DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
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$\mathbb{H}$-planar curves

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1. Summary

There are several geometric structures based on quaternions $\mathbb{H}$ (hypercomplex numbers). We will deal here with three of them: hypercomplex structures, almost quaternionic and Kähler quaternionic structures. These have the following structural groups $GL(n, \mathbb{H})$, $GL(n, \mathbb{H})Sp(1)$, $Sp(n)Sp(1)$. Almost quaternionic structure is the most general one, and the other two ones are its reductions, because $Sp(n)Sp(1) \subset GL(n, \mathbb{H})Sp(1)$ and $GL(n, \mathbb{H}) \subset GL(n, \mathbb{H})Sp(1)$.

Some geometries are parabolic, i.e. Cartan geometries of type (G, P) with a semisimple Lie group G and a parabolic subgroup P.

A class of curves with respect to a chosen affine connection ∇ is defined on a real manifold M endowed with an almost hypercomplex structure (I, J, K) in [**MNP**]. These are the curves $c : \mathbb{R} \to M$ whose covariant derivative $\nabla_{\dot{c}} \dot{c}$ along the whole curve c is lying in the vector space $\langle \dot{c}, I\dot{c}, J\dot{c}, K\dot{c} \rangle$. Here $\dot{c}$ is the tangent velocity vector field along the curve c, i.e. the curve $\dot{c} = \frac{\partial c}{\partial t}$ in TM (image of the unit vector field on $\mathbb{R}$ by Tc).

The almost hypercomplex structure is a reduction of an almost quaternionic structure to $GL(n, \mathbb{H})$ given by a choice of two complex units. The above defined class of curves does not depend on the choice (I, J, K) from the sphere given by this choice. This allows us to reformulate the definition of such curves also for almost quaternionic structures.

An almost quaternionic structure is already parabolic (we set $G = Sp(n + 1, \mathbb{H})$, $P = GL(n, \mathbb{H})Sp(1)$) and it is possible to use the technique of Weyl structures, [**ČS-03**]. Moreover the given structure is irreducible (the corresponding Lie algebra is $|1|-$ graded). We will show that the defined class of curves does not depend on the concrete choice of a Weyl connection. Moreover we shall see that we deal with all geodesics of all Weyl connections. Almost quaternionic structures have been discussed explicitly in connection with the distinguished curves on parabolic geometries, e.g. in [**ČSŽ**], [**Ž**] and [**BE**].

We have consistently used purely algebraic approach to geometries and this allows us to exploit similarity of the projective and quaternionic structures. We always suppose that M is a smooth manifold.

This paper is in final form and no version of it will be published elswhere.

2. Motivation and basic terms

2.1. Quaternionic geometry. A detailed description of various types of quaternionic geometries can be found in [**J**]. Here we remind some basic facts for the convenience of the reader.

Quaternions $\mathbb{H}$ are the asociative non-abelian real algebra $\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k\} \cong \mathbb{R}^4$. Imaginary quaternions are $Im\mathbb{H} = \langle i, j, k\rangle \cong \mathbb{R}^3$. Multiplication is given by: $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, $i^2 = j^2 = k^2 = -1$. For general $x = x_0 + x_1 i + x_2 j + x_3 k$, we define $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$ and $|x|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = x\bar{x}$.

By the identification of $\mathbb{H}^n$ with $\mathbb{R}^{4n}$ we get three complex structures I, J, K on $\mathbb{R}^{4n}$, which correspond to the left multiplication by numbers i, j, k on $\mathbb{H}^n$. This structure is called a **hypercomplex structure**.

A subgroup $GL(n, \mathbb{H}) \subset GL(4n, \mathbb{R})$ preserving I, J, K is called the **general quaternionic group**.

