions of variationality of a differential form and invariance of its exterior derivative. We introduce *contact symmetries* as vector fields, preserving contact differential forms (Garcia [4]). We show that the Lie derivative of a dynamical form $\varepsilon$ by a contact symmetry is variational if and only if $Z$ leaves invariant the Helmholtz form $H(\varepsilon)$ of $\varepsilon$, i.e., $\partial_Z H(\varepsilon) = 0$.

The next part of the paper is devoted to *fibered mechanics*, i.e., the variational theory on fibered manifolds over 1-dimensional bases. We give explicit formulas for contact symmetries, and derive equations for variational vector fields. In particular, we analyze a class of 2-forms, locally generated by *contact* 1-forms. Their fundamental property is that they correspond to systems of *second order* ordinary differential equations, *linear* in the second derivatives; moreover, *closed* 2-forms are in one-to-one correspondence with *variational* dynamical forms.

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Finally we discuss an example due to Douglas [3]. In particular, we show that there may exist systems of equations which are not variational, even do not possess variational multipliers, but admit variational contact symmetries.

Our discussion evidently opens several new questions that we do not touch in this paper. Among them, one of the most important concerns the relationship between solutions of the equations defined by the dynamical form $\varepsilon$, and those of the equations given by the transformed dynamical form $\partial_Z\varepsilon$ (here we refer to Prince [15], where similar features were observed).

## 2. Differential forms on a fibered manifold

Throughout the paper, $Y$ is a fibered manifold with base $X$ and projection $\pi$. We put $n = \dim X$, $n+m = \dim Y$. $J^rY$, $r \geqslant 0$, is the *$r$-jet prolongation* of $Y$, and $\pi^{r,s} : J^rY \to J^sY$, $\pi^r : J^rY \to X$ are the *canonical jet projections*. The points of $J^rY$ are $r$-jets $J^r_x\gamma$ of sections $\gamma$ of $Y$ at $x \in X$; the $r$-jet prolongation of $\gamma$ is the mapping $x \to J^r\gamma(x) = J^r_x\gamma$.

Any fibered chart $(V,\psi)$, $\psi = (x^i, y^\sigma)$ on $Y$ induces the associated charts $(U,\varphi)$, $\varphi = (x^i)$ on $X$, and $(V^r,\psi^r)$, $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_r})$ on $J^rY$, where $U = \pi(V)$, and $V^r = (\pi^r)^{-1}(V)$, $1 \leqslant i, j_1, j_2, \dots, j_r \leqslant n$, $1 \leqslant \sigma \leqslant m$.

A vector $\Xi$ at $y \in Y$ is *$\pi$-vertical* if $T_y\pi \cdot \Xi = 0$. A differential form $\rho$ on $Y$ is *$\pi$-horizontal* if it vanishes whenever one of its arguments is a $\pi$-vertical vector.

For any open set $W \subset Y$ we denote by $\Omega^rW$ the exterior algebra on $W^r = (\pi^r)^{-1}(W)$. $\Omega^r_0W$ and $\Omega^r_kW$ are the ring of smooth functions and the $\Omega^r_0W$-module of smooth $k$-forms on $W^r$, respectively. We also use some submodules, $\Omega^r_{k,X}W \subset \Omega^r_kW$, the submodule of $\pi^r$-horizontal forms, and $\Omega^r_{k,Y}W \subset \Omega^r_kW$, the submodule of $\pi^{r,0}$-horizontal forms.

We have a morphism of exterior algebras $h : \Omega^r_kW \to \Omega^{r+1}_{k,X}W$, defined by

$$hf = f\pi^{r+1,r}, \quad hdx^i = dx^i, \quad hdy^\sigma_{j_1j_2\dots j_l} = y^\sigma_{j_1j_2\dots j_lp}dx^p, \tag{2.1}$$

where $f : V^r \to \mathbb{R}$ is a function. Obviously, $J^r\gamma^*\rho = J^{r+1}\gamma^*h\rho$ for every section $\gamma$ of $\pi$. We call $h$ the *$\pi$-horizontalization*.

