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On the symmetry structure of the minimal surface equation

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ABSTRACT. An infinite sequence of commuting nonpolynomial contact symmetries of the two-dimensional minimal surface equation is constructed. Local and nonlocal conservation laws for n-dimensional minimal area surface equation are obtained by using the Noether identity.

Introduction. In this paper, we analyze the symmetry properties of the Euler equation

$$\mathcal{E}_{\min\Sigma} = \left\{\mathbf{E}_u\left(\left[(1 + (\operatorname{grad} u)^2)^{1/2}\,\mathrm{d}\mathbf{x}\right]\right) = 0\right\} \tag{1}$$

whose solutions describe the minimal area n-dimensional surfaces $\Sigma = \{x^0 = u(x^1, \ldots, x^n)\} \subset \mathbb{R}^{n+1}$. We obtain variational conservation laws (either local or nonlocal) for the equation $\mathcal{E}_{\min\Sigma}$ in case $n \geqslant 1$ is arbitrary and construct a denumerable set of nonpolynomial contact symmetries of the minimal surface equation if $n = 2$ (note that the equation $\mathcal{E}_{\min\Sigma}$ is the zero mean curvature equation in this case). We use the nonparametric representation of the minimal surfaces Σ such that at any regular point within an open domain in the ambient manifold $\mathbb{R}^{n+1}$ there is the fibre bundle $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ and any surface Σ is the graph of a certain section in this bundle. The minimal surface equation $\mathcal{E}_{\min\Sigma}$ is a restriction upon these sections. The latter equation itself is a submanifold $\mathcal{E}_{\min\Sigma} \subset J^2(\pi)$ of the second jet space of the fibre bundle π, see [**3**] for notation, definitions, and details. From now on, we assume that the symmetries, conservation laws, and other structures are restricted onto the infinite prolongation $\mathcal{E}^{\infty}_{\min\Sigma} \subset J^\infty(\pi)$; the superscipt ∞ will be omitted by default.

The analytic and topological properties of the minimal surfaces (*i.e.*, either the minimal area surfaces of the zero mean curvature surfaces) are considered in the vastest literature ([**8**]). Recently, classical results and concepts on this topic were enlarged by their applications in mathematical physics (*e.g.*, the harmonic maps theory and the σ-models, see [**4**]). Nevertheless, the symmetry structure of the minimal surface equation itself remains rather unveiled. In the paper [**2**], seven polynomial symmetries of the two-dimensional equation $\mathcal{E}_{\min\Sigma}$ (namely, the shift, two translations, three rotations, and the dilatation) were obtained — and a denumerable set of nonpolynomial contact symmetries ([**6**]) was missed at all.

The aim of this paper is to fill in the apparent gap in description of the symmetry algebra for the two-dimensional minimal surface equation. Also, we reconstruct

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several conservation laws for the n-dimensional equation $\mathcal{E}_{\min\Sigma}$. The knowledge of local and nonlocal conservation laws for the minimal surface equation is expected to contribute to further studies of the recursion operators for (not necessarily local part of) the symmetry algebra of this equation.

1. Conservation laws of the n-dimensional equation

First, we recall known symmetries of the minimal area surface equation.

PROPOSITION 1 ([**2**]). *The generators of the Lie algebra of point symmetries for the two-dimensional minimal surface equation*

$$\mathcal{E}_{\min\Sigma} = \{(1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} = 0\} \tag{2}$$

are

$$\begin{aligned}
&\varphi_1 = 1 && (\textit{the shift}),\\
&\varphi_2^i = u_{x^i} && (\textit{translations}),\\
&\varphi_3^{12} = yu_x - xu_y && \\
&\bar{\varphi}_3^i = x^i + uu_{x^i} && (\textit{rotations}),\\
&\varphi_4 = u - \langle \operatorname{grad} u, \mathbf{x}\rangle && (\textit{the dilatation}).
\end{aligned}$$

REMARK 1. *The sections φ_1, φ_2^i, $\varphi_3^{ij} = x^ju_{x^i} - x^iu_{x^j}$, $\bar{\varphi}_3^i$, and $\varphi_4 = u - \langle \operatorname{grad} u, \mathbf{x}\rangle$ are symmetries of the n-dimensional minimal area surface Euler equation $\mathcal{E}_{\min\Sigma} = \{\mathbf{E}_u(\mathcal{L}_{\min\Sigma}) = 0\}$ if $n \geqslant 1$ is arbitrary and $1 \leqslant i,\ j \leqslant n$; here the Lagrangian is $\mathcal{L}_{\min\Sigma} = [L_{\min\Sigma}\,\mathrm{d}\mathbf{x}]$ and $L_{\min\Sigma} = \sqrt{1+(\operatorname{grad} u)^2}$.*

Also, we note that the sections φ_1, φ_2^i, φ_3^{ij}, and $\bar{\varphi}_3^i$ generate the $\frac{1}{2}(n+1)(n+2)$-dimensional algebra of the Killing vector fields on the Euclidean space $\mathbb{R}^{n+1} \supset \Sigma$.