Lemma 1. *Let us choose certain hypercomplex structure I, J, K. Let $\bar{I}, \bar{J}, \bar{K}$ are such that $\bar{I} = a_0 + a_1 I + a_2 J + a_3 K$, $\bar{J} = b_0 + b_1 I + b_2 J + b_3 K$, $\bar{K} = c_0 + c_1 I + c_2 J + c_3 K$. Then for all $x = a, b, c$, $x_0 = 0$ and $x_1^2 + x_2^2 + x_3^2 = 1$.*

Proof. If we choose a different complex structure $\bar{I} = a_0 + a_1 I + a_2 J + a_3 K$, $\bar{J} = b_0 + b_1 I + b_2 J + b_3 K$, $\bar{K} = c_0 + c_1 I + c_2 J + c_3 K$, then the same relations must hold as for the original I, J, K. If we evaluate the equation $\bar{I}\bar{J} = -\bar{J}\bar{I}$ on the structure K, we obtain $a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0 = -(b_0 a_3 + b_1 a_2 - b_2 a_1 + b_3 a_0)$. After some calculations we get $a_0 b_3 + a_3 b_0 = -b_0 a_3 - b_3 a_0$ and finally $a_0 b_3 = -b_0 a_3$. If we collect coefficient of I, J, K we get $a_0 b_1 = -b_0 a_1$, $a_0 b_2 = -b_0 a_2$, $a_0 b_3 = -b_0 a_3$, respectively. At the same time the coefficient of the constant is $a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = 0$. If we suppose that $b_0 \neq 0$ we can express the coefficients a_1, a_2, a_3 and after evaluation and computation we get $a_0 b_0^2 + a_0 b_1^2 + a_0 b_2^2 + a_0 b_3^2 = 0$. Because we supposed that $b_0 \neq 0$ then inevitably $a_0 = 0$. But if $b_0 = 0$ we get the following equations $a_0 b_1 = 0$, $a_0 b_2 = 0$, $a_0 b_3 = 0$, which have two solutions. But $b_1 = b_2 = b_3 = 0$, doesn't satisfy the propery $\bar{J}^2 = -1$. Finally we have to come to $a_0 = 0$. This can be easily modified and we can get $b_0 = 0$ and $c_0 = 0$. If we use the next relation $\bar{I}^2 = -a_1^2 - a_2^2 - a_3^2 = -1$ we get, that the coefficients a_1, a_2, a_3 must form a sphere in $\mathbb{R}^3$. □

A simple observation is that the group $GL(n, \mathbb{H})$ remains the same for any chosen hypercomplex structure from given sphere. Indeed for $A \in GL(n, \mathbb{H})$ a $\bar{I}$ we have that $A\bar{I} = A(a_1 I + a_2 J + a_3 K) = a_1 AI + a_2 AJ + a_3 AK = (a_1 I + a_2 J + a_3 K)A = \bar{I}A$. The last but one equality is using the fact that $A \in GL(n, \mathbb{H})$.

It is natural to define a more general structure, such that instead of choosing certain hypercomplex structure I, J, K in each point we define the whole sphere $Z \subset Aut(\mathbb{R}^{4n})$, which we can obtain from some hypercomlex structure by the above procedure. This point of view can be found in [**AMP**]. Such structure is called **almost quaternionic structure**. When we look closer at the group of automorphism preserving Z, we can see that it got much bigger. Apart from the left action of $GL(n, \mathbb{H})$, we can rotate the sphere by right multiplying by the unit quaternion. By AXq^{-1} we shall denote the common action of $(A, q) \in GL(n, \mathbb{H}) \times Sp(1)$ on the element $X \in \mathbb{R}^{4n}$. But the group $GL(n, \mathbb{H}) \times Sp(1)$ is not the group of automorphisms we wanted, yet. It is a double covering, becouse $(-A, q)X = (A, -q)X$. We shall denote $GL(n,\mathbb{H})Sp(1) := GL(n,\mathbb{H}) \times_{\mathbb{Z}_2} Sp(1)$ the proper group of automorphisms, where $Sp(1)$ can be understood as the group of unit quaternions.

Using the matrices, we can write:

$$Ad\Big(\begin{pmatrix} q & 0 \\ 0 & A \end{pmatrix}\Big)\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & A \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ AXq^{-1} & 0 \end{pmatrix}.$$

Definition 1. *Let us have a manifold M. A reduction of a frame bundle on a smooth manifold M to the structure group $GL(n,\mathbb{H})Sp(1)$ is called an almost* quaternionic geometry.