We say that a form $\rho \in \Omega^r_kW$ is *contact* if $h\rho = 0$. For any fibered chart $(V,\psi)$, $\psi = (x^i, y^\sigma)$, the 1-forms

$$\omega^\sigma_{j_1j_2\dots j_l} = dy^\sigma_{j_1j_2\dots j_l} - y^\sigma_{j_1j_2\dots j_lp}dx^p, \tag{2.2}$$

where $0 \leqslant l \leqslant r-1$, are examples of contact forms on $V^r$. The system of forms

$$(dx^i, \omega^\sigma, \dots, \omega^\sigma_{j_1j_2\dots j_{r-1}}, dy^\sigma_{j_1j_2\dots j_r}) \tag{2.3}$$

is a basis of linear forms on $V^r$. By the *contact ideal* on $W$ we mean the ideal $\Theta^rW$ in the exterior algebra $\Omega^rW$, locally generated by the forms $\omega^\sigma_{j_1\dots j_l}, d\omega^\sigma_{j_1\dots j_l}$, $0 \leqslant l \leqslant r-1$. Since

$$d\omega^\sigma_{j_1\dots j_l} = -dy^\sigma_{j_1\dots j_lp} \wedge dx^p, \tag{2.4}$$

the contact ideal is also generated by the forms $\omega^\sigma, \omega^\sigma_{j_1}, \dots, \omega^\sigma_{j_1\dots j_{r-1}}, d\omega^\sigma_{j_1\dots j_{r-1}}$.

A form $\rho \in \Omega^r_kW$ has a unique decomposition

$$(\pi^{r+1,r})^*\rho = h\rho + p_1\rho + p_2\rho + \dots + p_k\rho, \tag{2.5}$$

in which $p_i\rho$ contains, in any fibered chart, exactly $i$ exterior factors $\omega^\sigma_{j_1j_2\dots j_l}$. Transformation properties of these forms guarantee invariance of the decomposition. $p_i\rho$

is the *$i$-contact component* of $\rho$. If $k \geqslant n+1$, we define $\rho \in \Omega^r_k W$ to be *strongly contact* if $p_{k-n}\rho = 0$.

By a *$\pi$-projectable* vector field we mean a vector field $\Xi$ on $Y$ such that there exists a vector field $\xi$ on $X$ satisfying $T\pi \cdot \Xi = \xi \circ \pi$. We denote by $J^r\Xi$ the *$r$-jet prolongation* of $\Xi$. We shall need the behavior of the projections $h, p_1, \ldots, p_k$ under the Lie derivative $\partial_{J^r\Xi}$. Since for any $\pi$-projectable vector field $\Xi$ the operator $\partial_{J^r\Xi}$ preserves contact forms, and $(\pi^{r+1,r})^* \partial_{J^r\Xi}\rho = \partial_{J^{r+1}\Xi}(\pi^{r+1,r})^*\rho$, we have $h\partial_{J^r\Xi}\rho = \partial_{J^{r+1}\Xi}\, h\rho$, and for all $i = 1, 2, \ldots, k$, $p_i \partial_{J^r\Xi}\rho = \partial_{J^{r+1}\Xi}\, p_i\rho$.

A *dynamical form* (of order $r$) is defined to be a 1-contact element $\varepsilon$ of the module $\Omega^r_{n+1,Y}W$, where $W$ is an open subset of $Y$ (Takens calls these forms *source forms* [17]). A *Lagrangian* (of order $r$) for $\pi$ is defined to be an element $\lambda \in \Omega^r_{n,X}W$. In a fibered chart $(V, \psi)$, $\psi = (x^i, y^\sigma)$,

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0, \tag{2.6}$$

and

$$\lambda = L\omega_0, \tag{2.7}$$

where $\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$. The component $L : V^r \to \mathbb{R}$ is a *Lagrange function*. The components $\varepsilon_\sigma$ of a dynamical form of order $r$ represent left-hand sides of a system of $m$ *differential equations* of order $r$ for sections of $\pi$ (ODE if $\dim X = 1$ and PDE if $\dim X = n > 1$).

A form $\rho \in \Omega^s_n W$ is called a *Lepage equivalent* of a Lagrangian $\lambda$ if $h\rho = \lambda$ (up to a canonical jet projection), and $p_1 d\rho \in \Omega^{s+1}_{n+1,Y}W$. The dynamical form $p_1 d\rho$ (depending only on $\lambda$) is called the *Euler–Lagrange form* of $\lambda$ and denoted by $E(\lambda)$. In a fibered chart

$$E(\lambda) = E_\sigma(L)\omega^\sigma \wedge \omega_0, \tag{2.8}$$

where

$$E_\sigma(L) = \sum_{l=0}^{r} (-1)^l d_{j_1} d_{j_2} \ldots d_{j_l} \frac{\partial L}{\partial y^\sigma_{j_1 j_2 \ldots j_l}}. \tag{2.9}$$

The components of $E(\lambda)$ are called the *Euler–Lagrange expressions*. The mapping

$$\Omega^r_{n,X}W \ni \lambda \to E(\lambda) \in \Omega^{2r}_{n+1,Y}W, \tag{2.10}$$

assigning to a Lagrangian its Euler–Lagrange form, is called the *Euler–Lagrange mapping*. Forms belonging to the *kernel* of the Euler–Lagrange mapping are called *variationally trivial*, elements of the *image* are called *variational forms*.

