DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 151 – 155

An alternative proof of the Bryant representation

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1. Introduction

In his famous 1987 paper, R. Bryant [**Bry**] established a meromorphic representation for the surfaces with constant mean curvature one in the hyperbolic 3-space $\mathbb{H}^3$. Afterwards, in 1993 M. Umehara and K. Yamada [**UY**] gave the definitive shape to the basis of this theory. Since then the study of such surfaces has become one of the topics of most actuality in submanifold geometry and has received a great number of important contributions (see [**CHR, HRR, UY2, UY3**] for instance).

The objective of this short note is to present an alternative proof of the Bryant representation in [**Bry**]. Our motivation comes from two facts.

On the one hand, the original proof by Bryant uses the techniques and notations of the general theory of submanifolds. However, these tools have not been used in the later study of mean curvature one surfaces in $\mathbb{H}^3$. In contrast, the proof that we propose here uses standard techniques of the theory.

On the other hand, the availability of a meromorphic representation for a class of surfaces is of great importance, since the powerful theorems from complex analysis allow to study very precisely the global geometry of such surfaces. So, a research line with an important number of contributions in the last few years has been to generalize the Bryant representation to other geometric theories (see [**AA, AA2, GMM, KTUY**] for instance). In this sense, the usefulness of an alternative proof of such representation is obvious, since it can suggest further generalizations of the theory.

The methods used in the present proof of the Bryant representation have been used in a more elaborated way in [**GMM, GaMi**]

2. Setup

Before dealing with the Bryant representation, we include here for the reader's convenience some basic notions about surface theory in hyperbolic 3-space. These concepts can be consulted in most references on mean curvature one surfaces in $\mathbb{H}^3$.

Let $\mathbb{L}^4$ be the Minkowski 4-space, that is, $\mathbb{R}^4$ with canonical coordinates (x_0, x_1, x_2, x_3) and the Lorentzian metric $\langle,\rangle = -dx_0^2+dx_1^2+dx_2^2+dx_3^2$. Then the hyperbolic 3-space is viewed as $\mathbb{H}^3 = \{x \in \mathbb{L}^4 : \langle x, x\rangle = -1, x_0 > 0\}$. We will use the Hermitian model

2000 *Mathematics Subject Classification.* 53C42.
Key words and phrases. constant mean curvature, Bryant surfaces, Liouville equation.
This paper is in final form and no version of it will be published elswhere.

for $\mathbb{L}^4$, i.e. we identify $\mathbb{L}^4 \equiv \mathrm{Herm}(2)$ as

$$(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathrm{Herm}(2).$$

The metric $\langle , \rangle$ on this model is determined by $\langle m, m\rangle = -\det(m)$ for all $m \in \mathrm{Herm}(2)$. In addition, the complex Lie group $\mathbf{SL}(2,\mathbb{C})$ acts on $\mathbb{L}^4$ through the isometric and orientation-preserving action

$$\Phi \in \mathbf{SL}(2,\mathbb{C}) \mapsto \Phi \cdot m = \Phi m \Phi^*, \quad m \in \mathrm{Herm}(2), \ \ \Phi^* = \bar{\Phi}^t.$$

This implies that the hyperbolic 3-space may be regarded as $\mathbb{H}^3 = \{\Phi\Phi^* : \Phi \in \mathbf{SL}(2,\mathbb{C})\}$. Besides, the positive null cone $\mathbb{N}^3 = \{x \in \mathbb{L}^4 : \langle x, x\rangle = 0, x_0 > 0\}$ is seen in the Hermitian model as

$$\mathbb{N}^3 = \{w\overline{w}^t : w^t = (w_1, w_2) \in \mathbb{C}^2 \setminus (0,0)\} \subset \mathrm{Herm}(2).$$

Here the vector $w \in \mathbb{C}^2$ is defined up to multiplication by $\lambda \in \mathbb{C}$, $|\lambda| = 1$. The space of null lines of $\mathbb{L}^4$ is the quotient $\mathbb{N}^3/\mathbb{R}^+$, which can be regarded as the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$, and is then identified with $\mathbb{C} \cup \{\infty\}$ by

$$[(x_0, x_1, x_2, x_3)] \in \mathbb{N}^3/\mathbb{R}^+ \longleftrightarrow \frac{x_1 + ix_2}{x_0 - x_3} \in \mathbb{C} \cup \{\infty\} \equiv \mathbb{S}^2_\infty.$$

In the Hermitian model, the projection $\mathbb{N}^3 \to \mathbb{N}^3/\mathbb{R}^+ \equiv \mathbb{C} \cup \{\infty\}$ becomes $w\overline{w}^t \to w_1/w_2$, for $w = (w_1, w_2)$.