Many information about the G-structure for structure group $G = Gl(n,\mathbb{H})Sp(1)$ can by found in [**S**]. As well as the construction of homogenous model of the G-structure as an n-dimensional quaternionic projective space $P^n\mathbb{H}$. Quaternionic structure is then curved version of a projective quaternionic structure.

2.2. F-planar curves. We remark that affinor structure on a manifold is a smooth tensor field of type $(1,1)$. The fundament of the following observations is the concept of F-planar curves and F-planar mappings on a manifold M.

If there is an affine conection ∇ and and affinor F_i^h on a manifold M, we can talk about F-planar curves. These are such curves $x^h = x^h(t) : \mathbb{R} \to M$, that the covariant derivation of their tangent vector $\nabla\lambda^h = \nabla\frac{dx^h}{dt}$ lies in the vector space determined by the tangent vector λ^h and its image $F_h^i\lambda^h$. Shortly we can write $\nabla\lambda^h \in \langle\lambda^h, F_j^h\lambda^j\rangle$ and the coordinate version is

$$\nabla\lambda^h = \frac{d\lambda^h}{dt} - \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \rho_1\lambda^h + \rho_2 F^h_\alpha\lambda^\alpha.$$

If we have two affine connections ∇, $\bar\nabla$ on a manifold M, we define a F-planar mapping $\gamma : M \to M$ as a mapping, which transforms the F-planar curves with respect to ∇ to the $\bar F$-planar curves with respect to $\bar\nabla$. The information about the F-planar curves together with the detailed description of the F-planar mappings can be found in [**MS**].

By a natural generalization we can come to the case when we have more affinors. There are three almost complex structures $\overset{1}{F}{}^h_\lambda := I, \overset{2}{F}{}^h_\lambda := J, \overset{3}{F}{}^h_\lambda := \overset{1}{F}{}^h_k \overset{2}{F}{}^k_\lambda$, on the almost hypercomlex structure. In the following by $\overset{0}{F}{}^h_\lambda$ we shall denote the identity.

Definition 2. *A unparameterized curve on M is* 4-planar *if there is a parameterization $\ell : x^h = x^h(t)$ such that the parallel translation of the tangent vector $\lambda^\alpha = \partial x^h/\partial t$ lies in four dimensional space generated by the tangent vector λ^α together with the vectors $\overset{1}{F}{}^h_\alpha\lambda^\alpha, \overset{2}{F}{}^h_\alpha\lambda^\alpha, \overset{3}{F}{}^h_\alpha\lambda^\alpha$.*

One can easily see from coordinate description

$$\nabla\lambda^h = \frac{d\lambda^h}{dt} - \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \rho_0\lambda^h + \rho_1\overset{1}{F}{}^h_\alpha\lambda^\alpha + \rho_2\overset{2}{F}{}^h_\alpha\lambda^\alpha + \rho_3\overset{3}{F}{}^h_\alpha\lambda^\alpha$$

that the defining condition is equivalent to $\nabla_{\dot c}\dot c \in \langle\dot c\rangle_{\mathbb{H}}$. In particular, this property clearly does not depend on the choice of the parameterization. Moreover, given a fixed hypercomplex structure from an almost quaternionic geometry on M, all other ones are given by linear combinations of I, J, and K. This proves the following lemma:

Lemma 2. *Curves which are 4-planar with respect to* $(\overset{1}{F}{}^h_\alpha, \overset{2}{F}{}^h_\alpha)$ *are also 4-planar with respect to all other hypercomplex structures* $(\overset{*1}{F}{}^h_\alpha, \overset{*2}{F}{}^h_\alpha)$ *from the given class.*

Thus the definition of the F-planar curves does not depend on the choice of the almost hypercomplex structure from the induced sphere. Let us also remind how to recognize the unparameterized geodesics:

Lemma 3. *Let* ∇ *be a connection on the manifold* M*. Then a curve is geodesic after a reparametrization if and only if* $\nabla_{\dot c}\dot c = \dot c \cdot k$*,* $k \in \Omega^0(M, \mathbb{R})$*.*