The *inverse problem of the calculus of variations* for a dynamical form $\varepsilon$ consists in finding a Lagrangian $\lambda$ such that $\varepsilon = E(\lambda)$. A weaker version of the inverse problem, the *variational multipliers problem* for a dynamical form $\varepsilon$, then means to find a regular matrix $(G^\nu_\sigma)$ such that $\bar{\varepsilon}$ with components $\bar{\varepsilon}_\sigma = G^\nu_\sigma \varepsilon_\nu$ is variational. In this weaker version $\varepsilon$ and $\bar{\varepsilon}$ have the same solutions.

## 3. Variational sequences

We recall the main steps of the construction of an exact sequence of sheaves, the *variational sequence*, in which the Euler–Lagrange mapping appears as a sequence morphism. By means of this sequence one obtains more information about the structure of the Euler–Lagrange mapping, and discovers new objects, describing its local and global properties.

Put $\Omega^r_{0,c} = \{0\}$, and let $\Omega^r_{k,c}$ be the sheaf of *contact* $k$-forms if $k \leqslant n$, or the sheaf of *strongly contact* $k$-forms if $k > n$, on $J^rY$. We set

$$\Theta^r_k = \Omega^r_{k,c} + d\Omega^r_{k-1,c}, \tag{3.1}$$

where $d\Omega^r_{k-1,c}$ is the image sheaf of $d\Omega^r_{k-1,c}$ by the exterior derivative $d$. It can be shown that we get an exact sequence of soft sheaves $0 \to \Theta^r_1 \to \Theta^r_2 \to \Theta^r_3 \to \dots$, where each of the morphisms is the exterior derivative, i.e., a subsequence of the *De Rham sequence* $0 \to \mathbb{R} \to \Omega^r_1 \to \Omega^r_2 \to \Omega^r_3 \to \dots$. The quotient sequence

$$0 \to \mathbb{R} \to \Omega^r_0 \to \Omega^r_1/\Theta^r_1 \to \Omega^r_2/\Theta^r_2 \to \Omega^r_3/\Theta^r_3 \to \dots \tag{3.2}$$

which is also exact, is called the *$r$-th order variational sequence on $Y$*. We denote the sequence (3.2) symbolically by $0 \to \mathbb{R} \to \mathcal{V}^r$, and the quotient mappings by

$$E_k : \Omega^r_k/\Theta^r_k \to \Omega^r_{k+1}/\Theta^r_{k+1}. \tag{3.3}$$

The variational sequence is an acyclic resolution of the constant sheaf $\mathbb{R}$ over $Y$. Let $\Gamma(Y, \mathcal{V}^r)$ denote the cochain complex of global sections of (3.2), i.e.,

$$0 \to \Gamma(Y, \mathbb{R}) \to \Gamma(Y, \Omega^r_0) \to \Gamma(Y, \Omega^r_1/\Theta^r_1) \to \Gamma(Y, \Omega^r_2/\Theta^r_2) \to \dots. \tag{3.4}$$

As a corollary to the Abstract De Rham Theorem we get the following identification of the cohomology groups $H^k(\Gamma(Y, \mathcal{V}^r))$ of this complex with the De Rham cohomology groups of the manifold $Y$:

$$H^k(\Gamma(Y, \mathcal{V}^r)) = H^kY. \tag{3.5}$$

To understand the meaning of variational sequences for global higher order variational theory, first note that the quotient sheaves $\Omega^r_k/\Theta^r_k$ are determined *up to natural isomorphisms* of Abelian groups. Thus, the classes in $\Omega^r_k/\Theta^r_k$ admit various equivalent characterizations. A simple analysis shows that the sections of the quotient sheaf $\Omega^r_n/\Theta^r_n$ can be identified, in a fibered chart, with some $n$-forms $\lambda = L\omega_0$, i.e., with some *Lagrangians.* Elements of $\Omega^r_{n+1}/\Theta^r_{n+1}$ can be identified with some $(n+1)$-forms $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$, i.e., with *dynamical forms.* More precisely, we can prove that the sheaf $\Omega^r_n/\Theta^r_n$ is isomorphic with a subsheaf of the *sheaf of Lagrangians* $\Omega^{r+1}_{n,X}$, and $\Omega^r_{n+1}/\Theta^r_{n+1}$ is isomorphic with a subsheaf of the sheaf of *dynamical forms* $\Omega^{2r+1}_{n+1,Y}$. The quotient mapping

