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Geodesics of Sub-Finslerian Geometry

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Abstract. In this paper the geodesics of the Grushin plane and Heisenberg group are obtained in the new context of sub-Finslerian geometry equipped with a special Randers metric. We have used the Pontryagin Maximum Principle for the Grushin plane and a new technique that is making use of Lie algebroids was applied for the Heisenberg group.

1. Sub-Finslerian structure

If M is a smooth n-dimensional manifold then a sub-Riemannian structure on M is a pair (D, g) where D is a distribution of rank m and g is a Riemannian metric on D. A sub-Riemannian manifold (M, D, g) is a smooth manifold M equipped with a sub-Riemannian structure [2],[14],[15].

A piecewise smooth curve $c : I \subset R \to M$ is called *horizontal* if its tangent vectors are in D, i.e. $\dot{c}\ (t) \in D_{c(t)} \subset TM$, for almost every $t \in I$. In sub-Riemannian geometry the *length* of a horizontal curve c is defined by $L(c) = \int_I \sqrt{g(\dot{c}\ (t))}dt$, where g is a Riemannian metric on D. The *distance* from a to b is $d(a, b) = \inf(L(c))$, where the infimum is taken over all horizontal curves connecting a to b. The distance is assumed to be infinite if there is no horizontal curve that connects these two points. If locally the distribution D of rank m is generated by X_i, $i = \overline{1, m}$, a sub-Riemannian structure on M is locally given by a control system

$$\dot{x} = \sum_{i=1}^{m} u_i(t) X_i(x), \tag{1}$$

of constant rank m, with controls $u(.)$. The *controlled paths* are obtained by integrating the above system. If D is assumed to be *bracket generating,* i.e. sections of D and iterated brackets span the entire tangent space TM, by a well-known theorem of Chow [5] the system (1) is *controllable*, that is for any two points a and b there exists a horizontal curve which connects these points (M is assumed to be connected).

The concept of sub-Riemannian geometry can be extended to a more general setting, by replacing the Riemannian metric with a Finslerian one. For the theory of optimal control this extension is equivalent to the change of the quadratic cost of a control affine system with a positive homogeneous cost. Also, the case when the

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rank of D is not constant may produce interesting examples. We do not intend to develop a comprehensive study of the sub-Finslerian geometry. This will be done in a forthcoming paper. In the present paper we introduce two particular sub-Finslerian geometries: the Grushin plane and the Heisenberg group, endowed with some special Randers metric. The geodesics of these geometries are obtained by using two different approaches: a direct application of the Pontryagin Maximum Principle for the Grushin plane and the same principle but combined with some results of Lie algebroids for the Heisenberg group.

DEFINITION 1. *A sub-Finslerian structure on M is a triple $(E,\sigma,\mathcal{F})$ where*
1) (E,π) is a vector bundle over M, with the projection map π.
2) $\sigma : E \to TM$ is a morphism of vector bundles.
3) $\mathcal{F}$ is a Finsler metric on E, i.e. $\mathcal{F} : E \to [0,\infty)$ and satisfies the following properties:
a) $\mathcal{F}$ is C^∞ on $E \setminus \{0\}$.
b) $\mathcal{F}(\lambda u) = \lambda\mathcal{F}(u)$ for $\lambda > 0$ and $u \in E_x, x \in M$.
c)For each $y \in E_x\backslash\{0\}$ the quadratic form

$$g_y(u,v) = \frac{1}{2}\frac{\partial\mathcal{F}^2}{\partial s\partial t}(y+su+tv)_{s,t=0}$$

$u, v \in E_x$, $x \in M$, is positive definite.