Proof. In any coordinate system x^h :
$(\nabla_{\dot c}\dot c)^h = (\ddot x)^h + \Gamma^h_{ij}\dot x^i \dot x^j$ where $c(t) = (x^h(t))$. For reparametrization $\bar c(t) = c \circ \varphi(t)$ we have $\bar x^h = x^h \varphi(t)$, after derivation by t we have $\dot{\bar x}^h = \dot x^h \dot\varphi$ a $\ddot{\bar x}^h = \ddot x^h \dot\varphi^2 + \dot x^h \ddot\varphi$. We substitute $(\nabla_{\dot{\bar c}}\dot{\bar c})^h = (\ddot{\bar x})^h + \Gamma^h_{ij}\dot{\bar x}^i \dot{\bar x}^j = \ddot x^h \dot\varphi^2 + \dot x^h \ddot\varphi + \dot\varphi^2 \Gamma^h_{ij}\dot x^i \dot x^j = \dot\varphi^2(\ddot x^h + \Gamma^h_{ij}\dot x^i \dot x^j) + \dot x^h(\ddot\varphi) = \dot\varphi^2(\dot x^h \cdot k) + \dot x^h \ddot\varphi = \dot x^h(\dot\varphi^2 \cdot k + \ddot\varphi)$. If we want this to be equal zero, we must have $\ddot\varphi + k \cdot \dot\varphi^2 = 0$. The general solution of this equation for smal δ_1, δ_2 is $\varphi = c_1 \int_{-\delta_1}^{\delta_1} \exp^{-\int_{-\delta_2}^{\delta_2} k dt} dt + c_2$ where $c_1 \neq 0$ and $c_1, c_2, \delta_1, \delta_2 \in \mathbb{R}$. □

3. Weyl structure for parabolic geometry

3.1. Weyl structures.

Definition 3. Parabolic geometry $(\mathcal{G}, \omega)$ of type (G/P) is a principal fiber bundle $\mathcal{G}$ with structure group P, equiped with a smooth one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying

(1) $\omega(\zeta_Z)(u) = Z$ for all $u \in \mathcal{G}$ and fundamental fields $\zeta_Z, Z \in \mathfrak{p}$
(2) $(r^b)^*\omega = Ad(b^{-1}) \circ \omega$ for all $b \in P$
(3) $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

Definition 4. Morphism *between parabolic geometries* $(\mathcal{G}, \omega)$ *and* $(\bar{\mathcal{G}}, \bar\omega)$ *is a principal fiber bundle morphism* $\varphi : \mathcal{G} \to \bar{\mathcal{G}}$ *satisfying* $\varphi^*\bar\omega = \omega$*.*

Definition 5. *Let* $(\mathcal{G} \to M, \omega)$ *be a parabolic geometry on the manifold* M *and* $\mathcal{G}_0 \to M$ *be the underlying frame bundle together with the projection* $\pi : \mathcal{G} \to \mathcal{G}_0$*. A* Weyl structure *for* $(\mathcal{G}, \omega)$ *is a global* G_0*-equivariant smooth section* $\sigma : \mathcal{G}_0 \to \mathcal{G}$ *of the projection* π*.*

In the case of the irreducible parabolic geometries the Cartan connection splits into three parts $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$. The σ-pullback of each of these components is an important part of the Weyl geometry structure. The *canonical form* of the G_0-structure is $\sigma^*\omega_{-1} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1})$. The *Rho-tensor* is $\mathrm{P} := \sigma^*\omega_1 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_1)$ and the *Weyl connection* is $\gamma^\sigma = \sigma^*\omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$.

There are several significant associated bundles connected to the parabolic geometry. Using the left multiplication we can easily associate the Lie groups G and the factor G/P to $\mathcal{G}$ as a principle bundle with a structure group P. We can do the same with the Lie algebra $\mathfrak{g}$ and vector space $\mathfrak{g}/\mathfrak{h}$ using the action Ad and $\underline{Ad}$. We get associated bundles:

(1) The principal bundle $\tilde{\mathcal{G}} \cong \mathcal{G} \times_P G$ with the structure group G and the principal conection, which is induced from Cartan conection.
(2) The **Cartan space** $S\mathcal{G} \cong \mathcal{G} \times_P G/P$
(3) The **tangent bundle** $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{h}$ on the manifold M.