$$E_n : \Omega^r_n/\Theta^r_n \to \Omega^r_{n+1}/\Theta^r_{n+1} \tag{3.6}$$

in this representation of sheaves coincides with the *Euler–Lagrange mapping.*

We say that a dynamical form $\varepsilon \in \Omega^s_{n+1,Y}$ is *associated* with a $(n+1)$-form $\rho \in \Omega^r W$ if $\varepsilon = [\rho]$ (here $[\rho]$ denotes the *class of* $\rho$, belonging to $\Omega^r_{n+1}/\Theta^r_{n+1}$). Then we call the class $E_{n+1}(\varepsilon) = [d\rho]$ the *Helmholtz class* of $\varepsilon = [\rho]$. The mapping

$$E_{n+1} : \Omega^r_{n+1}/\Theta^r_{n+1} \to \Omega^r_{n+2}/\Theta^r_{n+2} \tag{3.7}$$

is called the *Helmholtz mapping.* When we do not want to stress the context of the variational sequence, we also write for the Helmholtz class of $\varepsilon = [\rho]$

$$H(\varepsilon) = E_{n+1}(\varepsilon). \tag{3.8}$$

Now it is clear what kind of results are described by the variational sequence:

(i) Assume that a Lagrangian $\lambda = [\rho]$ satisfies $E_n(\lambda) = 0$. Then by exactness of (3.2), there always exists a class $[\eta]$ such that $E_{n-1}([\eta]) = [\rho] = [d\eta]$. This means that, locally, $\rho$ decomposes as the sum of a closed form and a contact form. Condition

$$E_n(\lambda) = 0 \tag{3.9}$$

is the *local variational triviality condition*. If, in addition, $H^n Y = \{0\}$, (3.5) says that that $\eta$ may be chosen *globally defined* on $J^r Y$. Condition (3.9) strongly determines the structure of Lagrangians whose Euler–Lagrange forms vanish identically.

(ii) Suppose that we have a dynamical form $\varepsilon = [\rho]$. Similarly as above, we have the *local variationality condition*

$$E_{n+1}(\varepsilon) = 0, \tag{3.10}$$

stating that $\varepsilon$ is locally variational if and only if the associated Helmholtz class vanishes. If $\varepsilon$ satisfies (3.10) then there exists a class $[\eta]$ such that $E_n([\eta]) = \varepsilon = [\rho] = [d\eta]$. Thus, locally, $\rho$ can be expressed as the sum of a closed form and a strongly contact form. If, in addition, $H^{n+1} Y = \{0\}$, (3.5) guarantees that $\eta$ may be chosen *globally defined* on $J^r Y$. Again, condition (3.10) strongly determines the structure of locally variational dynamical forms.

## 4. Variational vector fields

We say that a vector field $Z$ on $J^r Y$ *preserves contact forms* if for any contact form $\rho$ on $J^r Y$ the Lie derivative $\partial_Z \rho$ is again a contact form; we also say that $Z$ is a *contact symmetry*.

If $Z$ is a contact symmetry then for any two $k$-forms $\rho_1, \rho_2$ belonging to the same class in the variational sequence, the $k$-forms $\partial_Z \rho_1$, $\partial_Z \rho_2$ also belong to the same class. Thus we can define the *Lie derivative* of a class $[\rho]$ to be the class

$$\partial_Z [\rho] = [\partial_Z \rho]. \tag{4.1}$$

For any $\pi$-projectable vector field $\Xi$ on an open subset of $Y$, the $r$-jet prolongation $Z = J^r \Xi$ is a contact symmetry. This property of the vector field $J^r \Xi$ implies, among others, the commutativity of the Lie derivative $\partial_{J^r \Xi}$ and the Euler-Lagrange mapping,

$$\partial_{J^{2r} \Xi} E(\lambda) = E(\partial_{J^r \Xi} \lambda). \tag{4.2)}$$

(see Krupka [6], [7]). One can easily show that an analogous property holds for any contact symmetry and any morphism $E_k : \Omega^r_k / \Theta^r_k \to \Omega^r_{k+1} / \Theta^r_{k+1}$:

THEOREM 1. *Let $W \subset Y$ be an open set, and let a vector field $Z$, defined on $W$, be a contact symmetry. Then for all $k$,*

$$\partial_Z E_k([\rho]) = E_k(\partial_Z [\rho]) = E_k([i_Z d\rho]). \tag{4.3}$$

PROOF. Since the Lie derivative commutes with the exterior derivative, we have for any $k$-form $\rho$ on $J^r Y$

$$\partial_Z([d\rho]) = [\partial_Z d\rho] = [d \partial_Z \rho] = [d i_Z d\rho]. \tag{4.4}$$

Writing this formula in terms of the morphism $E_k$ we get (4.3). □

Our main goal is to introduce the concept of a variational vector field for a given dynamical form.

DEFINITION 1. *Let $\varepsilon \in \Omega^s_{n+1,Y}W$ be a dynamical form such that $\varepsilon = [\rho]$ for some $\rho \in \Omega^r_{n+1}W$. We say that a vector field $Z$ on $W^s \subset J^sY$ is a variational vector field for $\varepsilon = [\rho]$ if the Lie derivative $\partial_Z\varepsilon$ is a locally variational form.*

The proof of the following theorem is based on a simple observation explaining the meaning of identity (4.3) for the Helmholtz mapping $E_{n+1}$.

THEOREM 2. *Let $W \subset Y$ be an open set, let $\varepsilon$ be a dynamical form on $W^s \subset J^sY$, and $Z$ be a vector field on $W^s$. Suppose that $Z$ is a contact symmetry. Then the following two conditions are equivalent:*

- *$Z$ is a variational vector field for $\varepsilon$, i.e., $E_{n+1}(\partial_Z\varepsilon) = 0$.*
- *$Z$ leaves invariant the Helmholtz class, i.e. $\partial_Z E_{n+1}(\varepsilon) = 0$.*

PROOF. We choose $\rho \in \Omega^r_{n+1}W$ such that $\varepsilon = [\rho]$ and then apply Theorem 1. We obtain $\partial_Z E_{n+1}(\varepsilon) = E_{n+1}(\partial_Z\varepsilon)$. Theorem 2 is a direct consequence of this formula. □

## 5. Variational contact symmetries of second order dynamical forms

In what follows, we shall specify the above results for the case of mechanics, i.e. for second-order dynamical forms on fibered manifolds with 1-dimensional bases. A fibered chart on $Y$ is denoted by $(V,\psi)$, $\psi = (t, q^\sigma)$, the associated charts on $J^1Y$ and $J^2Y$ by $(V^1, \psi^1)$, $\psi^1 = (t, q^\sigma, \dot{q}^\sigma)$ and $(V^2, \psi^2)$, $\psi^2 = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$, respectively. In this notation,

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad \dot{\omega}^\sigma = d\dot{q}^\sigma - \ddot{q}^\sigma dt.$$

For any differentiable function $f : V^1 \to \mathbb{R}$ we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\nu}\dot{q}^\nu + \frac{\partial f}{\partial \dot{q}^\nu}\ddot{q}^\nu,$$

and we define the *cut formal derivative operator* $d'f/dt$ by

$$\frac{d'f}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\nu}\dot{q}^\nu. \tag{5.1}$$

The formal derivative $df/dt$ is defined on $V^2$, and $d'f/dt$ is a function on $V^1$.

The following simple observation gives us a geometric description of second order dynamical forms, depending linearly on the second derivatives.

LEMMA 1. *Let $\varepsilon$ be a dynamical form, expressed in a fibered chart by*

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge dt. \tag{5.2}$$

*The following two conditions are equivalent:*

(1) *The components of $\varepsilon$ are linear in the second derivatives,*

$$\varepsilon_\sigma = A_\sigma + C_{\sigma\nu}\ddot{q}^\nu. \tag{5.3}$$

(2) *On $V^1$, there exists a 2-form $\rho$, generated by the contact forms $\omega^\sigma$, such that $\varepsilon = [\rho]$.*

PROOF. If the components $\varepsilon_\sigma$ are of the form (5.3), we set

$$\rho = \omega^\sigma \wedge (A_\sigma dt + C_{\sigma\nu} d\dot{q}^\nu). \tag{5.4}$$

Then $(\pi^{2,1})^*\rho = A_\sigma\omega^\sigma \wedge dt + C_{\sigma\nu}\omega^\sigma \wedge (\dot{\omega}^\nu + \ddot{q}^\nu dt) = \varepsilon_\sigma\omega^\sigma \wedge dt + C_{\sigma\nu}\omega^\sigma \wedge \dot{\omega}^\nu$, showing that $\varepsilon = [\rho]$.