Consider now an immersed oriented surface $\psi : \Sigma \to \mathbb{H}^3$, where Σ is endowed with the Riemann surface structure associated to its induced metric. Let also $\eta : \Sigma \to \mathbb{L}^4$ be its unit normal in $\mathbb{H}^3$, so that $\langle \eta, \eta\rangle = 1$ and $\langle \psi, \eta\rangle = 0$. Then we can consider $\psi + \eta : \Sigma \to \mathbb{N}^3$, and thus define the ***hyperbolic Gauss map*** $G : [\psi + \eta] : \Sigma \to \mathbb{C} \cup \{\infty\}$. The following Lemma shows the importance of mean curvature one surfaces in $\mathbb{H}^3$:

LEMMA 1 (Bryant). *A surface $\psi : \Sigma \to \mathbb{H}^3$ has mean curvature one if and only if its hyperbolic Gauss map $G : \Sigma \to \mathbb{C} \cup \{\infty\}$ is meromorphic.* The surfaces in $\mathbb{H}^3$ with constant hyperbolic Gauss map are the horospheres. These are the only totally umbilical surfaces with mean curvature one in $\mathbb{H}^3$.

3. The Bryant representation

THEOREM 1 (Bryant representation). *Let $\mathcal{B} : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ be a holomorphic immersion from a Riemann surface Σ into $\mathbf{SL}(2,\mathbb{C})$ with $\det(d\mathcal{B}) = 0$. Then the map $\psi : \Sigma \to \mathbb{H}^3$ given by $\psi = \mathcal{B}\mathcal{B}^*$ is a mean curvature one surface.*

Conversely, all simply connected mean curvature one surfaces in $\mathbb{H}^3$ are constructed in this way.

PROOF. Let $\psi : \Sigma \to \mathbb{H}^3$ be a simply connected Bryant surface, and take a global complex parameter z on Σ so that $\langle d\psi, d\psi\rangle = \lambda|dz|^2$. The Hopf differential of the surface is $Q = q(z)dz^2$ where $q(z) = \langle \psi_{zz}, \eta\rangle$ and $\eta : \Sigma \to \mathbb{S}^3_1$ is the unit normal to ψ. As ψ has constant mean curvature, Q is a holomorphic 2-form on Σ, i.e. $q : \Sigma \to \mathbb{C}$ is holomorphic. If $q = 0$, the surface is a horosphere. From now on we will assume that $q \not\equiv 0$, and so q vanishes only at isolated points.

Besides, the Gauss equation for ψ indicates that

$$(\log \lambda)_{z\bar{z}} = 2|q|^2/\lambda. \tag{1}$$

Since q is holomorphic, by (1) we get that $\varphi = 4|q|^2/\lambda$ verifies Liouville's equation

$$\Delta \log \varphi = -2\varphi. \tag{2}$$

This equation can be explicitly solved (the resolution goes back to Liouville [**Lio**], but it can also be found in [**Bry**], for instance). So, there is a meromorphic function $g : \Sigma \to \mathbb{C} \cup \{\infty\}$ such that

$$\varphi = \frac{4|q|^2}{\lambda} = \frac{4|g_z|^2}{(1+|g|^2)^2}. \tag{3}$$

In addition, by the structure equations of ψ and the condition $H = 1$ it holds

$$(\psi + \eta)_z = -\frac{2q}{\lambda}\psi_{\bar{z}}. \tag{4}$$

As the hyperbolic Gauss map $G = [\psi + \eta]$ of the Bryant surface is meromorphic, there exist $A, B : \Sigma \to \mathbb{C}$ holomorphic functions with $G = [(A, B)]$, and a positive smooth function $\mu : \Sigma \to \mathbb{R}^+$ so that

$$\psi + \eta = \mu \begin{pmatrix} A\bar{A} & A\bar{B} \\ \bar{A}B & B\bar{B} \end{pmatrix}. \tag{5}$$

Therefore,

$$\langle (\psi+\eta)_z, (\psi+\eta)_{\bar{z}} \rangle = \frac{\mu^2}{2}|AB_z - BA_z|^2. \tag{6}$$