(4) The **adjoint tractor bundle** $\mathcal{A} \cong \mathcal{G} \times_P \mathfrak{g}$.

The following well known lemma about the one to one correspondence between the sections and the equivariant mappings is crucial.

LEMMA 4. *For asociated bundle $\mathcal{G} \times_H V$ to the principal bundle $\mathcal{G}$ with structure group H there is one to one correspondence between the smooth sections and H-equivariant smooth functions $\mathcal{G} \to V$.*

PROOF. For $x \in M$ the section s of $\mathcal{G} \times_H V$ determine the element $\{s(x), \bar{s}(s(x))\} \in \mathcal{G} \times_H V$ and function $\bar{s} : \mathcal{G} \to V$. Using property $\{u, g\} = \{uh, h^{-1}g\}$ we get $\bar{s}$ and finaly we get $\bar{s}(uh) = h^{-1}\bar{s}(u)$. Conversely, for H-equivariant function $\bar{s} : \mathcal{G} \to V$ we can define section of $\mathcal{G} \times_H V$ in this way: We choose section σ on $\mathcal{G} \to M$ and define $s(x) = \{\sigma(x), \bar{s}(\sigma(x))\}$. □

So for the case (4) we can induce the Lie bracket from $\mathfrak{g}$. Since the Lie bracket is Ad-equivariant, we can naturally define an algebraic bracket on $\mathcal{A}$:

$$\{,\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}.$$

One can find more information about Weyl structures in [**ČS-03**], [**ČS**] or in [**Ž**].

Any two Weyl connections differ by a one-form Υ. This means, given a G_0- equivariant function $\Upsilon : \mathcal{G}_0 \to \mathfrak{g}_1$, we get another Weyl structure $\bar{\sigma}(u) = \sigma(u) \exp(\Upsilon(u))$ and we denote it by $\bar{\sigma} = \sigma + \Upsilon$ (see [**ČS-03**], Theorem 3.1).

Every principal connection induces a linear connection. In the case of Weyl structure σ we get the principal connection $\sigma^*\omega_0$ and the corresponding affine connection ∇^σ. It is defined by $\nabla^\sigma_\xi s = Ts \cdot \gamma^\sigma(\xi)$. Here γ^σ is a horizontal lift of the corresponding principal connection $\sigma^*\omega_0$ and s is a vector field. Any choice of the Weyl strucure σ reduces its structure group to G_0 and identifies $TM \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$, this leads to the following claim, see [**ČS-03**].

LEMMA 5. *Let $(\mathcal{G}, \omega)$ be a parabolic geometry. Then there exist Weyl-structures. If σ and $\bar{\sigma}$ are two Weyl structures, then there is a unique smooth section $\Upsilon \in \mathcal{A}$ such that*

$$\nabla^{\sigma+\Upsilon}_\xi s = \nabla^\sigma_\xi s + \{\{\xi, \Upsilon\}, s\}.$$

PROOF. See [**ČS-03**]. □

3.2. Almost quaternionic structures. Our case is the real form $\mathfrak{sl}(n+1, \mathbb{H})$ of the complex algebra $\mathfrak{gl}(2n+2, \mathbb{C})$. Here is the corresponding Satake diagram:

$$\bullet\!-\!\circ\!-\!\bullet\!-\cdots-\!\bullet\!-\!\circ\!-\!\bullet$$

We obtain the appropriate parabolic subalgebra $\mathfrak{p}$ by setting $|1|$-gradation by crossing the second (first noncompact) root, i.e.

$$\bullet\!-\!\times\!-\!\bullet\!-\cdots-\!\bullet\!-\!\circ\!-\!\bullet$$

Let $\mathfrak{g} := \mathfrak{sl}(n+1, \mathbb{H})$ be the Lie algebra of quaternionic $(n+1) \times (n+1)$ matrices with zero real part of the trace. The parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ obtained by crossing of the second simple root is the stabilizer of the quaternionic line generated by the first vector from the standard basis. It is described by the following matrix:

$$\mathfrak{g} = \left\{ \begin{bmatrix} a & Z \\ X & A \end{bmatrix} \middle| X, Z^\top \in \mathbb{H}^n, a \in \mathbb{H}, A \in Mat_n(\mathbb{H}),\ \mathrm{Re}(a) +\ \mathrm{Re}(\mathrm{tr}(A)) = 0 \right\}.$$