Conversely, if $\rho$ is a 2-form generated by the contact forms $\omega^\sigma$, i.e., $\rho = \omega^\sigma \wedge (A_\sigma dt + B_{\sigma\nu}\omega^\nu + C_{\sigma\nu} d\dot{q}^\nu)$, then $[\rho] = (A_\sigma + C_{\sigma\nu}\ddot{q}^\nu)\omega^\sigma \wedge dt$. □

We now discuss the question under what conditions a dynamical form $\varepsilon$ with components (5.3) is variational. The above Lemma says that variationality of $\varepsilon$ is closely related with the properties of the form $\rho$ (5.4). For these forms the Helmholtz form reads (see Krupka [10], [11])

$$H(\varepsilon) = (F_{\sigma\nu}\omega^\sigma + G_{\sigma\nu}\dot{\omega}^\sigma + H_{\sigma\nu}\ddot{\omega}^\sigma) \wedge \omega^\nu \wedge dt, \tag{5.5}$$

where

$$\begin{aligned} F_{\sigma\nu} &= \frac{1}{2}\Big(\frac{\partial A_\nu}{\partial q^\sigma} - \frac{\partial A_\sigma}{\partial q^\nu}\Big) - \frac{1}{4}\frac{d}{dt}\Big(\frac{\partial A_\nu}{\partial \dot{q}^\sigma} - \frac{\partial A_\sigma}{\partial \dot{q}^\nu}\Big) + \frac{1}{2}\Big(\frac{\partial C_{\nu\tau}}{\partial q^\sigma} - \frac{\partial C_{\sigma\tau}}{\partial q^\nu}\Big)\ddot{q}^\tau \\ &\quad - \frac{1}{4}\frac{d}{dt}\Big(\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} - \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\Big)\ddot{q}^\tau - \frac{1}{4}\Big(\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} - \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\Big)\dddot{q}^\tau, \\ G_{\sigma\nu} &= \frac{1}{2}\Big(\frac{\partial A_\nu}{\partial \dot{q}^\sigma} + \frac{\partial A_\sigma}{\partial \dot{q}^\nu}\Big) - \frac{1}{2}\frac{d(C_{\nu\sigma} + C_{\sigma\nu})}{dt} + \frac{1}{2}\Big(\frac{\partial C_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial C_{\sigma\tau}}{\partial \dot{q}^\nu}\Big)\ddot{q}^\tau, \\ H_{\sigma\nu} &= \frac{1}{2}(C_{\nu\sigma} - C_{\sigma\nu}). \end{aligned} \tag{5.6}$$

Next, by straightforward computations using the definition, we obtain the following complete description of contact symmetries on $W^2$, where $W \subset Y$ is an open set:

Lemma 2. *(1) Let* $\dim Y = 2$, *and let* $Z$ *be a vector field on* $J^2 Y$,

$$Z = \zeta_0 \frac{\partial}{\partial t} + \zeta \frac{\partial}{\partial q} + \dot{\zeta}\frac{\partial}{\partial \dot{q}} + \ddot{\zeta}\frac{\partial}{\partial \ddot{q}}. \tag{5.7}$$

*The following three conditions are equivalent:*

(a) $Z$ *is a contact symmetry.*

(b) *There exists a function* $f = f(t, q, \dot{q})$ *such that*

$$\zeta_0 = -\frac{\partial f}{\partial \dot{q}}, \quad \zeta = f - \frac{\partial f}{\partial \dot{q}}\dot{q}, \quad \dot{\zeta} = \frac{d' f}{dt}, \quad \ddot{\zeta} = \frac{d'}{dt}\frac{df}{dt}. \tag{5.8}$$

(c) *The components* $\zeta_0$ *and* $\zeta$ *depend on* $t, q, \dot{q}$ *only, and satisfy*

$$\frac{\partial \zeta}{\partial \dot{q}} - \frac{\partial \zeta_0}{\partial \dot{q}}\dot{q} = 0, \tag{5.9}$$

*and*

$$\dot{\zeta} = \frac{d\zeta}{dt} - \frac{d\zeta_0}{dt}\dot{q}, \quad \ddot{\zeta} = \frac{d\dot{\zeta}}{dt} - \frac{d\zeta_0}{dt}\ddot{q}. \tag{5.10}$$