Now, putting together (3), (4) and (6) we get

$$\mu^2 |AdB - BdA|^2 = \frac{4|dg|^2}{(1+|g|^2)^2}. \tag{7}$$

From this relation, and as Σ is simply connected, there exists a meromorphic function $S : \Sigma \to \mathbb{C} \cup \{\infty\}$ verifying $S^2 = dg/(AdB - BdA)$. Consider now $C = AS$, $D = BS$. Then $G = [(C, D)]$, and by repeating the above computations with C, D instead of A, B we see that $CdD - DdC = dg$, and that (5) turns into

$$\psi + \eta = \varrho \begin{pmatrix} C\bar{C} & C\bar{D} \\ \bar{C}D & D\bar{D} \end{pmatrix} \tag{8}$$

for

$$\varrho = \frac{2}{1+|g|^2}. \tag{9}$$

Consider next the meromorphic curve $F : \Sigma \to \mathbf{SL}(2, \mathbb{C})$ given by

$$F = \begin{pmatrix} C & dC/dg \\ D & dD/dg \end{pmatrix}.$$

Then (8) writes down in terms of F as

$$\psi + \eta = F \begin{pmatrix} \varrho & 0 \\ 0 & 0 \end{pmatrix} F^*. \tag{10}$$

As there exists a meromorphic function θ on Σ such that

$$F^{-1}F_z = \begin{pmatrix} 0 & \theta \\ g_z & 0 \end{pmatrix}, \tag{11}$$

using (9), (11) and (10) we obtain

$$(\psi+\eta)_z = F\left[\mathcal{A}\begin{pmatrix} \varrho & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \varrho_z & 0 \\ 0 & 0 \end{pmatrix}\right]F^* = F\begin{pmatrix} \dfrac{-2\bar{g}g_z}{(1+|g|^2)^2} & 0 \\ \dfrac{2g_z}{1+|g|^2} & 0 \end{pmatrix}F^*.$$

Now, by (4) and (3), if we denote $h = q/g_z$, we get

$$\psi_z = F\begin{pmatrix} gh & -h(1+|g|^2) \\ 0 & 0 \end{pmatrix}F^*. \tag{12}$$

Besides, as $\mathbf{SL}(2,\mathbb{C})$ acts isometrically on $\mathbb{H}^3$, there exists a curve $\Omega : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ such that $\psi = F\Omega F^*$. If we write

$$\Omega = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

being $a, b : \Sigma \to \mathbb{R} \cup \{\infty\}$ and $b : \Sigma \to \mathbb{C} \cup \{\infty\}$, then by (12) we obtain the system

$$\begin{pmatrix} a_z + \bar{b}\theta & b_z + c\theta \\ \bar{b}_z + ag_z & c_z + bg_z \end{pmatrix} = \begin{pmatrix} gh & -h(1+|g|^2) \\ 0 & 0 \end{pmatrix}. \tag{13}$$

This differential system can be easily solved using the condition $\det(\Omega) = 1$, to obtain

$$\Omega = \begin{pmatrix} 1 & -\bar{g} \\ -g & 1+|g|^2 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & -ig \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & i\bar{g} \end{pmatrix}. \tag{14}$$

Therefore $\psi : \Sigma \to \mathbb{H}^3$ can be recovered as $\psi = \mathcal{B}\mathcal{B}^*$, being $\mathcal{B} : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ the meromorphic curve

$$\mathcal{B} = F\begin{pmatrix} 0 & i \\ i & -ig \end{pmatrix}. \tag{15}$$

Moreover, by $\psi = \mathcal{B}\mathcal{B}^*$ we see that $\mathcal{B} : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ is actually holomorphic, and a straightforward computation shows that $\mathcal{B}$ is an immersion and $\det(d\mathcal{B}) = 0$.

Conversely, let $\mathcal{B} : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ be a holomorphic immersion with $\det(d\mathcal{B}) = 0$. Then $\mathcal{B}^{-1}d\mathcal{B}$ has vanishing determinant and trace (as it takes its values in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$). This implies the existence of a meromorphic function g and a holomorphic 1-form ω so that

$$\mathcal{B}^{-1}d\mathcal{B} = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega. \tag{16}$$

Define finally $\psi = \mathcal{B}\mathcal{B}^* : \Sigma \to \mathbb{H}^3$, which is an immersion because so is $\mathcal{B}$, and let $N : \Sigma \to \mathbb{N}^3$ be

$$N = \frac{2}{1+|g|^2}F\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}F^*,$$

where F is given by (15). Then $\langle N, N\rangle = 0$, $\langle N, \psi\rangle = -1$, and by (12), $\langle \psi_z, N\rangle = 0$. At last, if η is the unit normal to ψ, it holds $N = \psi + \eta$. As N is conformal and F is holomorphic, ψ has mean curvature one by Lemma 1. This ends up the proof.

□

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