The corresponding gradation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ looks like:

$$\begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix} \in \mathfrak{g}_{-1}, \begin{bmatrix} a & 0 \\ 0 & A \end{bmatrix} \in \mathfrak{g}_0, \begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix} \in \mathfrak{g}_1.$$

We construct the group $SL(n+1,\mathbb{H})$ of all invertible quaternionic linear endomorphisms of $\mathbb{H}^{n+1}$, which we can consider as endomorphisms of $\mathbb{R}^{4n+4}$ with determinant equal one. We define $G := PSL(n+1,\mathbb{H})$ as the quotient of $SL(n+1,\mathbb{H})$ by its center $\{\pm id\}$. As a parabolic subgroup we obtain quotient of the stabilizer of the quaternionic line generated by the first vector from the standard basis:

$$G_0 := \Big\{ \begin{bmatrix} \gamma & 0 \\ 0 & \varphi \end{bmatrix} \Big| 0 \neq \gamma \in \mathbb{H}, \varphi \in GL(n,\mathbb{H})\colon |\gamma|^4 \cdot det(\varphi) = 1 \Big\}$$

$$P := \Big\{ \begin{bmatrix} \gamma & \rho \\ 0 & \varphi \end{bmatrix} \Big| 0 \neq \gamma \in \mathbb{H}, \rho \in \mathbb{H}^n, \varphi \in GL(n,\mathbb{H})\colon |\gamma|^4 \cdot det(\varphi) = 1 \Big\}$$

For complete description see 4.1.4. in [**ČS**].

Although one cannot define multiplication by the quaternionic scalars, we are able to recognize quaternionically colinear vectors in the tangent bundle. Indeed, an obvious choice is to use the coordinate (right) multiplication of a quaternionic collumn vector in $\mathfrak{g}_{-1}$ by a quaternion. Since $Ad(A,q)(X\cdot r) = (Ad(A,q)(X))\cdot r$ does not hold for all $X \in T_xM \cong \mathfrak{g}_{-1}, (A,q) \in G$, this does not define an operation on TM. However, for a vector $X \in T_xM \cong \mathfrak{g}_{-1} \cong \mathbb{H}^n$ a change of a basis $(a,A) \in G_0$ leeds to a vector $\bar{X} = Ad(a,A)(X) = AXa^{-1}$. Thus, for its right quaternionic multiple $Y = Xq$ we have $\bar{Y} = Ad(a,A)(Xq) = AXqa^{-1}$. The choice yields $\bar{q} = aqa^{-1}$, we have $\bar{X}\bar{q} = \bar{Y}$.

4. $\mathbb{H}$-planar curves

Let us have a manifold M endowed with an almost quaternionic structure. Let us remind, that a curve $c : \mathbb{R} \to M$ is called a geodesic up to reparametrization, if it suffices $\nabla_{\dot c}\dot c \in \langle \dot c\rangle_{\mathbb{R}}$ and it is called a 4-planar curve, if is suffices $\nabla_{\dot c}\dot c \in \langle \dot c\rangle_{\mathbb{H}}$. We showed that even thought we haven't defined the multiplication by a quaternion, we can write $\nabla_{\dot c}\dot c = \dot c \cdot q$ where the value of $q \in \mathbb{H}$ depends on the choice of a reduction to the hypercomplex structure.

THEOREM 1. *A curve which is 4-planar with respect to a Weyl structure σ is also 4-planar with respect to all Weyl structures.*

PROOF. For technical reasons we shall work in chosen coordinates and in chosen reduction on TM and use the right multiplication by quaternion. Therefore for $X \in T_xM \cong \mathfrak{g}_{-1}$ and $q \in \mathbb{H}$, $X \cdot q$ will denote the multiplication of a column of quaternions by a quaternion. This is not invariant because the multiplication is not $Ad(G_0)$-equivariant, but it is sufficient to see that $Ad(a,A)(Xq) \in \langle Xq\rangle_{\mathbb{H}}$.