*(2) Let* $\dim Y \geqslant 3$. *A vector field*

$$Z = \zeta_0 \frac{\partial}{\partial t} + \zeta^\sigma \frac{\partial}{\partial q^\sigma} + \dot{\zeta}^\sigma \frac{\partial}{\partial \dot{q}^\sigma} + \ddot{\zeta}^\sigma \frac{\partial}{\partial \ddot{q}^\sigma} \tag{5.11}$$

*is a contact symmetry if and only if the functions* $\zeta_0$ *and* $\zeta^\sigma$ *depend on* $t, q^\nu$ *only, and*

$$\dot{\zeta}^\sigma = \frac{d\zeta^\sigma}{dt} - \frac{d\zeta_0}{dt}\dot{q}^\sigma, \quad \ddot{\zeta}^\sigma = \frac{d^2\zeta^\sigma}{dt^2} - \frac{d^2\zeta_0}{dt^2}\dot{q}^\sigma - 2\frac{d\zeta_0}{dt}\ddot{q}^\sigma. \tag{5.12}$$

Definition 2. *If $Z$ is a contact symmetry and $f$ is a function such that conditions (5.8) are satisfied, we say that $Z$ is associated with $f$.*

Note that, by (5.8), $f = \zeta - \zeta_0 \dot{q}$. Formulas (5.10) for contact symmetries on 2-dimensional fibered manifolds $Y$ have the same form as (5.12). Thus, we can use formulas (5.12) independently of the dimension of $Y$, having in mind, however, differences in the dependence of $\zeta_0$ and $\zeta$ on the coordinates.

Lemma 3. *Let $\varepsilon$ be a dynamical form with components (5.3). Then the Lie derivative $\partial_Z \varepsilon$ by a contact symmetry $Z$ is of the same form, i.e.,*

$$\partial_Z \varepsilon = [\partial_Z \rho] = \varepsilon^Z_\sigma \omega^\sigma \wedge dt, \tag{5.13}$$

*where*

$$\varepsilon^Z_\sigma = A^Z_\sigma + C^Z_{\sigma\nu} \ddot{q}^\nu, \tag{5.14}$$

*with*

(5.15)

$$A^Z_\sigma = \partial_Z A_\sigma + \frac{\partial(\zeta^\tau - \zeta_0 \dot{q}^\tau)}{\partial q^\sigma} A_\tau + \Big(\frac{\partial \zeta_0}{\partial t} + \frac{\partial \zeta_0}{\partial q^\tau}\dot{q}^\tau\Big) A_\sigma + \Big(\frac{\partial \dot{\zeta}^\nu}{\partial t} + \frac{\partial \dot{\zeta}^\nu}{\partial q^\tau}\dot{q}^\tau\Big) C_{\sigma\nu},$$

$$C^Z_{\sigma\nu} = \partial_Z C_{\sigma\nu} + C_{\tau\nu}\frac{\partial(\zeta^\tau - \zeta_0 \dot{q}^\tau)}{\partial q^\sigma} + \frac{\partial \zeta_0}{\partial \dot{q}^\nu} A_\sigma + \frac{\partial \dot{\zeta}^\tau}{\partial \dot{q}^\nu} C_{\sigma\tau}.$$

Using Theorem 2 together with the above Lemmas we can find *equations* for a vector field to be a variational contact symmetry:

Theorem 3. *Let $\rho$ be a 2-form locally generated by the contact forms $\omega^\sigma$, expressed by (5.4). Then a contact symmetry $Z$ is variational for the dynamical form $\varepsilon = [\rho]$ if and only if*