EXAMPLE. *(i) $E = M \times R^m$,$\{X_1, ..., X_m\}$ is a system of m vector fields on M, $\sigma : E \to TM$,*

$$\sigma(x,u) = \sum_{i=1}^{m} u_i(t)X_i(x),$$

and $\mathcal{F}$ is a Minkowski norm on R^m.
(ii) $E = D$, $\sigma : D \to TM$ the inclusion and $\mathcal{F} = F$ a Finsler metric on D. We can associate to any sub-Finslerian structure $(E,\sigma,\mathcal{F})$ a Finsler metric on $im\sigma \subset TM$, defined as following: for each $v \in (im\sigma)_x \subset T_xM, x \in M$

$$F(v) = \inf_u\{\mathcal{F}(u)| \quad u \in E_x, \quad \sigma(u) = v\}. \tag{2}$$

DEFINITION 2. *A curve $u : I \to E$ is called admissible if there is an absolute continuous curve $c : I \to M$, called horizontal, such that $\pi(u(t)) = c(t)$ and $\sigma((u(t)) = \dot{c}(t)$, $t \in I$.* The length of an absolutely continuous horizontal curve $c(t)$, $t \in I$ is:

$$length(c) = \int_I \mathcal{F}(u(t))dt = \int_I F(\dot{c}\ (t))dt$$

We can also consider the sub-Finslerian distance

$$d(a,b) = \inf length(c)$$

where the infimum is taken over all horizontal curves connecting a and b . This distance is infinite if there is no admissible curve joining a and b. However, if we assume that imσ is bracket generated, Chow's theorem guarantees that the sub-Finslerian distance between points is finite.

DEFINITION 3. *A length minimizing geodesic or shortly a minimizer is an absolutely continuous horizontal curve on M that makes the distance between two points.*

REMARK 1. *The energy of a horizontal curve is* $E(c) = \frac{1}{2}\int_I F^2(\dot{c}\ (t))dt$ *and it can easily be proved that if a curve is parametrized to a constant speed, then it minimize the length integral if and only if it minimize the energy integral.* If we take the Lagrangians $L = \frac{1}{2}F^2$ and $\mathcal{L} = \frac{1}{2}\mathcal{F}^2$ we have $\mathcal{L} = L \circ \sigma$.

The Fenchel-Legendre dual of L is the Hamiltonian

$$H(p) = \sup_v \{\langle p, v\rangle - L(v)\} =$$

$$\sup_v \left\{ \langle p, v\rangle + \sup_u \{-\mathcal{L}(u); \sigma(u) = v\} \right\} =$$

$$\sup_{u,v} \{\langle p, v\rangle - \mathcal{L}(u); \sigma(u) = v\} =$$

$$\sup_u \{\langle p, \sigma(u)\rangle - \mathcal{L}(u)\} =$$

$$\sup_u \{\langle \sigma^\star(p), u\rangle - \mathcal{L}(u)\} = \mathcal{H}(\sigma^\star(p))$$

Hence

$$H(p) = \mathcal{H}(\mu), \quad \mu = \sigma^\star(p) \tag{3}$$

$p \in T_x^*M$, $\mu \in E_x^*$. The Hamiltonian H on T^*M is degenerate on $Ker\sigma^\star$.

The Hamiltonian H generates a system of differential equations which can be written in terms of canonical coordinates (x, p) in the standard form:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i} \tag{4}$$

Based on the Pontryagin Maximum Principle we can prove:

THEOREM 1. *Let $(x(t), p(t))$ be a solution of the Hamiltonian system (4). Then every sufficient short subarc of $x(t)$ is a minimizing sub-Finslerian geodesic. This subarc is the unique minimizer joining its end points.*

DEFINITION 4. *The projected curve $x(t)$ will be called normal geodesic or shortly geodesic.*

REMARK 2. *Contrary to Finslerian geometry, not every minimizer is the projection of a solution of (4) i.e. is a normal geodesic.Those minimizers that are not normal geodesics are called singular geodesics [14],[3].*