Now we assign continuity equations to the symmetries that preserve the Lagrangian.

REMARK 2. *The resulting conserved currents are either purely local or involve nonlocalities. This is owing to three possible ways for a Lagrangian $\mathcal{L}$ to be conserved w.r.t. the symmetries of the corresponding Euler equation $\mathcal{E}_{\mathrm{EL}}$:*

- If the functional $\mathcal{L}$ is conserved on the whole jet space $J^\infty(\pi) \supset \mathcal{E}_{\mathrm{EL}}^\infty$ along a symmetry $\varphi \in \operatorname{sym}\mathcal{E}_{\mathrm{EL}}$, then φ is a ***Noether symmetry*** of the Lagrangian $\mathcal{L}$ and the local conservation law is assigned to φ by the first Noether theorem ([**1, 3**]). Then we use the following lemmas.

LEMMA 1 (Noether's identity, [**5, 9**]). *Let $\mathcal{E}_{\mathrm{EL}} = \{F \equiv \mathbf{E}_u(\mathcal{L}) = 0\}$ be the Euler equation assigned to a Lagrangian $\mathcal{L} = [L\,\mathrm{d}\mathbf{x}]$ and let φ be a Noether symmetry: $\partial_\varphi(\mathcal{L}) = \mathrm{d}_h(\mu)$, where d_h is the standard horizontal differential. Then φ is the generating section of the conservation law $[\omega] = [\mu - \nu - \lambda]$: $\mathrm{d}_h(\omega) = \square(F)\,\mathrm{d}\mathbf{x}$ such that $\varphi = \square^*(1)$; here $\partial_\varphi(\mathcal{L}) = \langle \varphi, F\rangle + \mathrm{d}_h(\nu)$ and $\langle 1, \square(F)\rangle = \langle \varphi, F\rangle + \mathrm{d}_h(\lambda)$.*

PROOF. We have

$$\mathrm{d}_h(\mu) = \partial_\varphi(\mathcal{L}) = \langle 1, \ell_L^*(\varphi)\rangle = \langle \ell_L^*(1), \varphi\rangle + \mathrm{d}_h(\nu),$$

where $\ell_L^*(1) = \mathbf{E}_u(\mathcal{L})$ by definition. Therefore, $\mathrm{d}_h(\mu - \nu) = \langle 1, \square(F)\rangle + \mathrm{d}_h(\lambda)$ such that $\varphi = \square^*(1)$. $\square$

LEMMA 2 ([**5**]). *In local coordinates, the Noether identity is*

$$\ni_\varphi = \varphi \cdot \mathbf{E}_u + \sum\nolimits_{i=1}^{n} D_i \circ Q_{\varphi,i}, \tag{3}$$

where

$$Q_{\varphi,i} = \sum_{\tau} \sum_{\rho+\eta=\tau} (-1)^\eta D_\rho(\varphi) \cdot D_\eta \circ \partial/\partial u_{\tau+1_i}.$$

PROOF. Fix a multi-index σ; then the coefficient of $\partial/\partial u_\sigma$ in the r.h.s. of Eq. (3) is

$$\begin{aligned}
\varphi \cdot (-1)^\sigma D_\sigma + \sum_{i=1}^{n} D_i \circ \Big[\sum_{\tau+1_i=\sigma} \sum_{\rho+\eta=\tau} (-1)^\eta D_\rho(\varphi) \cdot D_\eta\Big] = \\
= \varphi \cdot (-1)^\sigma D_\sigma + \sum_{\substack{\rho'+\eta=\sigma \\ |\rho'|>0}} (-1)^\eta D_{\rho'}(\varphi) \cdot D_\eta - \sum_{\substack{\rho+\eta'=\sigma \\ |\eta'|>0}} (-1)^{\eta'} D_\rho(\varphi) \cdot D_{\eta'} = \\
= \varphi \cdot (-1)^\sigma D_\sigma + D_\sigma(\varphi) - (-1)^\sigma \varphi \cdot D_\sigma = D_\sigma(\varphi).
\end{aligned}$$

This completes the proof. □

- If $\mathcal{L}$ is conserved on the equation $\mathcal{E}_{\mathrm{EL}}$ only (*i.e.*, $[\ni_\varphi(\mathcal{L})] = \mathrm{const} \cdot \mathcal{L}$), then φ is a ***variational symmetry*** of $\mathcal{E}_{\mathrm{EL}}$ and the conservation law reconstructed by φ involves nonlocalities.