$$\begin{aligned}
&\frac{\partial A^Z_\nu}{\partial q^\sigma} - \frac{\partial A^Z_\sigma}{\partial q^\nu} - \frac{1}{2}\frac{d}{dt}\Big(\frac{\partial A^Z_\nu}{\partial \dot{q}^\sigma} - \frac{\partial A^Z_\sigma}{\partial \dot{q}^\nu}\Big) + \Big(\frac{\partial C^Z_{\nu\tau}}{\partial q^\sigma} - \frac{\partial C^Z_{\sigma\tau}}{\partial q^\nu}\Big)\ddot{q}^\tau \\
&\qquad - \frac{1}{2}\frac{d}{dt}\Big(\frac{\partial C^Z_{\nu\tau}}{\partial \dot{q}^\sigma} - \frac{\partial C^Z_{\sigma\tau}}{\partial \dot{q}^\nu}\Big)\ddot{q}^\tau - \frac{1}{2}\Big(\frac{\partial C^Z_{\nu\tau}}{\partial \dot{q}^\sigma} - \frac{\partial C^Z_{\sigma\tau}}{\partial \dot{q}^\nu}\Big)\dddot{q}^\tau = 0, \\
&\frac{\partial A^Z_\nu}{\partial \dot{q}^\sigma} + \frac{\partial A^Z_\sigma}{\partial \dot{q}^\nu} - \frac{d(C^Z_{\nu\sigma} + C^Z_{\sigma\nu})}{dt} + \Big(\frac{\partial C^Z_{\nu\tau}}{\partial \dot{q}^\sigma} + \frac{\partial C^Z_{\sigma\tau}}{\partial \dot{q}^\nu}\Big)\ddot{q}^\tau = 0, \\
&C^Z_{\nu\sigma} - C^Z_{\sigma\nu} = 0.
\end{aligned} \tag{5.16}$$

## 6. Symmetries of the Helmholtz form: Example

Let $Y = \mathbb{R}^3$, $X = \mathbb{R}$. Let us consider, in the canonical coordinates on $J^2\mathbb{R}^3$, the second order dynamical form $\varepsilon = (\varepsilon_1 \omega^1 + \varepsilon_2 \omega^2) \wedge dt$ given by

$$\varepsilon_1 = \dot{q}^2 + \ddot{q}^1, \quad \varepsilon_2 = q^2 + \ddot{q}^2. \tag{6.1}$$

We wish to find vertical contact symmetries $Z$ of the Helmholtz form $H(\varepsilon)$ of $\varepsilon$, and verify by a direct computation that the Lie derivative $\partial_Z \varepsilon$ is a variational dynamical form.

Expressions (6.1) were studied by Douglas [3] in connection with his analysis of variational integrators for systems of two second order ordinary differential equations. He proved that $\varepsilon$ is not variational and has no variational integrators (cf. also [1] and [16] for a geometric analysis of the problem).

We restrict ourselves to invariance with respect to vector fields $Z = J^2\Xi$, where

$$\Xi = \xi^\nu \frac{\partial}{\partial q^\nu}, \tag{6.2}$$

and $\xi^\nu = \xi^\nu(t, q^\sigma)$. The Helmholtz form $H(\varepsilon)$ is given by

$$H(\varepsilon) = \tfrac{1}{2}(\dot\omega^2 \wedge \omega^1 + \dot\omega^1 \wedge \omega^2) \wedge dt, \tag{6.3}$$

and it is easily seen that the condition $\partial_Z H(\varepsilon) = 0$ is equivalent with the system

$$\frac{\partial\dot\xi^2}{\partial q^2} - \frac{\partial\dot\xi^1}{\partial q^1} = 0, \quad \frac{\partial\xi^2}{\partial q^1} = 0, \quad \frac{\partial\xi^2}{\partial q^2} + \frac{\partial\xi^1}{\partial q^1} = 0, \quad \frac{\partial\xi^1}{\partial q^2} = 0. \tag{6.4}$$

Solving this system we obtain

$$\xi^1 = A + Cq^1, \quad \xi^2 = B - Cq^2, \tag{6.5}$$

where $A = A(t)$, $B = B(t)$ are arbitrary functions, and $C \in \mathbb{R}$. The dynamical form $\partial_Z\varepsilon$, with $Z = J^2\Xi$ defined by (6.5), is given as $\partial_Z\varepsilon = (\tilde\varepsilon_1\omega^1 + \tilde\varepsilon_2\omega^2) \wedge dt$, where

$$\tilde\varepsilon_1 = 2C\ddot q^1, \quad \tilde\varepsilon_2 = B - 2Cq^2 - 2C\ddot q^2. \tag{6.6}$$

From these formulas it immediately follows that $H(\partial_Z\varepsilon) = 0$, which proves that the dynamical form $\partial_Z\varepsilon$ is variational.

Clearly, the same result can be obtained from Theorem 3.

Note that in the family of vector fields $Z = J^2\Xi$ where $\Xi$ is defined by (6.5) there exist vector fields such that the systems $\varepsilon_1 = 0$, $\varepsilon_2 = 0$ and $\tilde\varepsilon_1 = 0$, $\tilde\varepsilon_2 = 0$ have common solutions. Indeed, setting $B = 0$, we can easily check that the functions $q^1 = at + b$, $q^2 = 0$, in which $a, b \in \mathbb{R}$, verify both the systems.

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