Let us compute the iterated Lie bracket:

$$\Big[\big[\dot c, \Upsilon\big], \dot c\Big] = \Big[\Big[\begin{bmatrix} 0 & 0 \\ \dot c & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Upsilon \\ 0 & 0 \end{bmatrix} \Big], \begin{bmatrix} 0 & 0 \\ \dot c & 0 \end{bmatrix} \Big] = \Big[\begin{bmatrix} -\Upsilon\dot c & 0 \\ 0 & \dot c\Upsilon \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \dot c & 0 \end{bmatrix} \Big] =$$

$$= \begin{bmatrix} 0 & 0 \\ 2\dot c \cdot \Upsilon(\dot c) & 0 \end{bmatrix} = 2\dot c \cdot \Upsilon(\dot c).$$

Hence if c is a 4-planar curve corresponding to σ, i.e. $\nabla^{\sigma}_{\dot{c}}\dot{c} = \dot{c}\cdot q$, we have:

$$\nabla^{\sigma+\Upsilon}_{\dot{c}}\dot{c} = \nabla^{\sigma}_{\dot{c}}\dot{c} + \Big[\Big[\dot{c},\Upsilon\Big],\dot{c}\Big] = \dot{c}\cdot q + \dot{c}\cdot 2\Upsilon(\dot{c}) = \dot{c}\cdot(q + 2\Upsilon(\dot{c})).$$

□

The following definition does not depend on the structure of σ.

If we choose an arbitrary Weyl structure σ, 4-planar curves of an induced linear connection ∇^{σ} are always the same. So the following deffinition makes sence.

DEFINITION 6. *A curve $c:\mathbb{R}\to M$ is called* ℍ-planar *if some Weyl connection σ satisfies:*

$$\nabla^{\sigma}_{\dot{c}}\dot{c} \in \langle\dot{c}\rangle_{\mathbb{H}}.$$

THEOREM 2. *The ℍ-planar curves are exactly all unparameterized geodesics of all Weyl connections.*

PROOF.

(1) Let us begin with the implication saying that for every ℍ-planar curve there is a Weyl structure σ, such that this curve is a geodesic of the linear connection ∇^{σ}. Let σ be a Weyl structure, we need to find such Υ, that $\nabla^{\sigma+\Upsilon}_{\dot{c}}\dot{c} = \dot{c}\cdot k$, where $k\in\Omega^0(M,\mathbb{R})$. We have seen in the proof of the previous theorem

$$\nabla^{\sigma+\Upsilon}_{\dot{c}}\dot{c} = \nabla^{\sigma}_{\dot{c}}\dot{c} + [[\dot{c},\Upsilon],\dot{c}] = \dot{c}\cdot q + \dot{c}\cdot 2\Upsilon(\dot{c}) = \dot{c}\cdot(q + 2\Upsilon(\dot{c})).$$

So we need the values of $q+\Upsilon(\dot{c})$ to be real. But we can easily reach this with the choice of Υ satisfying $2\Upsilon(\dot{c}) = -\bar{q}$ along the curve c.

(2) We need to show that every unparameterized geodesic of any Weyl connection is ℍ-planar. Every geodesic satisfies

$$\nabla^{\sigma}_{\dot{c}}\dot{c} = \dot{c}\cdot k,\ k\in\Omega^0(M,\mathbb{R})$$

and this is exactly what we required together with the property $k\in\Omega^0(M,\mathbb{R})$.

□

Summerizing, we have shown that the ℍ-planar curves are exactly the unparameterized geodesics of all linear connection with the fixed minimal torsion which are invariant with respect to the given almost quaternionic structure. So we can see that this a considerably large class curves.

5. Remark

In future work, I want to discuss the morphisms of almost quaternionic structures in relation with those mappings which send ℍ-planar curves to ℍ-planar curves. These mappings were explicitly described in [**MNP**] for hypercomplex structures.

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