Indeed, we have

$$\mathrm{d}_h(\mu) + \mathrm{const} \cdot \mathcal{L} = \ni_\varphi(\mathcal{L}) = \langle \mathbf{E}_u(\mathcal{L}), \varphi \rangle + \mathrm{d}_h(\nu) = \langle 1, \Box(F) \rangle + \mathrm{d}_h(\nu + \lambda),$$

where $\varphi = \Box^*(1)$. Therefore, if a set of nonlocalities $\boldsymbol{s}$ trivializes the Lagrangian: $\tilde{\mathrm{d}}_h(\eta) = \mathcal{L}$, here we put $\eta = \sum_i s^i \cdot (\partial/\partial x^i \lrcorner \, \mathrm{d}\mathbf{x})$, then we obtain a nonlocal conservation law $\Omega = \mu + \eta - \nu - \lambda$ whose generating section is φ. Generally, the compatibility conditions $\boldsymbol{s}_{x^i x^j} = \boldsymbol{s}_{x^j x^i}$ determine an infinite set of new variables.

- Finally, if the Lagrangian $\mathcal{L}$ is not conserved neither on the jet space $J^\infty(\pi)$ nor on the equation $\mathcal{E}_{\mathrm{EL}}$, then there is no conservation law for the symmetry φ.

We obtain the following assertion.

PROPOSITION 2.

(1) Under the notation of Proposition 1 *and Remark* 1, *the point symmetries* φ_1, φ_2, *and* φ_3^{ij} *are Noether's symmetries of the Lagrangian* $\mathcal{L}_{\min\Sigma}$. *Indeed, the equality* $[\ni_{\varphi_k}(\mathcal{L}_{\min\Sigma})] = 0$ *holds on the jet space* $J^\infty(\pi)$ *for* $1 \leqslant k \leqslant 3$. *The conserved currents assigned to these symmetries are such that the corresponding continuity equations are, respectively,*

$$\sum\nolimits_{i=1}^{n} \bar{D}_i(\partial L/\partial u_{x^i}) = 0,$$

$$\bar{D}_i\left(u_{x^i} \cdot \partial L/\partial u_{x^i} - L\right) + \sum\nolimits_{j\neq i} \bar{D}_j\left(u_{x^i} \cdot \partial L/\partial u_{x^j}\right) = 0,$$

$$\bar{D}_i(x^j L - \varphi_3^{ij} u_{x^i}/L) - \bar{D}_j(x^i L + \varphi_3^{ij} u_{x^j}/L) - \sum\nolimits_{k\neq i,j} \bar{D}_k(\varphi_3^{ij} \cdot u_{x^k}/L) = 0.$$

(2) The dilatation φ_4 preserves the Lagrangian $\mathcal{L}_{\min\Sigma}$ on the equation $\mathcal{E}_{\min\Sigma}$:

$$[\mathcal{D}_{\varphi_4}(\mathcal{L}_{\min\Sigma})] = n\,\mathcal{L}_{\min\Sigma},$$

and hence

$$\mathbf{E}_u(\mathcal{D}_{\varphi_4}(\mathcal{L}_{\min\Sigma})) = n\,\mathbf{E}_u(\mathcal{L}_{\min\Sigma}) = 0 \quad on \quad \mathcal{E}_{\min\Sigma} = \{\mathbf{E}_u(\mathcal{L}_{\min\Sigma}) = 0\}.$$

Let s^i, $1 \leqslant i \leqslant n$, be nonlocal variables such that their derivatives be $s^i_{x^i} = L_{\min\Sigma}$. Then the nonlocal continuity equation

$$\sum\nolimits_i \tilde{D}_i\Big(s^i - x^i\,L - \frac{u u_i}{\sqrt{1+\sum_j u_j^2}} + \frac{u_i\cdot\sum_j x^j u_j}{\sqrt{1+\sum_k u_k^2}}\Big) = 0$$

is assigned to the symmetry φ_4.