2. Grushin Plane

We consider the following distributional system with positive homogeneous cost (Grushin Plane):

$$\dot{x} = u^1 X_1 + u^2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbb{R}^2, \quad X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x^1 \end{pmatrix}$$

$$\min_{u(.)} \int_0^T \mathcal{F}(u(t))dt, \quad \mathcal{F}(u) = \|u\| + \langle b, u\rangle, \ b = (\varepsilon, 0)^t, \ u = (u^1, u^2)^t, \ 0 \leqslant \varepsilon < 1.$$

$$x(0) = 0, \quad x(T) = x_T.$$

We are looking for the geodesics starting from the origin and parametrized by arclength. The distribution $D =< X_1, X_2 >$ is bracket generating and has not a constant rank on $\mathbb{R}^2$. If take the regular Lagrangian $\mathcal{L}=\frac{1}{2}\mathcal{F}^2$ on $E=D$ and use a result from [9],[13] we obtain a regular Hamiltonian $\mathcal{H}$ on E^*

$$\mathcal{H}=\frac{1}{2}\left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{(1-\varepsilon^2)}}-\frac{\varepsilon\mu_1}{1-\varepsilon^2}\right)^2$$

and from (3) we get

$$\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & x^1\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}$$

and the corresponding the Hamiltonian H on T^*M is

$$H=\frac{1}{2}\left(\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2}+\frac{(p_2)^2(x^1)^2}{(1-\varepsilon^2)}}-\frac{\varepsilon p_1}{1-\varepsilon^2}\right)^2$$

From Hamilton equations (4) we obtain

$$\dot{x_1}=\frac{(1+\varepsilon)^2p_1}{(1-\varepsilon^2)^2}-\frac{\varepsilon}{1-\varepsilon^2}\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2}+\frac{a^2(x^1)^2}{1-\varepsilon^2}}-\frac{\varepsilon p_1^2}{(1-\varepsilon^2)^3}\frac{1}{\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2}+\frac{a^2(x^1)^2}{1-\varepsilon^2}}}$$

$$\dot{x_2}=\frac{(x^1)^2a}{1-\varepsilon^2}-\frac{\varepsilon a(x^1)^2p_1}{(1-\varepsilon^2)^2}\frac{1}{\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2}+\frac{a^2(x^1)^2}{1-\varepsilon^2}}} \tag{5}$$

$$\dot{p_1}=-\frac{x^1a^2}{1-\varepsilon^2}+\frac{\varepsilon p_1x^1a^2}{(1-\varepsilon^2)^2}\frac{1}{\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2}+\frac{a^2(x^1)^2}{1-\varepsilon^2}}},\qquad p_2=a=ct.$$

We make the following change of variables

$$\begin{cases} x^1=\frac{\sqrt{1-\varepsilon^2}}{a}r(t)\sin A\theta(t)\\ \\ p_1=(1-\varepsilon^2)r(t)\cos A\theta(t)\end{cases}$$

and from (5) we get

$$\frac{A(1-\varepsilon^2)(\sqrt{1-\varepsilon^2}}{a}\frac{d\theta}{dt}=(1-\varepsilon\cos A\theta)^2$$

$$\frac{(1-\varepsilon^2)(\sqrt{1-\varepsilon^2}}{a}\frac{dr}{dt}=\varepsilon r\sin A\theta(\varepsilon\cos A\theta-1)$$

Hence

$$r=\frac{1}{c(1-\varepsilon\cos A\theta)},\qquad c\in R$$

and

$$t=\frac{A(1-\varepsilon^2)\sqrt{1-\varepsilon^2}}{a}\int\frac{d\theta}{(1-\varepsilon\cos A\theta)^2}.$$

But the geodesics are parametrized by arclength, that corresponds to fix the level 1/2 of the Hamiltonian and we have

$$H=\frac{r^2}{2}(1-\varepsilon\cos A\theta)^2=\frac{1}{2c^2}$$

so $c=\pm1$ and therefore

$$r=\pm\frac{1}{1-\varepsilon\cos A\theta}$$

and finally we obtain

$$x^1 = \pm\sqrt{1-\varepsilon^2}\frac{\sin A\theta}{a(1-\varepsilon\cos A\theta)}$$

$$x^2 = \frac{A(1-\varepsilon^2)\sqrt{1-\varepsilon^2}}{a}\int\frac{\sin^2 A\theta}{(1-\varepsilon\cos A\theta)^3}d\theta.$$