The proof is straightforward.

2. Contact symmetries of the two-dimensional equation

From now on we assume $n = 2$. We claim that the symmetries $\varphi_1, \ldots, \varphi_4$ do not exhaust the set of generators of the contact symmetry Lie algebra for the minimal surface equation $\mathcal{E}_{\min\Sigma}$, see Eq. (2). Indeed, we have

Proposition 3. *The Lie algebra of contact symmetries for the minimal surface equation $\mathcal{E}_{\min\Sigma}$ is generated by the solutions $\varphi(u_x, u_y)$ of the equation*

$$(1+u_x^2)\,\frac{\partial^2\varphi}{\partial(u_x)^2} + 2u_xu_y\,\frac{\partial^2\varphi}{\partial u_x\partial u_y} + (1+u_y^2)\,\frac{\partial^2\varphi}{\partial(u_y)^2} = 0 \tag{4}$$

(e.g., the shift $\varphi_1 = 1$ and the translations $\varphi_2^i = u_{x^i}$ are solutions to Eq. (4)*) and the sections φ_3^{12}, $\bar\varphi_3^i$, and φ_4; here $i = 1, 2$.*

The proof is elementary by using the analytic transformations software ([**7**]) and therefore omitted. We also note that Eq. (4) and its solutions except the point symmetries φ_1 and φ_2^i were missed in [**2**].

Now we construct a denumerable set ([**6**]) contact symmetries of the minimal surface equation (2). Consider Eq. (4) and suppose that $\varphi(u_x, u_y)$ is polynomial in u_y (of course, all reasonings are preserved by the symmetry transformation $x \leftrightarrow y$). Then we obtain two distinct cases: the degree K of the polynomial at hand can be either even ($K = 2k$) or odd ($K = 2k+1$). In what follows, we treat these cases separately.

Case 1 ($K = 2k$). *We assume $\varphi_k^{\text{even}} = \sum_{\ell=0}^k f_\ell(u_x)\,u_y^{2\ell}$. Then from Eq.* (4) *we obtain the following chain of equations:*

(1) The homogeneous equation

$$(1+u_x^2)\,f_k'' + 4ku_x\,f_k' + 2k(2k-1)\,f_k = 0 \tag{5}$$

at the highest power u_y^{2k}.

(2) The nonhomogeneous equations

$$(1+u_x^2)\,f_\ell'' + 4\ell u_x\,f_\ell' + 2\ell(2\ell-1)\,f_\ell = -(2\ell+2)(2\ell+1)\,f_{\ell+1} \tag{6}$$

at the intermediate powers $u_y^{2\ell}$, $1 \leqslant \ell < k$; here the homogeneous components are Eq. (5) *with the index shift.*

(3) The terminal relation at the zero power u_y^0:

$$(1+u_x^2)\, f_0'' = -2\, f_1. \tag{7}$$

The solutions of system (5–7) are the following.

(1) The solution of homogeneous equation (5) is $f_k(u_x) = C_1\cdot(\boldsymbol{i}+u_x)^{-2k+1} + C_2\cdot(-\boldsymbol{i}+u_x)^{-2k+1}$, where the constants C_1, C_2 are arbitrary. The equivalent basis of real solutions to Eq. (5) is

$$f_k^1 = (1+u_x^2)^{-2k+1}\cdot\sum_{\ell=0}^{k-1}(-1)^\ell\binom{2k-1}{2\ell}u_x^{2k-2\ell-1},$$
$$f_k^2 = (1+u_x^2)^{-2k+1}\cdot\sum_{\ell=1}^{k}\binom{2k-1}{2\ell-1}u_x^{2k-2\ell}.$$

(2) The solutions to the intermediate nonhomogeneous equations (6) are

$$f_\ell(u_x) = (2\ell+2)(2\ell+1)\cdot\Bigl\{f_\ell^1(u_x)\cdot\int\frac{f_{\ell+1}\,f_\ell^2\,\mathrm{d}u_x}{f_\ell^1\,(f_\ell^2)'-(f_\ell^1)'\,f_\ell^2} - f_\ell^2(u_x)\cdot\int\frac{f_{\ell+1}\,f_\ell^1\,\mathrm{d}u_x}{f_\ell^1\,(f_\ell^2)'-(f_\ell^1)'\,f_\ell^2}\Bigr\}.$$

All integration constants can be omitted since they originate from the homogeneous equations that provide the symmetries polynomial in u_y whose degrees are less than $K=2k$.

(3) The quadrature for terminal equation (7) is

$$f_0(u_x) = -2\int^{u_x}\mathrm{d}\xi\int^{\xi}\frac{f_1(\eta)\,\mathrm{d}\eta}{1+\eta^2} + \alpha u_x + \beta, \qquad \alpha,\beta\in\mathbb{R}.$$

The coefficients of the integration constants α, β provide the translation u_x and the shift $1\in\operatorname{sym}\mathcal{E}_{\min\Sigma}$, respectively.

Case 2 ($K=2k+1$). *Now we have $\varphi_k^{\text{odd}} = \sum_{\ell=0}^{k} g_\ell(u_x)\,u_y^{2\ell+1}$. From Eq. (4) we obtain the chain of equations:*

(1) The homogeneous equation

$$(1+u_x^2)\,g_k'' + 2(2k+1)ku_x\,g_k' + 2k(2k+1)\,g_k = 0 \tag{8}$$

at the highest power u_y^{2k+1}.

(2) The nonhomogeneous equations

$$(1+u_x^2)\,g_\ell'' + 2(2\ell+1)u_x\,g_\ell' + 2\ell(2\ell+1)\,g_\ell = -(2\ell+3)(2\ell+2)\,g_{\ell+1} \tag{9}$$

at the intermediate powers $u_y^{2\ell+1}$, $1\leqslant\ell<k$; again, the homogeneous components are Eq. (8) with the index shift.

(3) The terminal relation at u_y:

$$(1+u_x^2)\,g_0'' + 2u_x\,g_0' = -6\,g_1. \tag{10}$$

The solutions of system (8–10) are the following.

(1) The solution of homogeneous equation (8) is

$$g_k(u_x) = C_3\cdot(1+u_x^2)^{-k}\cdot\left(\frac{\boldsymbol{i}+u_x}{-\boldsymbol{i}+u_x}\right)^k + C_4\cdot(1+u_x^2)^{-k}\cdot\left(\frac{-\boldsymbol{i}+u_x}{\boldsymbol{i}+u_x}\right)^k,$$

where C_3, C_4 are arbitrary constants. The equivalent basis of real solutions to Eq. (8) *is*

$$g_k^1 = (1+u_x^2)^{-2k} \cdot \sum_{\ell=0}^{k-1} (-1)^\ell \binom{2k}{2\ell} u_x^{2k-2\ell},$$

$$g_k^2 = (1+u_x^2)^{-2k} \cdot \sum_{\ell=1}^{k} \binom{2k}{2\ell-1} u_x^{2k-2\ell+1}.$$

(2) The solutions to the intermediate nonhomogeneous equations (9) *are*

$$g_\ell(u_x) = (2\ell+3)(2\ell+2) \cdot \Big\{ g_\ell^1(u_x) \cdot \int \frac{g_{\ell+1}\, g_\ell^2\, \mathrm{d}u_x}{g_\ell^1\,(g_\ell^2)' - (g_\ell^1)'\, g_\ell^2} - g_\ell^2(u_x) \cdot \int \frac{g_{\ell+1}\, g_\ell^1\, \mathrm{d}u_x}{g_\ell^1\,(g_\ell^2)' - (g_\ell^1)'\, g_\ell^2} \Big\}.$$

Again, the integration constants can be omitted since they originate from the homogeneous equations that provide the symmetries which are polynomial in u_y and whose degrees are less than $K = 2k+1$.

(3) The quadrature for terminal equation (10) *is*

$$g_0(u_x) = -6 \int^{u_x} \frac{\mathrm{d}\xi}{1+\xi^2} \int^{\xi} g_1(\eta)\,\mathrm{d}\eta + \gamma + \delta \arctan u_x, \qquad \gamma, \delta \in \mathbb{R}.$$

The coefficients of the constants γ, δ are the translation u_y and the first nonpolynomial symmetry $\varphi_5 = u_y \arctan u_x$, respectively.

The above reasonings provide two sequences of the symmetries $\varphi_k^{\text{even},\text{odd}} \in \operatorname{sym} \mathcal{E}_{\min\Sigma}$ that are polynomial in u_y and nonpolynomial (except three starting terms) in u_x.