REMARK 3. 1. *The above geodesics are the only minimizers of this sub-Finslerian geometry.*

2. *For $\varepsilon = 0$ we obtain the geodesics of the Grushin plane endowed with the standard Euclidian metric, i.e. a sub-Riemannian geometry (distributional systems with quadratic cost)*

$$x_1 = \pm\frac{\sin at}{a}, \quad x_2 = \frac{t}{2a} - \frac{\sin 2at}{4a^2}.$$

(see [2],[8]*)*

3. Heisenberg Group

Let us make a short introduction on Lie algebroids. For more details see[12],[16],[11].

DEFINITION 5. *A Lie algebroid over a manifold M is a vector bundle (E, π, M) equipped with a Lie algebra structure [,] on its space of sections and a map $\sigma : E \to TM$ (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from sections of E to vector fields on M, satisfying the compatibility conditions*

$$[s_1, fs_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2$$

where f is a smooth function on M and s_1, s_2 are sections of E. Therefore, we also have the relations

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2]$$

and

$$[s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0$$

for every s_1, s_2, s_3 sections of E.

A local system of coordinates (x^i) on M and a local basis $\{s_\alpha\}$ of sections of the bundle E generate the local coordinates (x^i, y^α) on E. These coordinates determine the local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M given by

$$\sigma(s_\alpha) = \sigma^i_\alpha\frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L^\gamma_{\alpha\beta}s_\gamma, \quad i, j = \overline{1, n}, \quad \alpha, \beta = \overline{1, m} \tag{6}$$

and satisfying the relations

$$\sigma^j_\alpha\frac{\partial\sigma^i_\beta}{\partial x^j} - \sigma^j_\beta\frac{\partial\sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta} \tag{7}$$

$$\sum_{(\alpha,\beta,\gamma)}\left(\sigma^i_\alpha\frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta}L^\eta_{\beta\gamma}\right) = 0.$$

Let $\mathcal{L}$ be a Lagrangian on E, i.e. a differentiable function $\mathcal{L} : E \to \mathbb{R}$. When $\mathcal{L} \in C^\infty(E)$ we can define a dynamical system on E. This was done by A. Weinstein following the structures on the dual [16]. The symplectic formalism was later introduced by E. Martinez [11] for Lagrangian and Hamiltonian by using Klein's results. The Euler -Lagrange equations on Lie algebroids [16] are:

$$\frac{dx^i}{dt} = \sigma^i_\alpha y^\alpha, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y^\alpha}\right) = \sigma^i_\alpha \frac{\partial \mathcal{L}}{\partial x^i} - L^\gamma_{\alpha\beta} y^\beta \frac{\partial \mathcal{L}}{\partial y^\gamma}$$

and from [11] we get the Hamilton-Jacoby equations

$$\frac{dx^i}{dt} = \sigma^i_\alpha \frac{\partial \mathcal{H}}{\partial \mu_\alpha}, \quad \frac{d\mu_\alpha}{dt} = -\sigma^i_\alpha \frac{\partial \mathcal{H}}{\partial x^i} - \mu_\gamma L^\gamma_{\beta\alpha} \frac{\partial \mathcal{H}}{\partial \mu_\beta}. \tag{8}$$

We consider the following distributional system with positive homogeneous cost (Heisenberg Group):

$$\dot{x} = u^1 X_1 + u^2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3, \ X_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{x^2}{2} \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{x^1}{2} \end{pmatrix},$$

$$\min_{u(.)} \int_0^T \mathcal{F}(u(t))dt, \quad \mathcal{F}(u) = \|u\| + \langle b, u \rangle, \ b = (\varepsilon, 0)^t, \ u = (u^1, u^2)^t, \ 0 \leqslant \varepsilon < 1.$$

We are looking for the geodesics starting from the origin and parametrized by arclength.

Here $X_1 = \frac{\partial}{\partial x^1} - \frac{x^2}{2}\frac{\partial}{\partial x^3}$, $X_2 = \frac{\partial}{\partial x^2} + \frac{x^1}{2}\frac{\partial}{\partial x^3}$, $X_3 = [X_1, X_2] = \frac{\partial}{\partial x^3}$ and $\langle X_1, X_2, X_3 \rangle \equiv T\mathbb{R}^3$, hence the distribution $D =< X_1, X_2 >$ of constant rank is strong bracket generating (see [14],[2]).

Remark 4. *We can try to work directly on the cotangent bundle by computing the Hamiltonian $H(x,p) = \mathcal{H}(x,\mu)$, $\mu = \sigma^*(p)$. Since*

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x^2/2 \\ 0 & 1 & x^1/2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

we obtain

$$H = \frac{1}{2}\left(\sqrt{\frac{\left(p_1 - \frac{p_3 x^2}{2}\right)^2}{(1-\varepsilon^2)^2} + \frac{\left(p_2 + \frac{p_3 x^1}{2}\right)^2}{1-\varepsilon^2}} - \frac{\varepsilon(p_1 - \frac{p_3 x^2}{2})}{1-\varepsilon^2}\right)^2$$

Unfortunately, with this Hamiltonian (4) is a very complicated system of implicit differential equations.

We will use a different approach. Let us take $M = \mathbb{R}^3$ and $E = TM$ with the basis $\{X_1, X_2, X_3\}$. E is a Lie algebroid over M with at least one structural function nonzero. The anchor $\sigma : E \to TM$ is just the identity and the matrix of σ with respect to the basis of E and TM basis is:

$$\sigma^i_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x^2}{2} & \frac{x^1}{2} & 1 \end{pmatrix}$$

Now the above control system can be written:

$$\dot{x} = u^1X_1 + u^2X_2 + 0X_3, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3,$$

$$\min_{u(.)} \int_0^T \mathcal{F}(u(t))dt = \min_{u(.)} \int_0^T \mathcal{L}(u(t))dt$$

where $\mathcal{F} = \sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1$ and $\mathcal{L} = \frac{1}{2}\mathcal{F}^2$. To solve this problem we form the augmented Lagrangian

$$\widetilde{\mathcal{L}} = \mathcal{L}(u) + \lambda u^3$$

(λ is Lagrange a multiplier) and still work via the maximum principle but at the level of the Lie algebroid. If set $\mu_i = \frac{\partial \widetilde{\mathcal{L}}}{\partial u^i}$ and $\mathcal{H} = \mu_i u^i - \widetilde{\mathcal{L}}$, we obtain

$$\mu_1 = \frac{\partial \mathcal{L}}{\partial u^1}, \ \mu_2 = \frac{\partial \mathcal{L}}{\partial u^2}, \ \mu_3 = \lambda$$

and therefore $\mathcal{H} = \mu_i u^i - \widetilde{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial u^1}u^1 + \frac{\partial \mathcal{L}}{\partial u^2}u^2 + \lambda u^3 - \mathcal{L} - \lambda u^3 = \mathcal{L}$ because $\mathcal{L}$ is 2-homogeneous. Again using a result from [9], the Hamiltonian on E^* is given by

$$\mathcal{H} = \frac{1}{2}\left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{(1-\varepsilon^2)}} - \frac{\varepsilon\mu_1}{1-\varepsilon^2}\right)^2, \quad \mu_3 = \lambda.$$