Example. *The initial terms of these sequences are*

$$\varphi_1 = 1, \quad \varphi_2^1 = u_x, \qquad\qquad \varphi_2^2 = u_y, \quad \varphi_5 = u_y \arctan u_x,$$

$$\varphi_6 = \frac{u_x u_y^2}{1+u_x^2} + \arctan u_x, \qquad\qquad \varphi_7 = \frac{u_y^2}{1+u_x^2} - u_x \arctan u_x,$$

$$\varphi_8 = \frac{u_x u_y^3}{(1+u_x^2)^2} + \frac{3}{2}\frac{u_x u_y}{1+u_x^2}, \qquad\qquad \varphi_9 = \frac{u_x^2-1}{(1+u_x^2)^2} \cdot u_y^3 - \frac{3u_y}{1+u_x^2}.$$

Remark 3. *Analogous reasonings can be applied to the formal power series solutions*

$$\varphi_k^{\text{even}} = \sum_{\ell \leqslant -k} f_\ell(u_x)\, u_y^{2\ell}, \qquad \varphi_k^{\text{odd}} = \sum_{\ell \leqslant -k} g_\ell(u_x)\, u_y^{2\ell+1},$$

where $k > 0$. The symmetries φ_k^{even} are therefore polynomial in u_y^{-1}.

Remark 4. *There are two different types of recursion operators that act on the contact symmetries $\varphi_k(u_x, u_y)$. Let $k \geqslant 0$, $i = 1, 2$, and suppose $\varphi_{k,i}$ is the symmetry such that f_k^i (resp., g_k^i) is the coefficient of the highest power of u_y. Then we have*

- *the operator $\Delta\colon \varphi_{k,1} \rightleftarrows \varphi_{k,2}$ that swaps the elements of the basis if the highest power that depends on k is fixed;*

- *the operator* $\nabla\colon \varphi_{k,i} \mapsto \varphi_{k+1,i}$ *that proliferates the symmetry heads along the sequence by the rules*

$$\nabla^{\text{even}}\colon f_k^1 \mapsto \frac{1}{(\boldsymbol{i}+u_x)^2}\, f_k^1 = f_{k+1}^1, \qquad \nabla^{\text{even}}\colon f_k^2 \mapsto \frac{1}{(-\boldsymbol{i}+u_x)^2}\, f_k^2 = f_{k+1}^2,$$
$$\nabla^{\text{odd}}\colon g_k^1 \mapsto \frac{-1}{(\boldsymbol{i}+u_x)^2}\, g_k^1 = g_{k+1}^1, \qquad \nabla^{\text{odd}}\colon g_k^2 \mapsto \frac{1}{(-\boldsymbol{i}+u_x)^2}\, g_k^2 = g_{k+1}^2.$$

LEMMA 3. *Assume that* $\varphi'(u_x,u_y)$ *and* $\varphi''(u_x,u_y)$ *are the generating sections of evolutionary vector fields* $\mathrm{Э}_{\varphi'}$ *and* $\mathrm{Э}_{\varphi''}$. *Then their Jacobi bracket* $\{\varphi',\,\varphi''\}$ *is always trivial.*

PROOF. In local coordinates, the Jacobi bracket $\{\varphi',\,\varphi''\} = \mathrm{Э}_{\varphi'}(\varphi'') - \mathrm{Э}_{\varphi''}(\varphi')$ is

$$\{\varphi',\varphi''\} = \frac{\partial\varphi'}{\partial u_x}u_{xx}\cdot\frac{\partial\varphi''}{\partial u_x} + \frac{\partial\varphi'}{\partial u_y}u_{xy}\cdot\frac{\partial\varphi''}{\partial u_x} + \frac{\partial\varphi'}{\partial u_x}u_{xy}\cdot\frac{\partial\varphi''}{\partial u_y} + \frac{\partial\varphi'}{\partial u_y}u_{yy}\cdot\frac{\partial\varphi''}{\partial u_y} - \text{v. v.} = 0.$$

□

COROLLARY 1. *The contact symmetries* $\varphi_k(u_x,\,u_y)$ *of the minimal surface equation* $\mathcal{E}_{\min\Sigma}$ *commute.*

Conclusion. We see that the contact symmetry structure of the two-dimensional minimal surface equation is intermediate between its subalgebra of point symmetries ([**2**]) and the formalism of higher symmetries. The nature of the recursion operator ∇ above that proliferates the contact symmetries is quite unusual. Indeed, this operator generates two infinite sequences of commuting symmetries that depend on the first-order derivatives u_x and u_y only; the proliferation is based on growth of the degrees of the polynomials in u_y.

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