From the structure equations of Lie algebroids (7) and the relation $[X_\alpha, X_\beta] = L^\gamma_{\alpha\beta}X_\gamma$ we obtain the non-zero structural functions

$$L^3_{12} = 1, \quad L^3_{21} = -1.$$

Now, using the Hamilton-Jacoby equations on Lie algebroids (8) we get the following systems of differential equations

$$\text{(9)} \qquad \begin{cases} \dot{x}^1 = \frac{\partial \mathcal{H}}{\partial \mu_1} \\ \dot{x}^2 = \frac{\partial \mathcal{H}}{\partial \mu_2} \\ \dot{x}^3 = -\frac{x^2}{2}\frac{\partial \mathcal{H}}{\partial \mu_1} + \frac{x^1}{2}\frac{\partial \mathcal{H}}{\partial \mu_2} = \frac{x^1\dot{x}^2}{2} - \frac{\dot{x}^1 x^2}{2} \end{cases}$$

and

$$\text{(10)} \qquad \begin{cases} \dot{\mu}_1 = -L^3_{12}\mu_3\frac{\partial \mathcal{H}}{\partial \mu_2} = -\lambda\frac{\partial \mathcal{H}}{\partial \mu_2} \\ \dot{\mu}_2 = -L^3_{21}\mu_3\frac{\partial \mathcal{H}}{\partial \mu_1} = \lambda\frac{\partial \mathcal{H}}{\partial \mu_1} \\ \dot{\mu}_3 = 0 \Rightarrow \lambda = ct. \end{cases}$$

where

$$\frac{\partial \mathcal{H}}{\partial \mu_1} = \frac{\left(1+\varepsilon^2\right)\mu_1}{(1-\varepsilon^2)^2} - \frac{\varepsilon\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}}{1-\varepsilon^2} - \frac{\varepsilon\mu_1^2}{(1-\varepsilon^2)^3\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}}$$

$$\frac{\partial \mathcal{H}}{\partial \mu_2} = \frac{\mu_2}{1-\varepsilon^2} - \frac{\varepsilon\mu_1\mu_2}{(1-\varepsilon^2)^2\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}}.$$

We may use to the following transformations

$$\begin{aligned} \mu_1(t) &= (1-\varepsilon^2)r(t)(a\cos A\theta(t) - b\sin A\theta(t)) \\ \mu_2(t) &= \sqrt{1-\varepsilon^2}r(t)(a\sin A\theta(t) + b\cos A\theta(t)) \end{aligned} \tag{11}$$

such that $a^2+b^2=1$ and we get $\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{(1-\varepsilon^2)}} = |r|$. Also, from (10) we obtain

$$t = \frac{(1-\varepsilon^2)\sqrt{1-\varepsilon^2}A}{\lambda}\int \frac{d\theta}{(1-\varepsilon\,(a\cos A\theta - b\sin A\theta))^2}$$

and

$$r = \frac{1}{c\,(1-\varepsilon\,(a\cos A\theta - b\sin A\theta))}$$

But the geodesics are parametrized by arclength, that corresponds to fix the level 1/2 of the Hamiltonian and we have

$$\mathcal{H} = \frac{r^2}{2}\left(1-\varepsilon\,(a\cos A\theta - b\sin A\theta)\right)^2 = \frac{1}{2c^2}$$

so $c=\pm 1$ and therefore

$$r = \pm\frac{1}{1-\varepsilon\,(a\cos A\theta - b\sin A\theta)}$$

From (11) we obtain

$$\begin{aligned} \mu_1(t) &= \pm\frac{(1-\varepsilon^2)(a\cos A\theta - b\sin A\theta)}{1-\varepsilon\,(a\cos A\theta - b\sin A\theta)} \\ \mu_2(t) &= \frac{\sqrt{1-\varepsilon^2}(a\sin A\theta + b\cos A\theta)}{1-\varepsilon\,(a\cos A\theta - b\sin A\theta)} \end{aligned}$$

But $\dot{\mu}_2 = \lambda\,\dot{x}^1$ so $x^1(\theta) = \frac{\mu_2}{\lambda} - a_1$. Since we are looking for geodesics starting from the origin we have $a_1 = \frac{\sqrt{1-\varepsilon^2}b}{\lambda(1-\varepsilon a)}$ and therefore

$$x^1(\theta) = \frac{\sqrt{1-\varepsilon^2}(a\sin A\theta + b\cos A\theta)}{\lambda\,(1-\varepsilon\,(a\cos A\theta - b\sin A\theta))} - \frac{\sqrt{1-\varepsilon^2}b}{\lambda(1-\varepsilon a)}$$

and from $\dot{\mu}_1 = -\lambda\,\dot{x}^2$ we obtain

$$x^2(\theta) = \frac{(1-\varepsilon^2)(b\sin A\theta - a\cos A\theta)}{\lambda\,(1-\varepsilon\,(a\cos A\theta - b\sin A\theta))} + \frac{(1-\varepsilon^2)a}{\lambda(1-\varepsilon a)}$$

Finally, from $\dot{x}^3 = \frac{x^1\dot{x}^2}{2} - \frac{\dot{x}^1 x^2}{2}$, by straightforward computation we get

$$x^3(\theta) = \frac{(1-\varepsilon^2)\sqrt{1-\varepsilon^2}A}{2\lambda^2(1-\varepsilon a)}\int \frac{1-\cos A\theta}{(1-\varepsilon\,(a\cos A\theta - b\sin A\theta))^2}d\theta$$

REMARK 5. *For $\varepsilon = 0$ we obtain the sub-Riemannian case (distributional systems with quadratic cost) with the solutions*

$$x(t) = \frac{a\sin\lambda t - b(1-\cos\lambda\theta)}{\lambda}$$

$$y(\theta) = \frac{b\sin\lambda t + a(1-\cos\lambda t)}{\lambda}$$

$$z(\theta) = \frac{\lambda t - \sin \lambda t}{2\lambda^2}, \quad a^2 + b^2 = 1$$

(see also [2],[14],[8]*).*

References

[1] Bao, D., Chern S.S., and Shen Z., *An introduction to Riemann-Finsler Geometry*, Springer-Verlag, New York, 2000.

[2] Bellaiche A., Risler J.J., (editors) *Sub-Riemannian geometry*, Birkhauser 144, 1996.

[3] Bryant, R., Hsu, L., *Rigidity of integral curves of rank 2 distributions*, Invent. Math. 114(1993), 435-461

[4] Brockett A., *Control Theory and Singular Riemannian Geometry,* New Directions in Applied Mathematics, Springer, Berlin 1981, 11-27.

[5] Chow, W.L., *Uber Systeme von linearen partiellen Differentialgleichungen erster Ordnung,* Math. Ann. 117 (1939), 98-105.

[6] Clelland J., Moseley C., *Sub-Finsler Geometry in dimension three*, preprint 2004

[7] Cortes, J., Martinez E., *Mecanical control systems on Lie algebroids*, preprint 2004

[8] Faizullin, R.R., *On the connection between the nonholonomic metric on the Heisenberg group and the Grushin metric*, Sibirsk.Mat.Zh.44(2003),1085-1090

[9] Hrimiuc, D., Shimada, H., *On the L-duality between Lagrange and Hamilton Manifold*, Nonlinear World, 3 (1996), 613-641.

[10] Martinez, E., *Lagrangian mechanics on Lie algebroids*, Acta Applicadae Mathematicae, 67, (2001), 295-320.

[11] Martinez, E., *Geometric formulation of mechanics on Lie algebroids*, Proc. of the VIII Workshop on Geometry and Physics (Medina del Campo, 1999), vol. 2 of Publ. R. Soc. Mat. Esp. (2001) 209-222.

[12] Mackenzie, K., *Lie grupoids and Lie algebroids in differential geometry*, 124 London Mathematical Society Lecture Note Series, Cambridge, 1987.

[13] Miron, R., Hrimiuc, D., Shimada, H., Sabau, S., *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, no. 118, 2001.

[14] Montgomery, R., *A Tour of Subriemannian Geometries, their Geodesics and Applications*, AMS, 91, 2002.

[15] Strichartz, R.S., *Subriemannian Geometry*, Journal of Differential Geometry, 24 (1984) 221-263.

[16] Weinstein, A., *Lagrangian mechanics and grupoids*, Fields Institute Communications, 7, 206-231, 1996.

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